Math 441 Exam 1

1. (8 pts) a) Given a vector $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ find A so that $A\bar{x} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$. (Describe the form and state the dimensions of the matrix A)

We have $A\bar{x} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$ with A the indicated $(n-1) \times n$ matrix.

b) Express the vector $\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$ as a linear combination

$\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix} = x_1 \begin{bmatrix} \bar{u}_1 \\ \\ \\ \end{bmatrix} + x_2 \begin{bmatrix} \bar{u}_2 \\ \\ \\ \end{bmatrix} + \ldots + x_n \begin{bmatrix} \bar{u}_n \\ \\ \\ \end{bmatrix}$ (find the vectors $\bar{u}_1, \bar{u}_2, .., \bar{u}_n$)

We have $\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ \vdots \\ x_n - x_{n-1} \end{bmatrix} = x_1 \begin{bmatrix} -1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_{n-1} \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \\ -1 \end{bmatrix} + x_n \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

c) What is the connection between part a) and part b)?

The vectors $\bar{u}_1, \bar{u}_2, .., \bar{u}_n$ are the columns of the matrix A in part a)

2. (12 pts) Using Guassian elimination along with back substitution on the augmented matrix, solve the linear system

$$A\bar{x} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}.$$

$$\left[A \vdots \bar{b}\right] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 3 & -1 & 2 & 2 \\ -2 & 1 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 5 \\ 0 & -1 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 5/2 & -5/2 \end{bmatrix} = \left[U \vdots \bar{b}'\right]$$

$z = -1$,

$2y - z = 5$, $y = 2$

$x - y + z = -1$, $x = 2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

a) Without doing any further calculations, write the LDU decomposition of the matrix A.

The multipliers, in order, were $\ell_{21} = 3$, $\ell_{31} = -2$, $\ell_{32} = -1/2$ so

$$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$ (the U here has had its pivots normalized to 1 by division, the pivots go into D)

b) What sequence of elimination matrices $E_1, E_2, .., E_k$ multiplying A on the left in turn, takes us from A to U in part a) (so that $E_k E_{k-1} \cdot \cdot E_1 A = U$)

Noting the multipliers used above, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} = U$$

3. (8 pts) If A is a 4×4 matirx,

a) What matrix $E,$ upon the multiplication $EA,$ has the effect on A of subtracting 3 times row 2 of A from row 4 of A?

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}.$$

b) What is the effect on A if we perform the operation AE using the same matrix E from part a)

AE has the effect on A of subtracting 3 times column 4 of A from column 2 of A

c) Without doing any calculations, what is E^{-1} ?

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$ since adding back in 3*row2 takes us back to the identity.

4. (12 pts) Using Gauss-Jordan elimination, in the space below find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ and check your answer. Then, considering only those calculations, answer the questions that follow:

$$\left[A \,\vdots\, I\right] = \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i) When we go from A via Gaussian elimination to the upper triangular matrix U, what single matrix M, multiplying on the left, would turn A into U? Why? What is the relationship of this M to L from the LU decomposition.

The right side of the augmented matrix keeps track of the cumulative multiplications:

$M\left[A \,\vdots\, I\right] = \left[MA \,\vdots\, M\right] = \left[U \,\vdots\, M\right]$ so $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$. Now $MA = U$ so $A = M^{-1}U$ and $M = L^{-1}$.

ii) Changing a_{23} to what value would cause A^{-1} not to exist?

If $a_{23} = -2$ we obtain

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{bmatrix}$$ and A is singular.

5. (8 pts) If we know that $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, solve

$$A\bar{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

(Hint: first multiply both sides of $A\bar{x} = \bar{b}$ on the left by $P,$ do not calculate A !)

We have $PA = LU.$ Now if $A\bar{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ we have $PA\bar{x} = P\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$LU\bar{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Consider $L\bar{c} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \bar{c} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix};$ We have $\bar{c} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ so

$U\bar{x} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{x} = \bar{c} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ so $\bar{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$