Math 441 Exam 3

1. (8 pts) Subspaces of R^n are generally described either as the column space of some given matrix, or the null space of some given matrix.

a) If we define a subspace V as the nullspace of the matrix $A = \begin{bmatrix} -1 & 1 & -2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix}$ express V instead as the column space of some matrix B. Explain how/why this works in general.

Find the special solutions of $A\bar{x} = \bar{0}$ and put them in a matrix N :

$\begin{bmatrix} -1 & 1 & -2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$. Every vector in the null space is a linear combination of the special solutions, (the special solutions are a basis for the null space). This means that the column space of N is V. So our matrix B is the matrix N of special solutions of $A\bar{x} = \bar{0}$.

b) If we define a subspace V as the column space of the matrix $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -2 & 1 \\ 1 & -2 \end{bmatrix}$ express V instead as the null space of some matrix B. Explain how/why this works in general.

The column space of A is the orthogonal complement of the left null space of A, which is the null space of A^T. The null space of A^T comes from the solutions of $A^T\bar{x} = \bar{0}$, which, conveniently enough, were calculated above as the vectors in the column space of $N = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$. Now the orthogonal complement of the column space of N can be obtained as the null space of $N^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 0 & -1 & 1 \end{bmatrix}$. So V is the null space of $N^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 0 & -1 & 1 \end{bmatrix}$. In general we can say the the column space of A can be characterized as the null space of the matrix whose rows are composed of a basis of the left null space of A.

2. (8 pts) Find an orthonormal basis for the subspace V spanned by

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

This is just standard Gram-Schmidt.

$$\bar{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \bar{u}_2 = \bar{v}_2 - \frac{\bar{u}_1 \bullet \bar{v}_2}{\bar{u}_1 \bullet \bar{u}_1}\bar{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\bar{u}_3 = \bar{v}_3 - \frac{\bar{u}_1 \bullet \bar{v}_3}{\bar{u}_1 \bullet \bar{u}_1}\bar{u}_1 - \frac{\bar{u}_2 \bullet \bar{v}_3}{\bar{u}_2 \bullet \bar{u}_2}\bar{u}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ -1 \end{bmatrix}$$

(check your calculations by confirming orthogonality - it works)
Finally, normalize each vector:

$$\bar{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \bar{q}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \ \bar{q}_3 = \frac{1}{\sqrt{5/2}}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ -1 \end{bmatrix}$$

3. (5 points) The normal equations $A^T(\bar{b} - A\bar{x}) = \bar{0}$ arise how/why in calculating the projection $\bar{p}$ of a vector $\bar{b}$ onto the column space of A. What property do the normal equations express?

The projection $\bar{p}$ must have the form $A\bar{x}$ because it is in the column space of A. It is characterized by the requirement that the error vector $\bar{e} = \bar{b} - \bar{p}$ is orthogonal to the column space. This is expressed by the equation $A^T(\bar{b} - A\bar{x}) = \bar{0}$ which says that $\bar{b} - A\bar{x}$ is in the left null space of A, $N(A^T)$, which is the orthogonal complement of the column space.

4. (8 pts) Use the (amazing) cofactor matrix to calculate the inverse of $\begin{bmatrix} -1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ is an $n \times n$ matrix, and $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$ is its matrix of cofactors, then $\det\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ can be expressed as the dot product of __$\bar{x}$____ with _row 1 of C___.

The cofactor matrix is $C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 2 \\ -1 & 3 & -4 \end{bmatrix}$. The determinant of A can be calculated directly, or by noting that $\det A$ is the dot product of any row/column of C with the corresponding row/column of A. In any case $\det A = -2$ and

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{(-2)} \begin{bmatrix} 1 & 0 & -1 \\ -1 & -2 & 3 \\ 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{3}{2} \\ 0 & -1 & 2 \end{bmatrix}$$

As for the second part, expanding by the first row of the matrix with the $x's$, the cofactors appear in the first row of the matrix C so we get the determinant from the dot product of $\bar{x}$ with row 1 of C.

5. (10 pts) Complete the following by filling in the blanks:
If $A\bar{x} = \bar{b}$ then $\bar{b}$ is in the __column__ space of __A_.
If $\bar{y}$ is in the orthogonal complement of the column space of A, then $\bar{y}$ is in the _left null space of A or the null space of A^T.
If $\bar{x}$ is in the row space of A, then $\bar{x}$ is in the orthogonal complement of the __null_ space of _A_.
If V is a subspace of R^n of dimension k, then $V^\perp$ has dimension _$n-k$_. A basis for $V^\perp$ can be calculated from a basis of V as the special solutions of $A\bar{x} = \bar{0}$ where A contains _the basis of V as its rows_ and the rank of A is _k_ .

6. (5 points) If $A\bar{x} = \bar{b}$ has a solution, then the solution $\bar{x}$ of shortest length is in the row space of A. Why is that true? Explain.

Any solution $\bar{x}$, as a vector in R^n, can be expressed as $\bar{x} = \bar{x}_{\text{row}} + \bar{x}_{null}$. Now $A\bar{x} = A\bar{x}_{\text{row}} + A\bar{x}_{null} = A\bar{x}_{\text{row}}$ so $\bar{x}_{\text{row}}$ is a solution by itself. Any solution must be of the form $\bar{x} = \bar{x}_{\text{row}} + \bar{n}$ where $\bar{n}$ is in $N(A)$. However, note that $\|\bar{x}\|^2 = \|\bar{x}_{\text{row}}\|^2 + \|\bar{n}\|^2$ which is minimum when $\bar{n} = \bar{0}$ so that $\bar{x} = \bar{x}_{\text{row}}$ gives the shortest length.

7. (10 points) Find the values of a and b so that the vector $\bar{p} = a\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ is as close as possible to the vector $\bar{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. How would you describe the vector $\bar{p}$ using the idea of projection?

This is a projection problem, $\bar{p} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}\bar{x}$ where $\bar{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ so $\bar{p}$ is a vector in the column space of $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$ and the $\bar{p}$ closest to $\bar{b}$ is the projection of $\bar{b}$ onto the column space of A. We calculate from the normal equations $A^TA\bar{x} = A^T\bar{b}$, $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\bar{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. We can solve by elimination:

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{2}{3} \end{bmatrix} \text{ so } \bar{x} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$