Parity Edge-Coloring of Graphs

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Parity Vectors

Consider a graph $G$ whose edges $E(G)$ are assigned colors from a set $C$. Let $f : E(G) \rightarrow C$ denote the coloring.
Parity Vectors

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- Let $W$ be a walk in $G$. The parity vector $\pi_f(W)$ records, for each $c \in C$, the parity of the number of times $W$ traverses an edge with color $c$. 

![Graph with vertices $V_1, V_2, V_3, V_4, V_5$ and edges connecting them with different colors.](image)
Consider a graph $G$ whose edges $E(G)$ are assigned colors from a set $C$. Let $f : E(G) \rightarrow C$ denote the coloring.

Let $W$ be a walk in $G$. The parity vector $\pi_f(W)$ records, for each $c \in C$, the parity of the number of times $W$ traverses an edge with color $c$.

We also abuse notation and use $\pi_f(W)$ as the set of colors that appear an odd number of times in $W$. 
Parity Vectors

Example

\[ W = v_1 v_2 v_5 v_1 v_3 v_2 \]

\[ \pi(W) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]
Parity Vectors

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\[ \mathcal{W} = v_1 v_2 v_5 v_1 v_3 v_2 \]

\[ \pi(\mathcal{W}) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \]
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Example

\[ W = v_1 v_2 v_5 v_1 v_3 v_2 \]

\[ \pi(W) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \]

= \{ \text{blue, red, yellow} \}
Definition

A parity walk is a walk $W$ with $\pi(W) = \vec{0}$.
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- Parity walks can be closed ...
Parity Vectors

Definition

A parity walk is a walk $W$ with $\pi(W) = \overrightarrow{0}$.

- Parity walks can be closed ...
- ... or open.
Hypercubes and Parity Walks

Definition

The hypercube $Q_k$ is the graph with vertex set $\{0, 1\}^k$ with an edge between $u$ and $v$ iff $u$ and $v$ differ in 1 coordinate.
Theorem (Havel, Movárek (1972))

Let $G$ be a connected graph. $G$ is a subgraph of $Q_k$ iff there is an edge-coloring of $G$ using at most $k$ colors such that

$$\forall W \quad W \text{ is a parity walk} \iff W \text{ is closed}$$
Introduction

Hypercubes and Parity Walks

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- Some graphs (e.g. odd cycles, $K_{2,3}$) are not subgraphs of any hypercube
- All graphs have an edge-coloring in which every parity walk is closed
Theorem (Havel, Movárek (1972))

Let $G$ be a connected graph. $G$ is a subgraph of $Q_k$ iff there is an edge-coloring of $G$ using at most $k$ colors such that

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Definition

A strong parity edge-coloring (spec) is an edge-coloring such that

\[ \forall W \quad W \text{ is a parity walk } \implies W \text{ is closed.} \]

The strong parity edge chromatic number $\hat{p}(G)$ is the least $k$ such that $G$ has a spec using only $k$ colors.
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The strong parity edge chromatic number $\hat{p}(G)$ is the least $k$ such that $G$ has a spec using only $k$ colors.

Corollary

If $T$ is a tree, $\hat{p}(T)$ is the least $k$ so that $T \subseteq Q_k$. 
What is $\hat{\rho}(K_n)$?

Example

- $\hat{\rho}(K_1) = 0$

Proposition

If $n = 2^k$, then $\hat{\rho}(K_n) = n - 1$.

Proof.

Label the vertices from $\{0, 1\}$ and color an edge $uv$ with $u + v$. We call this the canonical coloring.
What is $\hat{\rho}(K_n)$?

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- $\hat{\rho}(K_5) \in \{4, 5, 6, 7\}$

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Label the vertices from $\{0, 1\}^k$ and color an edge $uv$ with $u + v$. We call this the canonical coloring.
Main Theorem

Theorem

\[ \hat{\rho}(K_n) = 2^{\lceil \log n \rceil} - 1 \]
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Lemma (Augmentation)

If \( n \) is not a power of two, then \( \hat{p}(K_n) = \hat{p}(K_{n+1}) \).
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*If* \( n \) *is not a power of two, then* \( \hat{p}(K_n) = \hat{p}(K_{n+1}) \).

- Strategy: add vertex, color new edges without introducing an open parity walk.
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Lemma (Augmentation)

*If n is not a power of two, then* \( \hat{p}(K_n) = \hat{p}(K_{n+1}) \).

- Strategy: add vertex, color new edges without introducing an open parity walk.
- We have a lot to worry about.
Spec Characterization Lemma

Lemma (Spec Characterization)

Fix an edge-coloring of $K_n$. There is an open parity walk iff there is a closed walk $W$ with $|\pi(W)| = 1$. 
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Proof.

$(\Rightarrow)$.

- Let $W'$ be an open parity $uv$-walk
**Lemma (Spec Characterization)**

Fix an edge-coloring of $K_n$. There is an open parity walk iff there is a closed walk $W$ with $|\pi(W)| = 1$.

**Proof.**

$(\implies)$. Let $W'$ be an open parity $uv$-walk. Let $W = W'vu$.
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**Lemma (Spec Characterization)**

*Fix an edge-coloring of $K_n$. There is an open parity walk iff there is a closed walk $W$ with $|\pi(W)| = 1$.***

**Proof.**

($\Rightarrow$).

- Let $W'$ be an open parity $uv$-walk.
- Let $W = W'vu$.
- $\pi(W) = \{a\}$. 
Lemma (Spec Characterization)

Fix an edge-coloring of $K_n$. There is an open parity walk iff there is a closed walk $W$ with $|\pi(W)| = 1$.

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$K_n$
Lemma (Spec Characterization)

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- Let $W$ be a closed walk with $\pi(W) = \{a\}$
- Let $vu$ be an edge in $W$ of color $a$
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- Let $W'$ be the $uv$-walk obtained by removing $vu$. 
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- Let $W$ be a closed walk with $\pi(W) = \{a\}$.
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- Let $W'$ be the $uv$-walk obtained by removing $vu$.
- $W'$ is an open parity walk.
**Spec Characterization Lemma**

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Fix an edge-coloring of $K_n$. There is an open parity walk iff there is a closed walk $W$ with $|\pi(W)| = 1$.

Augmentation only worries about introducing closed walks $W$ with $|\pi(W)| = 1$.
Lemma (Spec Characterization)

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- Augmentation only worries about introducing closed walks $W$ with $|\pi(W)| = 1$
- Linear algebra means we can worry even less!
The Parity Space

Proposition

Let $f$ be an edge-coloring of a connected graph $G$. The parity space of $f$ is

$$L_f = \{ \pi(W) : W \text{ is closed} \}.$$  

$L_f$ is a linear subspace of $\mathbb{F}_2^k$. 

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Proof.

- Let $W = u$. $\pi(W) = \overrightarrow{0} \in L_f$
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- Let $W_1, W_2$ be closed walks

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Proof.

- Let $W_1, W_2$ be closed walks
- Let $P$ be a path from $W_1$ to $W_2$
- Let $W = W_1PW_2\overline{P}$
- $\pi(W) = \pi(W_1) + \pi(W_2) \in L_f$
Lemma (Span Lemma)

Let $f$ be an edge-coloring of a graph $G$ with a dominating vertex $v$. Then

$$\{\pi(T) : T \text{ is a triangle containing } v\}$$

spans $L_f$. 

Augmentation only worries about triangles at $v$.

Attack from other direction

Argue $K_n$ has a rich parity space, before augmentation.
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Let $f$ be an edge-coloring of a graph $G$ with a dominating vertex $v$. Then

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A Parity Space Spanning Set

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- Attack from other direction
- Argue $K_n$ has a rich parity space, before augmentation
Lemma (Triple Color Lemma)

Let $f$ be a minimum spec of $K_n$. Then for every pair of colors $\{a, b\}$, there is a third color $c$ and a closed walk $W$ with $\pi(W) = \{a, b, c\}$.
Lemma (Triple Color Lemma)

Let $f$ be a minimum spec of $K_n$. Then for every pair of colors $\{a, b\}$, there is a third color $c$ and a closed walk $W$ with $\pi(W) = \{a, b, c\}$.

Proof.

- Collapse $a$ and $b$ to new color $d$ to form coloring $g$.
Lemma (Triple Color Lemma)

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Proof.

- Collapse $a$ and $b$ to new color $d$ to form coloring $g$
- $g$ is not a spec
**Triple Color Lemma**

**Lemma (Triple Color Lemma)**

Let $f$ be a minimum spec of $K_n$. Then for every pair of colors $\{a, b\}$, there is a third color $c$ and a closed walk $W$ with $\pi(W) = \{a, b, c\}$.

**Proof.**

- Collapse $a$ and $b$ to new color $d$ to form coloring $g$
- $g$ is not a spec
- Let $W'$ be a parity $uv$-walk
**Lemma (Triple Color Lemma)**

Let $f$ be a minimum spec of $K_n$. Then for every pair of colors $\{a, b\}$, there is a third color $c$ and a closed walk $W$ with $\pi(W) = \{a, b, c\}$.

**Proof.**

\[ \pi_g(W') = \emptyset \]
Lemma (Triple Color Lemma)

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Lemma (Triple Color Lemma)

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Proof.

- $\pi_g(W') = \emptyset$
- $\pi_f(W') = \{a, b\}$
- Let $c = f(uv)$, let $W = W'vu$
**Lemma (Triple Color Lemma)**

Let $f$ be a minimum spec of $K_n$. Then for every pair of colors $\{a, b\}$, there is a third color $c$ and a closed walk $W$ with $\pi(W) = \{a, b, c\}$.

**Proof.**

- $\pi_g(W') = \emptyset$
- $\pi_f(W') = \{a, b\}$
- Let $c = f(uv)$, let $W = W'vu$
- $c \notin \{a, b\}$
Uniqueness of Perfect Specs of $K_n$

Lemma (Augmentation)

If $n$ is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$. 
Uniqueness of Perfect Specs of $K_n$

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If $n$ is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$.

Theorem

A spec of $G$ is perfect if it uses $\Delta(G)$ colors. If $f$ is a perfect spec of $K_n$, then $n$ is a power of two and $f$ is the canonical coloring.
Uniqueness of Perfect Specs of $K_n$

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Proof (sketch).

Starting with a single vertex, the proof finds larger and larger canonically colored subgraphs of $K_n$ inductively.
Uniqueness of Perfect Specs of $K_n$

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**Theorem**

A spec of $G$ is **perfect** if it uses $\Delta(G)$ colors. If $f$ is a perfect spec of $K_n$, then $n$ is a power of two and $f$ is the canonical coloring.

- If $n$ is not a power of two, each vertex misses a color
Augmentation Lemma

Lemma (Augmentation)

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Augmentation Lemma

**Lemma (Augmentation)**

If $n$ is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$.

**Proof.**

- Choose a vertex $v$
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Choose a vertex $v$

Because $n$ is not a power of two, $v$ is not incident to some color $a$.
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**Lemma (Augmentation)**

If $n$ is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$.

**Proof.**

- Choose a vertex $v$.
- Because $n$ is not a power of two, $v$ is not incident to some color $a$.
- Introduce a new vertex $u$. Color $uv$ with $a$. 

**Introduction**

- **Cliques**
- **Open Problems**
Augmentation Lemma

Lemma (Augmentation)

If $n$ is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$.

Proof.

Choose another vertex $w$. How do we color $uw$?

$K_n$
Augmentation Lemma

Lemma (Augmentation)

If \( n \) is not a power of two, then \( \hat{p}(K_n) = \hat{p}(K_{n+1}) \).

Proof.

- Choose another vertex \( w \). How do we color \( uw \)?
- Let \( b = f(vw) \)
**Augmentation Lemma**

**Lemma (Augmentation)**

*If $n$ is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$.*

**Proof.**

- Choose another vertex $w$. How do we color $uw$?
- Let $b = f(vw)$
- By Triple Color Lemma, there is a closed walk $W$ with $\pi_f(W) = \{a, b, c\}$. 
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- Color $uw$ with $c$. 
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Proof.

- Choose another vertex $w$. How do we color $uw$?
- Let $b = f(vw)$
- By Triple Color Lemma, there is a closed walk $W$ with $\pi_f(W) = \{a, b, c\}$.
- Color $uw$ with $c$.
- Let $g$ be the coloring of $K_{n+1}$. 
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Lemma (Augmentation)

If $n$ is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$.

Proof.
- We show that $g$ is a spec.
Augmentation Lemma

Lemma (Augmentation)

If $n$ is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$.

Proof.

- We show that $g$ is a spec.
- By Spec Characterization Lemma, it suffices to show that $L_g \subseteq L_f$. 
Augmentation Lemma

Lemma (Augmentation)

If $n$ is not a power of two, then $\tilde{p}(K_n) = \tilde{p}(K_{n+1})$.

Proof.

- We show that $g$ is a spec.
- By Spec Characterization Lemma, it suffices to show that $L_g \subseteq L_f$.
- By Span Lemma, it suffices to show, for each triangle $T$ containing $v$, $\pi_g(T) \in L_f$. 

Diagram: $K_n$ with vertices $u$, $v$, and $w$. $u$ is connected to both $v$ and $w$, and $v$ and $w$ are connected.
Augmentation Lemma

**Lemma (Augmentation)**

*If n is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$.***

**Proof.**

- We show that $g$ is a spec.
- By Spec Characterization Lemma, it suffices to show that $L_g \subseteq L_f$.
- By Span Lemma, it suffices to show, for each triangle $T$ containing $v$, $\pi_g(T) \in L_f$.
- If $u \notin T$, then $\pi_g(T) = \pi_f(T) \in L_f$. 
Augmentation Lemma

**Lemma (Augmentation)**

*If n is not a power of two, then \( \hat{p}(K_n) = \hat{p}(K_{n+1}) \).*

**Proof.**

Otherwise, \( T = uvwu \) for some \( w \) in \( K_n \) and \( \pi_g(T) = \pi_f(W) \in L_f \) for some closed walk \( W \) by definition of \( g \).
Lemma (Augmentation)

If $n$ is not a power of two, then $\hat{p}(K_n) = \hat{p}(K_{n+1})$.

Proof.

- Otherwise, $T =uvwu$ for some $w$ in $K_n$ and $\pi_g(T) = \pi_f(W) \in L_f$ for some closed walk $W$ by definition of $g$.
- Hence, $g$ is a spec.
An Application

**Theorem (Daykin, Lovász (1974))**

Let \( \mathcal{F} \) be a family of \( n \) finite sets, and let

\[
\mathcal{G} = \{ A_1 \triangle A_2 : A_1 \neq A_2 \text{ and } A_1, A_2 \in \mathcal{F} \}.
\]

Then \( |\mathcal{G}| \geq n - 1 \). If \( n \) is not a power of two, then \( |\mathcal{G}| \geq n \).
An Application

Theorem (Daykin, Lovász (1974))

Let $\mathcal{F}$ be a family of $n$ finite sets, and let

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Then $|\mathcal{G}| \geq n - 1$. If $n$ is not a power of two, then $|\mathcal{G}| \geq n$.

Quotation (with changes in notation)

“The example where $\mathcal{F}$ is all subsets of a [finite set] show that the theorem is best possible. Closer examination of the proof shows that if $|\mathcal{G}| = n - 1$ then $\mathcal{F}$ is very similar to the former example, but details are omitted.”
An Application

Corollary

Let $\mathcal{F}$ be a family of $n$ finite sets, and let

$$\mathcal{G} = \{A_1 \triangle A_2 : A_1 \neq A_2 \text{ and } A_1, A_2 \in \mathcal{F}\}.$$

Then $|\mathcal{G}| \geq 2^{\lceil \lg n \rceil} - 1$. 
Corollary

Let $\mathcal{F}$ be a family of $n$ finite sets, and let

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Proof.

View $\mathcal{F}$ as the vertex set of $K_n$. Coloring an edge $A_1A_2$ with the symmetric difference of $A_1$ and $A_2$, we obtain a spec of $K_n$ using only colors from $\mathcal{G}$. The bound on $|\mathcal{G}|$ follows.
Tournaments

Proposition

If $T$ is an $n$-vertex tournament, then $\hat{\rho}(T) \geq \lceil \lg n \rceil$. 
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Question

- What is the maximum of $\hat{\rho}(T)$ when $T$ is an $n$-vertex tournament?
Proposition

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Question

- What is the maximum of $\hat{\rho}(T)$ when $T$ is an $n$-vertex tournament?
- Is it $O(\log n)$?
Proposition

\[ \hat{\rho}(G \square H) \leq \hat{\rho}(G) + \hat{\rho}(H) \]
Graph Products

Proposition

$\hat{\rho}(G \square H) \leq \hat{\rho}(G) + \hat{\rho}(H)$

Question

For which graphs $G, H$ does equality hold?
Graph Products

Proposition
\[ \hat{p}(G \Box H) \leq \hat{p}(G) + \hat{p}(H) \]

Question
- For which graphs $G, H$ does equality hold?
- Does it hold for all graphs?
What is $\hat{\rho}(K_{m,n})$?

**Theorem**

Let $m \leq n$ and $m' = 2^{\lceil \lg m \rceil}$. Then

$$\hat{\rho}(K_{m,n}) \leq m' \left\lceil \frac{n}{m'} \right\rceil.$$ 

Further,

$$\hat{\rho}(K_{2,n}) = n + (n \mod 2).$$
What is $\hat{p}(K_{m,n})$?

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- What is $\hat{p}(K_{m,n})$? Is the upper bound tight?
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**Question**

- What is $\hat{\rho}(K_{m,n})$? Is the upper bound tight?
- Does $\hat{\rho}(K_{n,n}) = 2^{\lceil \lg n \rceil}$? Note: $\hat{\rho}(K_{5,5}) = 8$ and $\hat{\rho}(K_{9,9}) \in \{14, 15, 16\}$. 
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Let $m \leq n$ and $m' = 2^{\lceil \lg m \rceil}$. Then

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**Question**

- What is $\hat{p}(K_{m,n})$? Is the upper bound tight?
- Does $\hat{p}(K_{n,n}) = 2^{\lceil \lg n \rceil}$? Note: $\hat{p}(K_{5,5}) = 8$ and $\hat{p}(K_{9,9}) \in \{14, 15, 16\}$.
- Lower bounds apply to $|\{A_1 \triangle A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}|$ with $m = |\mathcal{F}_1|$ and $n = |\mathcal{F}_2|$. 
Stability of the Canonical Coloring

Question (Dhruv Mubayi)

Is there a (strong) parity edge-coloring of $K_{2^k}$ which uses only $(1 + o(1))2^k$ colors but is “far” from the canonical coloring?
Many other open problems in our paper.
Thank You.