MATH 441.001 NAME (print): ____________________

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TEST # 2 Review

To be solved in class.

Test will start with: "Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)"

Ex. 1. Find a basis for a vector space V generated by the following vectors:

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 2 \\ -2 \end{bmatrix}$.

(**pts)** *Solution:* The space V is equal to the column space $C(A)$ of $A = [\vec{v}_1\vec{v}_2\vec{v}_2]$. Since A reduces as follows:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 2 & 2 \\ 5 & 4 & -2 \end{bmatrix} \begin{matrix} \\ \times 1/2 \\ -\vec{r}_1 \\ -5\vec{r}_1 \end{matrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \begin{matrix} -\vec{r}_2 \\ \\ -\vec{r}_2 \\ +\vec{r}_2 \end{matrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$$

the pivot columns of A are the same as of R: columns #1 and #2. Therefore these columns of **matrix A** form the basis of $V = C(A)$.

Answer: $\{\vec{v}_1, \vec{v}_2\}$ is a basis of V.

Ex. 2. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 2 & 2 \\ 5 & 4 & -2 \end{bmatrix}$. (a) For what value of number c vector $\vec{w} = \begin{bmatrix} 2 \\ -4 \\ 0 \\ c \end{bmatrix}$ belongs to the column space of A? (b) Find the most general solution of $A\vec{x} = \vec{w}$ for this value of c. (c) Without further calculation, identify a basis for a row space of A (i.e., for $C(A^T)$).

(**pts)** *Solution:* (a) Vector $\vec{w}$ belongs to $C(A)$ precisely when system $A\vec{x} = \vec{w}$ has a solution. Thus, we need to solve it. We will do it by reducing augmented matrix of this system:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 4 & -4 \\ 1 & 2 & 2 & 0 \\ 5 & 4 & -2 & c \end{bmatrix} \begin{matrix} \\ \times 1/2 \\ -\vec{r}_1 \\ -5\vec{r}_1 \end{matrix} \to \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & -1 & -2 & c-10 \end{bmatrix} \begin{matrix} -\vec{r}_2 \\ \\ -\vec{r}_2 \\ +\vec{r}_2 \end{matrix} \to \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c-12 \end{bmatrix}$$

For this system to have the solution we need to have $c - 12 = 0$, that is, $\mathbf{c = 12}$.

(b) We already have done all reduction work for this problem in part (a). The free variable is x_3, since the third column is the only non-pivot column. From the first reduced equation $x_1 - 2x_3 = 4$, we get $x_1 = 4 + 2x_3$. The second equation $x_2 + 2x_3 = -2$ gives $x_2 = -2 - 2x_3$.

Answer: The most general solution of $A\vec{x} = \vec{w}$ with $c = 12$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

(c) The basis is formed by pivot raws of either A or its reduced form. (But not augmented form, that is, we need to ignore the last column in our calculation.) Thus, the basis can be given either as $\{(1,1,0), (0,2,4)\}$ or as $\{(1,0,-2), (0,1,2)\}$.

Ex. 3. Let A be an 7×11 matrix with the rank $r = 4$. What is the dimension of the following four spaces. (a) Column space $C(A)$ of A. (b) Raw space $C(A^T)$ of A. (c) Null space $N(A)$ of A. (d) Left-null space $N(A^T)$ of A.

(pts) *Solution:* Answers: (a) $= r = 4$. (b) $= r = 4$. (c) $= n - r = 11 - 4 = 7$. (d) $= m - r = 7 - 4 = 3$. Name "left-null" comes from the fact that y is in $N(A^T)$ when it is a solution of $A^T\vec{y} = 0$, which is equivalent to $(A^T\vec{y})^T = 0^T$, that is, to $\vec{y}^T A = 0$.

Ex. 4. Let A be an $m \times n$ matrix of rank r. Describe precisely the possible number of solutions of a system $A\vec{x} = \vec{b}$ under the following assumptions:

(a) $r < m$ and $r < n$

(b) $r = m$ and $r < n$

(c) $r < m$ and $r = n$

(d) $r = m = n$

(pts) *Solution:* Answers: (a) none or infinitely many; (b) one or infinitely many; (c) none or one; (d) one.

Ex. 5. Finish the following sentence. Give a short explanation for your answer.

An $n \times n$ matrix A is invertible if, and only if, the dimension of its column space $C(A)$ is . . .

(pts) *Solution:* Answer: "the dimension of $C(A)$ is equal n."

This is the case, since A is invertible when every column of A is its pivot column, that is, when rank r of A is equal n. But we know, that the dimension of $C(A)$ is equal to r.

In the test, there will be also a bonus exercise.