General ideas:
Linear independence, span, basis dimension
Linear independence: $c_1\bar{v}_1 + \ldots + c_k\bar{v}_k = \bar{0}$ only when $c_1 = 0$, ..., $c_k = 0$
Equivalent statements (if $\bar{v}'s$ are column vectors):

$A\bar{x} = \bar{0}$ only if $\bar{x} = \bar{0}$ (A contains $\bar{v}'s$ as columns)

$rankA = k$

Span: (two usages) The span of a set of vectors is all the possible linear combinations of the vectors in the set (it is a subspace). We say a set S of vectors spans a subspace V if (with the previous usage) the span of the set S is the subspace V - Strang described it as: All the linear combinations of the vectors in S fill W.
you can look up the others

Definitions (and equivalent ways of characterizing) the four subspaces, and their dimensions
Example:
$C(A)$ is:
the set of vectors $A\bar{x}$ for all $\bar{x}$ in R^n ⇔ set of vectors $\bar{b}$ in R^m such that $A\bar{x} = \bar{b}$ has a solution ⇔ the span of the columns of A

$\dim C(A) = \dim C(A^T) = r$, $\dim N(A) = n - r$, $\dim N(A^T) = m - r$

Basic things to explain for general vectors, such as:

If a vector $\bar{v}$ is in the span of the vectors $\bar{v}_1, .., \bar{v}_k$ then the set $\{\bar{v}_1, .., \bar{v}_k, \bar{v}\}$ is linearly dependent. If $\bar{v} = c_1\bar{v}_1 + \ldots + c_k\bar{v}_k$, then $c_1\bar{v}_1 + \ldots + c_k\bar{v}_k - \bar{v} = \bar{0}$. This is a linear combination of the vectors in the set $\{\bar{v}_1, .., \bar{v}_k, \bar{v}\}$ that results in the zero vector, without all coeffs being zero (the coeff of $\bar{v}$ is -1). This means that the set $\{\bar{v}_1, .., \bar{v}_k, \bar{v}\}$ is linearly dependent.

If the zero vector is in a set, the set is linearly dependent. $1\bar{0} = \bar{0}$, is a nonzero linear combination of the vectors in the set resulting in the zero vector.

If the vector $\bar{w}$ is not in the span of the independent set of vectors $\bar{v}_1, .., \bar{v}_k$, then $\{\bar{v}_1, .., \bar{v}_k, \bar{w}\}$ is independent. Suppose $c_1\bar{v}_1 + \ldots + c_k\bar{v}_k + c\bar{w} = \bar{0}$. Now if $c = 0$ all the other $c's$ must be zero because the vectors $\bar{v}_1, .., \bar{v}_k$ are linearly independent. If $c \neq 0$ then $c_1\bar{v}_1 + \ldots + c_k\bar{v}_k + c\bar{w} = \bar{0}$ means that $\bar{w} = -\frac{1}{c}(c_1\bar{v}_1 + \ldots + c_k\bar{v}_k)$ - but that can't happen because $\bar{w}$ is not in the span of the $\bar{v}'s$

Matrix problems formulated in vector form:
$A\bar{x} = \bar{b}$ ⇔ $\bar{b}$ is in the span of the columns of A ⇔ $\bar{b}$ is in $C(A)$ ⇔ $\bar{b}$ is a linear combination of the columns of A
$A\bar{x} = \bar{b}$ has a solution for each $\bar{b}$ ⇔ $C(A) = R^m$ ⇔ $rank(A) = m$ ⇔ $\dim C(A) = m$ ⇔ rows of A are linearly independent
$A\bar{x} = \bar{0}$ has only $\bar{x} = \bar{0}$ as a solution ⇔ columns of A are linearly independent ⇔ rank$(A) = n$ ⇔ $\dim C(A) = n$ ⇔ $N(A) = Z = \{\bar{0}\}$

Matrix spaces:
Obtained from a row reduction

$$\begin{bmatrix} 3 & 2 & -5 & 0 & 5 & -2 \\ -1 & 1 & 5 & 5 & 1 & 0 \\ 2 & 1 & -4 & -1 & 1 & -3 \\ 1 & 2 & 1 & 4 & 5 & 0 \\ 1 & 3 & 3 & 7 & 12 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Give: Basis of column space, basis of row space, basis of null space
Pivot columns in A are a basis of col space: Cols 1,2,5 in A
Nonzero rows of R (or U) are a basis of row space.
Null space basis: the special solutions (you find them)

Express each column of A as a linear combination of your column space basis vectors
The required coeffis are in R in the free columns.
Express each row of A as a linear combination of your row space basis vectors
This can be done by inspection.

Write the general solution of

$$\begin{bmatrix} 3 & 2 & -5 & 0 & 5 \\ -1 & 1 & 5 & 5 & 1 \\ 2 & 1 & -4 & -1 & 1 \\ 1 & 2 & 1 & 4 & 5 \\ 1 & 3 & 3 & 7 & 12 \end{bmatrix} \bar{x} = \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 5 \end{bmatrix}$$

The work has already been done above. We use $\bar{x} = \bar{x}_p + \bar{x}_n$ where $\bar{x}_p$ is any particular solution (usually chosen so that the free variables are zero) and $\bar{x}_n$ represents a general vector in the null space, usually written as a linear combination of the special solutions of $A\bar{x} = \bar{0}$.. In this case we have

$$\bar{x} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering the four subspaces, which of them are the same for A and for R? Why?
$N(A)$ and $N(R)$ are the same because Gaussian elimination guarantees that the solutions of $A\bar{x} = \bar{0}$ and $R\bar{x} = \bar{0}$ are the same.
The two row spaces $C(A^T)$ and $C(R^T)$ are the same because every row in R is generated via linear combination from the rows of A, and every row of A can be regenerated from the rows of R.
The other spaces are not the same.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Answer without calculations; explain

What is the rank of A?

There are two independent rows, but not three, so $r = 2$

What is a basis of the column space, the row space, the null space, the left null space. How do you know?

Clearly cols 1 and 2 are independent, and since $\dim C(A) = r = 2,$ these must constitute a basis of $C(A)$

Likewise rows 1,2 are independent and form a basis of the row space.

We have $\dim N(A) = n - r = 3 - 2 = 1.$ We observe that the first two columns sum to the third column, so that $A\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \bar{0}$ so $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a basis for the null space (the dimension is right).

For the left null space, we are looking at how to combine the rows to get zero. In this case the first and third rows are the same so that

$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} A = \bar{0}$, and since the dimension of 1, we must have $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is a basis.

Note that "pivot rows" (rows that wind up with pivots at the end) are also a basis for row space, but you can't easily tell which those are simply from the final matrix R.

The submatrix consisting of the r pivot columns and r pivot rows is invertible (has rank r)

Matrix proof: What would be the row reduced echelon form for this submatrix (it is $r \times r$ with r pivots). Ans: It would be the identity matrix and so it would have rank r and the submatrix would be invertible.

Vector space proof: A subset of the columns, size $r,$ is a basis of the column space. If we take the submatrix consisting of these r columns, the resulting matrix has rank r (number of independent columns) and so there are r independent rows. Taking only those r independent rows leaves us with an $r \times r$ matrix of rank r.

Consider the vectors (x, y, z) in R^3 satisfying $2x - y + z = 0$

Is this a subspace? Why?

Yeah its the subspace $\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \bar{x} = \bar{0},$ i.e. the null space of $\begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$. Now that this subspace has dimension 2, since the matrix has rank 1.

Explain how you know that the vectors $\{(1, 1, -1)\ ,\ (2, 1, -3)\}$ are a basis of this subspace.

As the null space, the subspace in question has dimension $3 - r = 2$ and the given 2 vectors are in the nullspace and linearly independent, so they must be a basis.

If A is 6×4,

the dimension of the left null space is at least ______
the dimension of the column space is at most _______

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Answer yes/no/could be.
If you have six vectors in R^4
Do they span R^4? could be.
Are they linearly independent? No.
Are any four of them a basis of R^4? could be
(think: if they were all chosen at random, any 4 would clearly be a basis)

How do we know that $rank(AB) \leq rank(A)$ and $rank(AB) \leq rank(B)$
We discussed this in prior notes. Every vector $AB\bar{x} = A(B\bar{x})$ is a combination of the columns of A and so in the column space. So $rankAB = \dim C(AB) \leq \dim C(A) = rankA$. Similar arguments work on the rows.

What is the column space of I? The row space? The null space?