Appendix A

Axioms of set theory

Axiom 0 *(Set Existence)* There exists a set:

$$\exists x(x = x).$$

Axiom 1 *(Extensionality)* If x and y have the same elements, then x is equal to y:

$$\forall x \forall y\, [\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y].$$

Axiom 2 *(Comprehension scheme* or *schema of separation)* For every formula $\varphi(s, t)$ with free variables s and t, for every x, and for every parameter p there exists a set $y = \{u \in x \colon \varphi(u, p)\}$ that contains all those $u \in x$ that have property φ:

$$\forall x \forall p \exists y\, [\forall u(u \in y \leftrightarrow (u \in x \;\&\; \varphi(u, p)))].$$

Axiom 3 *(Pairing)* For any a and b there exists a set x that contains a and b:

$$\forall a \forall b \exists x(a \in x \;\&\; b \in x).$$

Axiom 4 *(Union)* For every family $\mathcal{F}$ there exists a set U containing the *union* $\bigcup \mathcal{F}$ of all elements of $\mathcal{F}$:

$$\forall \mathcal{F} \exists U \forall Y \forall x[(x \in Y \;\&\; Y \in \mathcal{F}) \rightarrow x \in U].$$

Axiom 5 *(Power set)* For every set X there exists a set P containing the set $\mathcal{P}(X)$ (the *power set*) of all subsets of X:

$$\forall X \exists P \forall z\, [z \subset X \rightarrow z \in P].$$

To make the statement of the next axiom more readable we introduce the following abbreviation. We say that y is a successor of x and write $y = S(x)$ if $S(x) = x \cup \{x\}$, that is,

$$\forall z[z \in y \leftrightarrow (z \in x \vee z = x)].$$

Axiom 6 *(Infinity)* (Zermelo 1908) There exists an infinite set (of some special form):

$$\exists x\, [\forall z(z = \emptyset \rightarrow z \in x) \;\&\; \forall y \in x \forall z(z = S(y) \rightarrow z \in x)].$$

Axiom 7 *(Replacement scheme)* (Fraenkel 1922; Skolem 1922) For every formula $\varphi(s, t, U, w)$ with free variables s, t, U, and w, every set A, and every parameter p if $\varphi(s, t, A, p)$ defines a function F on A by $F(x) = y \Leftrightarrow \varphi(x, y, A, p)$, then there exists a set Y containing the range $F[A] = \{F(x)\colon x \in A\}$ of the function F:

$$\forall A \forall p\, [\forall x \in A \exists! y \varphi(x, y, A, p) \rightarrow \exists Y \forall x \in A \exists y \in Y \varphi(x, y, A, p)],$$

where the quantifier $\exists! x \varphi(x)$ is an abbreviation for "there exists precisely one x satisfying φ," that is, is equivalent to the formula

$$\exists x \varphi(x) \;\&\; \forall x \forall y(\varphi(x) \;\&\; \varphi(y) \rightarrow x = y).$$

Axiom 8 *(Foundation* or *regularity)* (Skolem 1922; von Neumann 1925) Every nonempty set has an $\in$-minimal element:

$$\forall x\, [\exists y(y \in x) \rightarrow \exists y(y \in x \;\&\; \neg \exists z(z \in x \;\&\; z \in y))].$$

Axiom 9 *(Choice)* (Levi 1902; Zermelo 1904) For every family $\mathcal{F}$ of disjoint nonempty sets there exists a "selector," that is, a set S that intersects every $x \in \mathcal{F}$ in precisely one point:

$$\forall \mathcal{F}\, [\forall x \in \mathcal{F}(x \neq \emptyset) \;\&\; \forall x \in \mathcal{F} \forall y \in \mathcal{F}(x = y \vee x \cap y = \emptyset)]$$

$$\rightarrow \exists S \forall x \in \mathcal{F} \exists! z(z \in S \;\&\; z \in x),$$

where $x \cap y = \emptyset$ is an abbreviation for

$$\neg \exists z(z \in x \;\&\; z \in y).$$

Using the comprehensive schema for Axioms 0, 3, 4, and 5 we may easily obtain the following strengthening of them, which is often used as the original axioms.

Axiom 0′ *(Empty set)* There exists the *empty set* $\emptyset$:

$$\exists x \forall y \neg (y \in x).$$

Axiom 3′ *(Pairing)* For any a and b there exists a set x that contains *precisely* a and b.

Axiom 4′ *(Union)* (Cantor 1899; Zermelo 1908) For every family $\mathcal{F}$ there exists a set $U = \bigcup \mathcal{F}$, the *union* of all subsets of $\mathcal{F}$.

Axiom 5′ *(Power set)* (Zermelo 1908) For every set x there exists a set $Y = \mathcal{P}(x)$, the *power set* of x, that is, the set of all subsets of x.

The system of Axioms 0–8 is usually called Zermelo–Fraenkel set theory and is abbreviated by ZF. The system of Axioms 0–9 is usually denoted by ZFC. Thus, ZFC is the same as ZF+AC, where AC stands for the axiom of choice.

Historical Remark (Levy 1979) The first similar system of axioms was introduced by Zermelo. However, he did not have the axioms of foundation and replacement. Informal versions of the axiom of replacement were suggested by Cantor (1899) and Mirimanoff (1917). Formal versions were introduced by Fraenkel (1922) and Skolem (1922). The axiom of foundation was added by Skolem (1922) and von Neumann (1925).

Notice that Axioms 2 and 7 are in fact the schemas for infinitely many axioms, one for each formula φ. Thus theory ZFC has, in fact, infinitely many axioms. Axiom 1 of extensionality is the most fundamental one. Axioms 0 (or 0′) of set existence (of empty set) and 6 of infinity are "existence" axioms that postulate the existence of some sets. It is obvious that Axiom 0 follows from Axiom 6. Axioms 2 of comprehension and 7 of replacement are schemas for infinitely many axioms. Axioms 3 (or 3′) of pairing, 4 (or 4′) of union, and 5 (or 5′) of power set are conditional existence axioms. The existing sets postulated by 3′, 4′, and 5′ are unique. It is not difficult to see that Axiom 3 of pairing follows from the others.

The axiom of choice AC and the axiom of foundation also have the same conditional existence character. However, the sets existing by them do not have to be unique. Moreover, the axiom of foundation has a very set-theoretic meaning and is seldom used outside abstract set theory or logic. It lets us build "hierarchical models" of set theory. During this course we very seldom make use of it. The axiom of choice, on the other hand, is one of the most important tools in this course. It is true that its nonconstructive character caused, in the past, some mathematicians to reject it (for example, Borel and Lebesgue). However, this discussion has been for the most part resolved today in favor of accepting this axiom.

It follows from the foregoing discussion that we can remove Axioms 0 and 3 from 0–8 and still have the same theory ZF.