$\mathcal{I}$-DENSITY CONTINUOUS FUNCTIONS

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Abstract: The $\mathcal{I}$-density topology is a generalization of the ordinary density topology to the setting of category instead of measure. This work involves functions which are continuous when combinations of the $\mathcal{I}$-density, deep-$\mathcal{I}$-density, density and ordinary topology are used on the domain and range. In the process of examining these functions, the $\mathcal{I}$-density and deep-$\mathcal{I}$-density topologies are deeply explored and the properties of these function classes as semigroups are considered.

Key words and phrases. density topology, $\mathcal{I}$-density, Baire classes, semigroups.

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Introduction

A persistent area of study in real analysis during the last half century is an effort to explain the striking parallels between two sets of theorems on the real line: those concerned only with sets of measure zero and those concerned only with sets of the first category. Indeed, a general rule of thumb seems to be that the following "substitutions" can be made in many theorems to result in another theorem.

$$\left.\begin{array}{r}\text{sets of measure zero}\\ \text{measurable sets}\\ \text{measurable functions}\end{array}\right\}\longleftrightarrow\left\{\begin{array}{l}\text{first category sets}\\ \text{Baire sets}\\ \text{Baire functions}\end{array}\right.$$

An excellent overview containing many examples of this is contained in the book *Measure and Category* by J. C. Oxtoby [**55**].

A natural question to ask is whether this parallel can be extended to a larger class of sets. The major stumbling block in any such attempt seems to be the Lebesgue Density Theorem, which has no obvious parallel in the category sense. To see the difficulty here, recall that a point a is a Lebesgue density point of the measurable set A if, and only if,

$$\lim_{h\to 0^+}\frac{\mathrm{m}((a-h,a+h)\cap A)}{2h}=1,$$

where $\mathrm{m}(S)$ is the Lebesgue measure of the set S. From this definition, one can see that a is a density point of A if, in some sense, the set A *dynamically becomes of full measure nearby* a *as* $h\to 0^+$. Until recently, there was no clear parallel whereby A could "become full" in the category sense nearby a; a set is either first or second category–there is no middle ground. This changed in 1985 when W. Wilczyński [**68, 57**] introduced a category analogue of density which he called "$\mathcal{I}$-density."

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Lebesgue density plays a central role in the study of real functions, usually in the form of approximate limits and the density topology. The density topology consists of all sets $S \subset \mathbb{R}$ such that every point of S is a density point of S. The limit induced by this topology is the approximate limit. Wilczyński used these ideas to introduce the $\mathcal{I}$-density topology and the $\mathcal{I}$-approximate limit, which have many properties in common with the ordinary density topology and ordinary approximate limit. Since then, many other papers by Wilczyński and others have appeared, exploring the parallels and differences between the ordinary case and the category case more fully in the $\mathcal{I}$-density setting. Most of those papers are listed in the bibliography at the end of this work.[1]

In fact, this work began with our efforts to extend some recent theorems concerning the ordinary density topology to the $\mathcal{I}$-density topology. These theorems concern what we call *density continuous functions*; i.e., functions $f\colon \mathbb{R} \to \mathbb{R}$ which are continuous when both the domain and range are given the density topology. Such functions are quite badly behaved and our exploration of their properties in the ordinary case requires delicate arguments. The corresponding $\mathcal{I}$-density continuous functions proved even more difficult to handle, and new facts about the $\mathcal{I}$-density topology were needed to complete our objectives. These new facts opened up new avenues of investigation, which required new facts ... So, in that sense, this work is a "tale that grew in the telling."[2]

We expect readers of this work to have a thorough knowledge of basic analysis and topology on the real line. Much of this material is covered in the monograph by Bruckner [**7**]. Given the nature of the material, this understanding must, of course, include a comfortable familiarity with Lebesgue measure and the Baire category theorem. In addition to this, we have included lengthy discussions of several lesser-known topics which are used to motivate our work on the $\mathcal{I}$-density topology, such as the ordinary density topology, approximately continuous functions and density continuous functions. However, much of this material is stated without proof. Places where the proofs can be found are referenced.

Chapter 1 is a catch-all for material leading up to the definition of $\mathcal{I}$-density. It begins with a short section on an early, but naíve attempt to generalize the approximate limit to the category case, called the qualitative limit. The next three sections contain a brief overview of the standard facts about ordinary density which are needed in later chapters. In these sections most of the theorems are stated without proof. Those which are proved are usually done so either because the proof is different from the standard one, or because it sheds light on a corresponding proof in the $\mathcal{I}$-density case given in Chapter 2. Chapter 1 ends with the introduction of the very general $\mathcal{J}$-topologies on the real line. Some theorems and examples show in a somewhat startling way how much care must

[1] Credit for particular results is given in bibliography sections at the end of each chapter and sometimes in the text of the chapter.

[2] Borrowed from J. R. R. Tolkien.

be taken when defining these topologies.

Chapter 2 is devoted to proving the basic parallels between the $\mathcal{I}$-density topology and the ordinary density topology. It begins with some useful technical lemmas which are applied heavily in later sections. The first six sections contain a reasonably complete survey of everything known to date about the $\mathcal{I}$-density topology and $\mathcal{I}$-approximately continuous functions, including a number of new theorems. Everything is proved, even when the proposition has already been proved elsewhere.

A deficiency of the $\mathcal{I}$-density topology is that, unlike the ordinary density topology, it is not completely regular. This problem is examined in Section 2.6 with the introduction of the deep-$\mathcal{I}$-density-topology, a topology which is coarser than the $\mathcal{I}$-density topology and is completely regular. Section 2.6 also contains a short proof of the fact that the $\mathcal{I}$-density topology is not generated.

Section 2.4 is interesting in its own right as an alternative method, introduced by L. Zajíček [**72**], of viewing the $\mathcal{I}$-density topologies from the viewpoint of porosity. This porosity definition is not used very much in the later chapters, but we consider it an excellent tool for obtaining an intuitive understanding of the structure of $\mathcal{I}$-density open sets.

Chapter 2 ends with a presentation of the relationships between the ordinary, density, $\mathcal{I}$-density and deep-$\mathcal{I}$-density topologies.

In Chapter 3, we finally begin the study promised in the title to this work. $\mathcal{I}$-density continuous functions, deep-$\mathcal{I}$-density-continuous functions and $\mathcal{I}$-density preserving homeomorphisms are deeply examined. Virtually all properties known about the functions in the corresponding ordinary case are extended to the category case, when possible, and counter examples are given showing the limits of the parallel between the ordinary and category cases. The chapter contains an exhaustive examination of the containment relationships between the various classes of continuous functions studied here. Of particular note is the deep study of $\mathcal{I}$-density continuous and $\mathcal{I}$-density preserving homeomorphisms contained in Sections 2 and 3 and the Baire classification methods in Section 5.

The purpose of Chapter 4 is to consider the semigroup properties of several function classes introduced earlier. The basic question treated in this chapter is whether any of the function classes introduced earlier, which also happen to be semigroups under the composition operation, have the inner automorphism property. In the process of doing this, some of the natural subsemigroups of these spaces are also considered. In particular, the consequences of requiring the functions in these classes to also be differentiable, approximately differentiable, $\mathcal{I}$-approximately differentiable or homeomorphisms are examined. This material has interest in the general theory of topological semigroups because of the result from Chapter 2 that the $\mathcal{I}$-density topology is not generated. The spaces considered in Chapter 4 provide some of the few known examples of richly structured semigroups of continuous functions $f : X \to X$, where the underlying topological space X is not generated, but the semigroups have the inner automorphism property.

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CHAPTER 1

The Ordinary Density Topology

1.1. A Simple Category Topology

In studying the behavior of real functions, it is often helpful to redefine the concept of limit in such a way that sets which are "small" in some sense can be ignored. As a simple example, let $\mathbb{Q}$ be the set of rational numbers and $f\colon \mathbb{R} \to \mathbb{R}$ be a real function. If

$$i\!-\!\lim_{y\to x} f(y) = \lim_{y\to x, y\notin\mathbb{Q}} f(y)$$

exists, then $i\!-\!\lim_{y\to x} f(y)$ is the *irrational limit* of f at x. With this limit, the rational numbers are the small set which is ignored. Of course, the definition of *irrational continuity* is natural from this definition.

To gain some insight into what is happening with limits like this, it is useful to generalize this idea to a topological setting.

A nonempty family $\mathcal{J} \subset \mathcal{P}(X)$ of subsets of X is an *ideal on* X if $A \subset B$ and $B \in \mathcal{J}$ imply that $A \in \mathcal{J}$ and if $A \cup B \in \mathcal{J}$ provided $A, B \in \mathcal{J}$. An ideal $\mathcal{J}$ on X is said to be a σ*-ideal on* X if $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{J}$ for every family $\{A_n\colon n \in \mathbb{N}\} \subset \mathcal{J}$.[1]

Let $\mathcal{J}$ be an ideal on $\mathbb{R}$ and $\mathcal{T}_{\mathcal{O}}$ be the ordinary topology on $\mathbb{R}$. The set

$$\mathcal{T}(\mathcal{J}) = \{G \setminus J : G \in \mathcal{T}_{\mathcal{O}}, J \in \mathcal{J}\}$$

is a topology on $\mathbb{R}$ which is finer than $\mathcal{T}_{\mathcal{O}}$. The following proposition is evident from the definitions.

Proposition 1.1.1. *Let $\mathcal{J}$ be a σ-ideal on $\mathbb{R}$ and $\mathcal{T}(\mathcal{J})$ be as above. For $f : (\mathbb{R}, \mathcal{T}(\mathcal{J})) \to (\mathbb{R}, \mathcal{T}_{\mathcal{O}})$ and $x_0 \in \mathbb{R}$ the following statements are equivalent to each other.*

(i): *f is continuous at x_0.*

(ii): *Given $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\{x \in (x_0 - \delta, x_0 + \delta) : |f(x) - f(x_0)| \geq \varepsilon\} \in \mathcal{J}.$$

[1] For more information on ideals see [**33**].

(iii): *There exists a $J \in \mathcal{J}$ such that*

$$f(x_0) = \lim_{x \to x_0, x \notin J} f(x).$$

The continuity given by Proposition 1.1.1 is called *$\mathcal{J}$-continuity.* As the following easy examples show, $\mathcal{J}$-continuity can be used to define irrational continuity and ordinary continuity.

EXAMPLE 1.1.2. *If $\mathcal{P}(\mathbb{Q})$ is the power set of $\mathbb{Q}$, then $\mathcal{P}(\mathbb{Q})$-continuity is the same as irrational continuity.*

EXAMPLE 1.1.3. *If $\mathcal{J} = \{\emptyset\}$, then $\mathcal{J}$-continuity is just ordinary continuity.*

For the following theorem, some new terminology is needed. Let $\mathcal{J}$ be an ideal of subsets of $\mathbb{R}$. A proposition is said to be *true $\mathcal{J}$-a.e.* if the set of points at which it does not hold is a member of $\mathcal{J}$; more formally, a Boolean function P is true $\mathcal{J}$-a.e. if, and only if,

$$\{x : \neg P(x)\} \in \mathcal{J}.$$

THEOREM 1.1.4. *Let $\mathcal{J}$ be a σ-ideal and $f \colon \mathbb{R} \to \mathbb{R}$. The following statements are equivalent.*

(i): *The function f is $\mathcal{J}$-continuous $\mathcal{J}$-a.e.*

(ii): *There exists a set $K \in \mathcal{J}$ such that the restricted function $f|_{K^c}$ is continuous.*

Furthermore, if the ideal $\mathcal{J}$ contains no interval, then the following statement is equivalent to (i) and (ii)

(iii): *There exists a function $g \colon \mathbb{R} \to \mathbb{R}$ such that $f = g$ $\mathcal{J}$-a.e. and g is continuous in the ordinary sense $\mathcal{J}$-a.e.*

PROOF. The fact that (ii) implies (i) is obvious. Suppose (i) is true and let f be $\mathcal{J}$-continuous on a set $M = J^c$ for $J \in \mathcal{J}$. For each $n \in \mathbb{N}$ and $x \in M$, by Proposition 1.1.1(ii) there is an open interval $I(n,x)$ and a $J(n,x) \in \mathcal{J}$ such that

$$x \in I(n,x) \setminus J(n,x) \subset f^{-1}((f(x) - 1/n, f(x) + 1/n)).$$

For each fixed n, there must be a countable sequence $x_{n,m} \in M$ such that

$$M \subset \bigcup_{m \in \mathbb{N}} I(n, x_{n,m}).$$

Let

$$K = J \cup \bigcup_{n,m \in \mathbb{N}} J(n, x_{n,m}) \in \mathcal{J}.$$

If $x \in K^c$ and $\varepsilon > 0$, then there must exist natural numbers n and m such that $2/n < \varepsilon$ and $x \in I(n, x_{n,m})$. Then $|f(x) - f(x_{n,m})| < 1/n$ so that

$$f(x) \in (f(x_{n,m}) - 1/n, f(x_{n,m}) + 1/n) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$$

and

$$I(n, x_{n,m}) \cap K^c \subset f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon)).$$

Hence, $f|_{K^c}$ is continuous at x.

To prove the last part of the theorem, note first that (iii) implies (ii) even without the restriction that $\mathcal{J}$ contains no interval. Now suppose that $\mathcal{J}$ contains no interval and that f, K are as in (ii). Define

$$G(x) = \limsup_{t \to x, t \in K^c} f(t) \tag{1}$$

and

$$g(x) = \begin{cases} G(x) & \text{when } G(x) \text{ is finite,} \\ f(x) & \text{otherwise.} \end{cases} \tag{2}$$

In particular, it follows from (ii) that $f|_{K^c} = g|_{K^c}$. Let $x \in K^c$ and $\varepsilon > 0$. According to (ii) there is a $\delta > 0$ such that

$$|g(y) - g(x)| = |f(y) - f(x)| < \varepsilon/2 \tag{3}$$

whenever $y \in (x - \delta, x + \delta) \cap K^c$. If $z \in (x - \delta, x + \delta) \cap K$, then the assumption that K can contain no nonempty open set implies the existence of a sequence

$$\{z_n : n \in \mathbb{N}\} \subset (x - \delta, x + \delta) \cap K^c$$

such that $f(z_n) \to G(z)$. Hence, by (3), $G(z)$ is finite, so $g(z) = G(z)$ and $|g(z) - g(x)| \leq \varepsilon/2 < \varepsilon$. Therefore, g is continuous at x. □

The following example is interesting in light of the previous theorem.

Example 1.1.5. *Let $\mathcal{I}$ be the σ-ideal consisting of all first category subsets of $\mathbb{R}$. $\mathcal{I}$-continuity is often called* qualitative continuity [**26**]. *It is well-known in this case that f is a Baire function if, and only if, f is qualitatively continuous $\mathcal{I}$-a.e.*

In particular, combining Example 1.1.5 with Theorem 1.1.4 yields the following well-known corollary, which will be useful in the sequel.

Corollary 1.1.6. *Let $f : \mathbb{R} \to \mathbb{R}$. The following statements are equivalent.*

- **(i):** *f is a Baire function.*
- **(ii):** *There exists a residual set K such that $f|_K$ is continuous.*[2]
- **(iii):** *f is qualitatively continuous $\mathcal{I}$-a.e.*

In the case of Lebesgue measure, the following is true.

[2] A set is *residual* if its complement is first category. This is often called *comeager*.

EXAMPLE 1.1.7. *If $\mathcal{J} = \mathcal{N}$, the σ-ideal of Lebesgue measure zero sets, then $\mathcal{N}$-continuity corresponds to what is sometimes called* measure continuity*. There are measurable functions which are not $\mathcal{N}$-continuous a.e. An easy example of this is the characteristic function of a nowhere dense perfect set with positive measure.*

If condition (i) in Theorem 1.1.4 is strengthened to everywhere, the following corollary results.

COROLLARY 1.1.8. *Let $\mathcal{J}$ be a σ-ideal which contains no nonempty open set. A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous everywhere if, and only if, it is $\mathcal{J}$-continuous everywhere.*

PROOF. If f is continuous, then it is clearly $\mathcal{J}$-continuous. So, suppose f is $\mathcal{J}$-continuous everywhere, $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Using Proposition 1.1.1(ii), there must be an ordinary open neighborhood G_0 of x_0 such that

$$F_0 = \{x \in G_0 : |f(x) - f(x_0)| > \varepsilon\} \in \mathcal{J}.$$

Suppose there is an $x_1 \in F_0$. Choose $\delta > 0$ such that

$$\delta < |f(x_1) - f(x_0)| - \varepsilon.$$

As before, there exists an ordinary open neighborhood $G_1 \subset G_0$ of x_1 such that

$$F_1 = \{x \in G_1 : |f(x_1) - f(x)| > \delta\} \in \mathcal{J}.$$

It is clear that $G_1 \subset F_0 \cup F_1 \in \mathcal{J}$, because $|f(x_1) - f(x_0)| > \varepsilon + \delta$. But, this implies $\mathcal{J}$ contains a nonempty open set, which contradicts the condition placed on $\mathcal{J}$ in the statement of the corollary. This contradiction shows that $F_0 = \emptyset$. $\square$

The preceding corollary demonstrates that global $\mathcal{J}$-continuity may not be a very useful concept. In particular, it is worthwhile noting for future reference that global $\mathcal{I}$-continuity and global $\mathcal{N}$-continuity are no different than ordinary continuity.

In the remainder of this work, we will introduce refinements of the ordinary topology which are strictly finer than the $\mathcal{T}(\mathcal{J})$ topologies introduced above. These topologies are defined similarly to the ordinary density topology and have a rich and complicated structure which is still not fully understood. In particular, the functions continuous with respect to these topologies on the domain and the range are not necessarily continuous in the ordinary sense. The primary purpose of this book is to study the continuity properties of functions with respect to two of these fine topologies, the $\mathcal{I}$-density topology and the deep-$\mathcal{I}$-density topology. In the process of this investigation, many properties of both topologies are presented.

Since many of the propositions concerning the $\mathcal{I}$-density topology and the deep-$\mathcal{I}$-density topology are motivated by similar results concerning the ordinary density topology, the next few sections are devoted to an overview of standard facts concerning that topology. Much of this material is not proved, although the proofs can be found in several standard sources [**7, 28, 70**]. Finally, near the end of this chapter, the definition of abstract density topologies is given.

1.2. Definition of the Density Topology

If A is a subset of the reals, we denote its inner (outer, ordinary) Lebesgue measure by $\mathrm{m}_i(A)$ ($\mathrm{m}_o(A)$, $\mathrm{m}(A)$). Let $\mathcal{L}$ stand for the family of Lebesgue measurable subsets of $\mathbb{R}$ and let $\mathcal{N}$ denote the ideal of subsets of $\mathbb{R}$ with Lebesgue measure zero. A number x, not necessarily in A, is a *density point* of A if

$$\lim_{h\to 0^+} \frac{\mathrm{m}_i(A\cap(x-h,x+h))}{2h} = 1. \tag{4}$$

This definition is often made more tractible by the following proposition.

PROPOSITION 1.2.1. *Let $A \subset \mathbb{R}$. Equation (4) holds if, and only if,*

$$\lim_{n\to\infty} \frac{m_i(A\cap(x-\frac{1}{n},x+\frac{1}{n}))}{2/n} = 1. \tag{5}$$

PROOF. It is obvious that (4) implies (5). For simplicity we prove the opposite implication only for $x = 0$. Let $h_n < 1$ be any sequence of positive numbers converging to 0. For each n, let

$$\underline{h_n} = \max\{1/k : k \in \mathbb{N},\ 1/k \le h_n\}$$

and

$$\overline{h_n} = \min\{1/k : k \in \mathbb{N},\ 1/k \ge h_n\}.$$

It is easy to see that $\lim_{n\to\infty} \underline{h_n}/h_n = 1 = \lim_{n\to\infty} \overline{h_n}/h_n$. The proposition is now an immediate consequence of (5) and

$$\frac{\underline{h_n}}{h_n}\frac{\mathrm{m}_i(A\cap(-\underline{h_n},\underline{h_n}))}{2\underline{h_n}} \le \frac{\mathrm{m}_i(A\cap(-h_n,h_n))}{2h_n} \le \frac{\overline{h_n}}{h_n}\frac{\mathrm{m}_i(A\cap(-\overline{h_n},\overline{h_n}))}{2\overline{h_n}}.$$

We let $\Phi_{\mathcal{N}}(A)$ be the set of all density points of $A \subset \mathbb{R}$. It is a consequence of the Lebesgue Density Theorem [**64**, p. 107] that a set A is measurable if, and only if, almost every point of A is a density point of A. Therefore, whenever A is measurable, then so is $\Phi_{\mathcal{N}}(A)$ and $\mathrm{m}(A \triangle \Phi_{\mathcal{N}}(A)) = 0$.[3] □

Let $\mathcal{T}_{\mathcal{N}} = \{A \subset \mathbb{R} : A \subset \Phi_{\mathcal{N}}(A)\}$.

THEOREM 1.2.2. *$\mathcal{T}_{\mathcal{N}}$ is a topology on $\mathbb{R}$. Further, if $A \in \mathcal{T}_{\mathcal{N}}$, then A is Lebesgue measurable; i.e.,*

$$\mathcal{T}_{\mathcal{N}} = \{A \in \mathcal{L} : A \subset \Phi_{\mathcal{N}}(A)\} \subset \mathcal{L}.$$

[3] $A\triangle B = (A \cup B) \setminus (A \cap B)$ is the *symmetric difference* between A and B.

Proof. It is obvious that $\emptyset, \mathbb{R} \in \mathcal{T}_{\mathcal{N}}$ and that $\mathcal{T}_{\mathcal{N}}$ is closed under arbitrary unions. It is also easy to see that $\mathcal{T}_{\mathcal{N}}$ is closed under finite intersections. Therefore, $\mathcal{T}_{\mathcal{N}}$ is a topology on $\mathbb{R}$.

To show $\mathcal{T}_{\mathcal{N}} \subset \mathcal{L}$, let $A \subset \mathbb{R}$ be such that $A \subset \Phi_{\mathcal{N}}(A)$. For $x \in A$ and $n \in \mathbb{N}$ let $F_x^n \subset A$ be an F_σ set such that

$$\mathrm{m}(F_x^n \cap (x - \frac{1}{n}, x + \frac{1}{n})) = \mathrm{m}_i(A \cap (x - \frac{1}{n}, x + \frac{1}{n})).$$

Then, $F_x = \bigcup_{n\in\mathbb{N}} F_x^n$ is an F_σ set such that x is a density point of F_x. Construct, by transfinite induction on $\zeta < \Omega$, the first uncountable ordinal number, an increasing sequence $\{F_\zeta \colon \zeta < \Omega\}$ of $\mathbf{F}_\sigma$ subsets of A such that for $\zeta < \Omega$

$$F_\zeta = F_x \cup \bigcup_{\xi<\zeta} F_\xi$$

provided there exists an $x \in A$ such that $\mathrm{m}(F_x \setminus \bigcup_{\xi<\zeta} F_\xi) > 0$ and put $F_\zeta = \bigcup_{\xi<\zeta} F_\xi$ otherwise. Notice, that there exists $\eta < \Omega$ such that $F_\eta = F_\xi$ for every $\eta \le \xi < \Omega$, since otherwise we would have an uncountable family $\{F_{\zeta+1} \setminus F_\zeta \colon \zeta < \Omega\}$ of pairwise disjoint subsets of $\mathbb{R}$ with positive measure. Hence, $\mathrm{m}(F_x \setminus F_\eta) = 0$ for every $x \in A$. But this means that $x \in \Phi_{\mathcal{N}}(F_x) \subset \Phi_{\mathcal{N}}(F_\eta)$ for every $x \in A$; i.e., $F_\eta \subset A \subset \Phi_{\mathcal{N}}(F_\eta)$. But, by the Lebesgue Density Theorem, $\Phi_{\mathcal{N}}(F_\eta) \in \mathcal{L}$ and $\mathrm{m}(\Phi_{\mathcal{N}}(F_\eta) \setminus F_\eta) = 0$. So A is Lebesgue measurable. □

$\mathcal{T}_{\mathcal{N}}$ is called the *density topology* on $\mathbb{R}$. The following theorem summarizes some of its properties.

Theorem 1.2.3. *The topology $\mathcal{T}_{\mathcal{N}}$ on $\mathbb{R}$ has the following properties:*

- **(i):** *$\mathcal{T}_{\mathcal{O}} \subset \mathcal{T}_{\mathcal{N}}$* [**31**] *and the inclusion is proper; in particular $\mathcal{T}_{\mathcal{N}}$ is Hausdorff* [**29**]*;*
- **(ii):** *a subset C of $\mathbb{R}$ is closed and discrete with respect to $\mathcal{T}_{\mathcal{N}}$ if, and only if, $C \in \mathcal{N}$* [**61**]*;*
- **(iii):** *$\mathcal{T}_{\mathcal{N}}$ is neither separable* [**61**] *nor does it have has the Lindelöf property* [**29**]*;*
- **(iv):** *$\mathcal{T}_{\mathcal{N}}$ is completely regular, but not normal* [**28**]*;*
- **(v):** *every subinterval of $\mathbb{R}$ is connected in $\mathcal{T}_{\mathcal{N}}$* [**29**]*;*
- **(vi):** *a set A is compact with respect to $\mathcal{T}_{\mathcal{N}}$ if, and only if, it is finite* [**61**]*;*
- **(vii):** *$\mathcal{T}_{\mathcal{N}}$ is not generated*[4] [**11**]*; and,*
- **(viii):** *if the Continuum Hypothesis holds, then $(\mathbb{R}, \mathcal{T}_{\mathcal{N}})$ is not a Blumberg space*[5] [**65**].

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[4] A topological space X is said to be *generated* provided the family

$$\{\left(f^{-1}(\{x\})\right)^c : x \in X \text{ and } f \colon X \to X \text{ is continuous}\}$$

forms a subbase for X. (See Chapter 4.)

[5] A topological space X is Blumberg if for every function $f \colon X \to \mathbb{R}$ there exists a dense set $D \subset X$ such that $f|_D$ is continuous [**63, 65**].

Properties (vii) and (viii) of the density topology often serve as counterexamples in general topology. In particular, the property that the density topology is not generated is especially interesting in conjunction with the fact that the semigroup of its continuous selfmaps has the inner automorphism property [**54**]. This particular topic will be explored more fully in Chapter 4.

For use in Chapter 2, it is desirable to reformulate the definition of the density topology in a radically different, but equivalent manner. To do this let us consider the following sequence of equivalences for a measurable set A [**57**].

(A): 0 is a density point of A;
(B): $\lim_{n\to\infty} \frac{\mathrm{m}(A\cap(-1/n,1/n))}{2/n} = 1$;
(C): $\lim_{n\to\infty} \mathrm{m}(nA \cap (-1,1)) = 2$;
(D): $\chi_{nA\cap(-1,1)}$ converges to $\chi_{(-1,1)}$ in measure; and,
(E): for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{n_{m_p}A\cap(-1,1)} = \chi_{(-1,1)} \text{ a.e.}$$

The equivalences (A)–(D) are straightforward.

The equivalence of (D) and (E) follows from a well-known theorem of Riesz concerning convergence in measure (often called *stochastic convergence*) [**2**, Theorem 2.11.6].

The equivalence from Proposition 1.2.1 can be used to slightly generalize these five statements to the following equivalences.

(A): 0 is a density point of A;
(B′): for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity

$$\lim_{n\to\infty} \frac{\mathrm{m}(A\cap(-1/t_n,1/t_n))}{2/t_n} = 1;$$

(C′): for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity

$$\lim_{n\to\infty} \mathrm{m}(t_nA \cap (-1,1)) = 2;$$

(D′): for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity

$$\chi_{t_nA\cap(-1,1)} \quad \text{converges to} \quad \chi_{(-1,1)} \quad \text{in measure; and,}$$

(E′): for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity there exists a subsequence $\{t_{n_p}\}_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{t_{n_p}A\cap(-1,1)} = \chi_{(-1,1)} \text{ a.e.}$$

Using the translation invariance of Lebesgue measure, the same sequences of equivalences can be rewritten for any density point of A.

It is important to note that we argued for equivalence of (A), (E) and (E′) only in the case of measurable sets. It will be seen in Section 1.5 that for general sets this equivalence is false.

The significance of the equivalences of (E) and (E′) with (A) is that it clearly shows the measure function itself is not vital to the definition of a density point. What are needed is the σ-algebra $\mathcal{L}$ of measurable sets and the σ-ideal $\mathcal{N}$ of measure zero sets. We will return to this idea in Section 1.5. Right now, some more properties of the density topology will be examined.

1.3. Approximate Continuity

Using the various combinations of $\mathcal{T}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{N}}$ on the domain and range, there are four different ways that a function $f : \mathbb{R} \to \mathbb{R}$ can be continuous. Of course, the most studied is when the ordinary topology is used on both the domain and the range, giving ordinary continuity. The class of all ordinary continuous functions will be denoted by $\mathcal{C}_{\mathcal{O}\mathcal{O}}$. It is straightforward to show that if the density topology is put on the range and the ordinary topology is put on the domain, then only the constant functions are continuous.[6] If both the domain and range have the density topology, we have *density continuity*, which is the topic of Section 1.4. The remaining case is when the ordinary topology is used on the range and the density topology is used on the domain. Any function continuous under this setting is *approximately continuous*. The collection of all functions $f : \mathbb{R} \to \mathbb{R}$ which are approximately continuous is denoted as $\mathcal{C}_{\mathcal{N}\mathcal{O}}$.

The notion of approximate continuity was introduced by Denjoy in 1915 [**24**], surprisingly, several decades before the density topology was introduced. Denjoy said that a function f is approximately continuous at a point x if given any $\varepsilon > 0$, x is a density point of

$$\{t \in \mathbb{R} : f(t) \in (f(x) - \varepsilon, f(x) + \varepsilon)\}.$$

His definition implies the existence of a density neighborhood U of x such that $f|_U$ is continuous at x. It allows one to ignore a set which has density 0 at x. It is also easy to see from this definition that every continuous function is approximately continuous. The construction of functions which are approximately continuous, but not continuous is relatively easy as can be seen from Example 3.6.4, so the inclusion $\mathcal{C}_{\mathcal{O}\mathcal{O}} \subset \mathcal{C}_{\mathcal{N}\mathcal{O}}$ is proper.

The following theorem contains some of the most important properties of $\mathcal{C}_{\mathcal{N}\mathcal{O}}$ [**7**].

THEOREM 1.3.1. *Following are some of the properties of* $\mathcal{C}_{\mathcal{N}\mathcal{O}}$.

(i): $\mathcal{C}_{\mathcal{N}\mathcal{O}}$ *is closed under pointwise addition and multiplication.*

[6] If $f : (\mathbb{R}, \mathcal{T}_{\mathcal{O}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{N}})$ is continuous and $a < b$, then $f([a, b])$ must be a nonempty compact and connected set with respect to $\mathcal{T}_{\mathcal{N}}$. Parts (v) and (vi) of Theorem 1.2.3 show that $f([a, b])$ is a single point. (See also Theorem 3.8.2.)

(ii): *$\mathcal{C}_{\mathcal{N}\mathcal{O}}$ is closed under uniform convergence; hence, the bounded approximately continuous functions form a Banach space.*
(iii): *If $f \in \mathcal{C}_{\mathcal{N}\mathcal{O}}$, then f is a Darboux function of the first Baire class.*
(iv): *Every bounded approximately continuous function is a derivative.*

Approximate continuity has proved especially useful in the study of derivatives and integrals. The deep study by Z. Zahorski [**70**] of the derivatives of real functions is based around the twin ideas of approximate continuity and density points. (Also see [**7**].)

Let $\mathcal{F}_m$ stand for the collection of measurable functions $f : \mathbb{R} \to \mathbb{R}$. Theorem 1.3.1(iii) shows that $\mathcal{C}_{\mathcal{N}\mathcal{O}} \subset \mathcal{F}_m$. It is easy to see that this inclusion is proper. The following theorem, due to Lusin, gives the full relationship between approximate continuity and $\mathcal{F}_m$ [**7**, Theorem 5.2, p. 19].

THEOREM 1.3.2. *A function $f \in \mathcal{F}_m$ if, and only if, it is approximately continuous almost everywhere.*

1.4. Density Continuity

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *density continuous*, if it is continuous with respect to the density topology on the domain and the range. We denote the class of density continuous functions $f : \mathbb{R} \to \mathbb{R}$ by $\mathcal{C}_{\mathcal{N}\mathcal{N}}$. Since the density topology is finer than the ordinary topology, it is immediate that $\mathcal{C}_{\mathcal{N}\mathcal{N}} \subset \mathcal{C}_{\mathcal{N}\mathcal{O}}$. The containment is proper, and density continuity is apparently a much more delicate notion than approximate continuity. This statement is strikingly demonstrated by the following example [**13, 8**].

EXAMPLE 1.4.1. *There is a $\mathcal{C}^\infty$ function which is not in $\mathcal{C}_{\mathcal{N}\mathcal{N}}$.*

There are several differences which make the density continuous functions more intractible than the approximately continuous functions. For example, $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ is not closed under uniform convergence [**54**] (this follows immediately from Example 1.4.1 and Theorem 1.4.3, given below) or pointwise addition. In fact, there is a $\mathcal{C}^\infty$ density continuous function f such that $f(x)+x$ is not density continuous [**13, 8**].

Some positive things can be proved about $\mathcal{C}_{\mathcal{N}\mathcal{N}}$. For example, since approximately continuous functions are Darboux and in the first Baire class, so are the functions in $\mathcal{C}_{\mathcal{N}\mathcal{N}}$. This shows that every density continuous function must be continuous on a residual set.

This result has been improved in the following manner.

In an effort to generalize a well-known characterization of the Baire 1 functions, O'Malley [**51**] introduced the notion of a Baire*1 function. He defined a function f to be in Baire*1 if for every perfect set P, there exists a portion[7]

[7]That is, a nonempty set of the form $(a, b) \cap P$.

Q of P such that $f|_Q$ is continuous. An equivalent notion, denoted [CG], and known as *generalized continuity*, was used also by Ellis [**25**].

It is well-known that a function f is in Baire 1 if, and only if, for every perfect set P, the restricted function $f|_P$ has a point of continuity [**7**]. The Baire*1 functions are clearly Baire 1 functions. It also follows from the definition of Baire*1 that every Baire*1 function is continuous on a dense open set. The class of the Darboux Baire*1 functions will be denoted by $\mathcal{DB}_1^*$.

We have the following theorem [**21**].

THEOREM 1.4.2. $\mathcal{C}_{\mathcal{N}\mathcal{N}} \subset \mathcal{DB}_1^*$.

The relationship between the density continuous functions and the continuous functions is somewhat difficult. It follows from Example 1.4.1 that $\mathcal{C}_{\mathcal{O}\mathcal{O}} \not\subset \mathcal{C}_{\mathcal{N}\mathcal{N}}$. It is also not difficult to see that $\mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}_{\mathcal{O}\mathcal{O}}$. As noted above, every density continuous function is continuous on a dense open set. But, it turns out that if the continuous functions on $[0,1]$ are endowed with the uniform metric, then $\mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \mathcal{C}_{\mathcal{O}\mathcal{O}}$ is a first category subset of this Banach space. In fact, $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ with the uniform metric turns out to be first category in itself. (See Theorem 3.7.5($\mathcal{N}$8).) Furthermore, it can be proved that the "typical" continuous function[8] is nowhere density continuous [**21**]. Nevertheless, it can be shown that several familiar classes of functions are density continuous. Consider the following theorem [**13**].

THEOREM 1.4.3. *If f is convex on an interval (a,b), then f is density continuous on (a,b). In particular, polynomials and real analytic functions are density continuous.*

Another class of density continuous functions is obtained by the following [**6, 13**].

THEOREM 1.4.4. *If h is a homeomorphism such that h and h^{-1} fulfill a local Lipschitz condition, then h and h^{-1} are density continuous.*

All density continuous functions obtained from the above theorems are piecewise monotone. The existence of nowhere monotone density continuous function is a consequence of the following example [**20**].

EXAMPLE 1.4.5. *The first coordinate of the classical Peano area-filling curve is continuous, density continuous and nowhere approximately differentiable.*

[8]A subset of a Baire topological space is "typical", if its complement is first category.

1.5. Abstract Density Topologies

Let us now return to the idea introduced at the end of Section 1.2. We use this idea to introduce an entire class of fine topologies on $\mathbb{R}$ exhibiting behavior which is more complicated than the standard density topology.

As before, let $\mathcal{N}$ denote the ideal of Lebesgue null sets. The sequence of equivalences at the end of Section 1.2 shows that the Lebesgue density points can be defined using only $\mathcal{N}$ and not the full machinery of Lebesgue measure theory. This leads us to the following definitions, which are meant to mimic the situation with ordinary density points.

Let $\mathcal{J}$ be an ideal of subsets of $\mathbb{R}$. If f_n is a sequence of real-valued functions, we say that f_n *converges ($\mathcal{J}$)* to a function f, if for every subsequence f_{n_p} of f_n, there exists a further subsequence $f_{n_{p_q}}$ such that $f_{n_{p_q}}$ converges pointwise to f $\mathcal{J}$-a.e. In the case when $\mathcal{J} = \mathcal{N}$ and all functions f_n are equal outside of some set of finite measure, this is the usual notion of stochastic convergence, or convergence in measure.

A point a is a *$\mathcal{J}$-density point* of a set $S \subset \mathbb{R}$ if $\chi_{n(S-a)\cap(-1,1)}$ converges ($\mathcal{J}$) to $\chi_{(-1,1)}$. We let $\Phi_{\mathcal{J}}(S)$ denote the set of all $\mathcal{J}$-density points of the set S.

In this way we have defined a $\mathcal{J}$-density which is analogous to ordinary Lebesgue density. In fact, when the ideal $\mathcal{J}$ is taken to be the ideal of Lebesgue null sets, $\mathcal{N}$, and the set S is measurable, we saw in Section 1.2 that the $\mathcal{J}$-density points are precisely the points of ordinary Lebesgue density. To continue the analogy, let

$$\mathcal{T}'_{\mathcal{J}} = \{S \subset \mathbb{R} : S \subset \Phi_{\mathcal{J}}(S)\}.$$

It is quite easy to see that the family $\mathcal{T}'_{\mathcal{J}}$ is a topology on $\mathbb{R}$ and that in the case $\mathcal{J} = \mathcal{N}$, it contains the ordinary density topology; i.e., that $\mathcal{T}_{\mathcal{N}} \subset \mathcal{T}'_{\mathcal{N}}$.

At this point, one might suspect that the analogy will continue and that the properties of $\mathcal{T}'_{\mathcal{J}}$ could be developed very naturally along the same lines as the ordinary density topology. The following example shows this is not the case [**19**].

Example 1.5.1. *There exists a nonmeasurable set $A \subset \mathbb{R}$, which does not have the Baire property, such that* $\lim_{n\to\infty} \chi_{n(A-a)\cap(-1,1)} = \chi_{(-1,1)}$ *for every* $a \in A$.

Proof. Let B be a Hamel basis which is also a Bernstein set; i.e., a linear basis of $\mathbb{R}$ over $\mathbb{Q}$, such that B intersects every nonempty perfect set.[9] For $x \in \mathbb{R}$ let $\rho'(x) = \sum_{i=1}^{n} |\alpha_i|$ where $x = \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n$ is the representation of x in the base B. Define

$$A = \{x \in \mathbb{R} \colon \rho'(x) < s\},$$

[9] To construct such a basis $B = \{b_\zeta : \zeta < c\}$, it is enough to define it by transfinite induction on ζ. We can simply choose b_ζ from $P_\zeta \setminus \mathbb{Q}(\{b_\xi : \xi < \zeta\})$, where $\{P_\zeta : \zeta < c\}$ is a fixed enumeration of the nonempty perfect subsets of $\mathbb{R}$ and $\mathbb{Q}(\{b_\xi : \xi < \zeta\})$ stands for the subfield of $\mathbb{R}$ generated by $\{b_\xi : \xi < \zeta\}$. Compare also [**32**].

where $s \in (1/2, 1)$ is any irrational number. Obviously, if $P \neq \emptyset$ is perfect, then $2P \neq \emptyset$ is also perfect, so B intersects $2P$ and $\frac{1}{2}B$ intersects P. But,

$$\frac{1}{2}B \subset A,$$

so that A intersects every nonempty perfect set. Also, $B \subset \mathbb{R} \setminus A$, and thus the complement of A also intersects every nonempty perfect set. This proves that A is neither Lebesgue measurable, nor does it have the Baire property.

Let $a \in A$. We will prove that

$$\lim_{n\to\infty} \chi_{n(A-a)\cap(-1,1)} = \chi_{(-1,1)}$$

everywhere.

Let $x \in (-1, 1)$. There is a natural number n_0 such that

$$\frac{1}{n}x + a \in A$$

for all $n \geq n_0$, $n \in \mathbb{N}$. In fact, for $x = \beta_1 b_1 + \beta_2 b_2 + \ldots + \beta_k b_k$ and $a = \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_k b_k$ it suffices to choose n_0 such that

$$\sum_{i=1}^{k} |\alpha_i| + \frac{1}{n_0} \sum_{i=1}^{k} |\beta_i| < s.$$

Then, for every $n \geq n_0$,

$$x = n\left(\left(\frac{1}{n}x + a\right) - a\right) \in n(A - a).$$

This ends the proof. $\square$

Let $\mathcal{O} = \{\emptyset\}$, the empty ideal. Using the notation of the previous proof, it is easy to see that, by the irrationality of s, the complement A^c of A satisfies the same condition as A in the above example. Thus

Corollary 1.5.2. *Let $\mathcal{J}$ be an arbitrary ideal. There exists a nonmeasurable set $A \subset \mathbb{R}$, which does not have the Baire property, such that $A, A^c \in \mathcal{T}'_{\mathcal{O}} \subset \mathcal{T}'_{\mathcal{J}}$. In particular, $\mathcal{T}'_{\mathcal{J}}$ is disconnected.*

The set A in the previous example is a "universal" open and closed set in all the topologies $\mathcal{T}'_{\mathcal{J}}$ as defined above. In particular, it allows us to conclude $\mathcal{T}_{\mathcal{N}} \neq \mathcal{T}'_{\mathcal{N}}$.

So the logical question is, why is it avoided in the ordinary density topology? A careful reading of the equivalences used in Section 1.2 shows that those equivalences are only valid when the set is assumed to be Lebesgue measurable. Thus, the ordinary density topology consists of all sets S such that $S \subset \Phi_{\mathcal{N}}(S)$ *and* S is measurable, which excludes the set defined in Example 1.5.1. Theorem 1.2.2 shows that this restriction to measurable sets is a consequence of the normal definition of the density topology, contained in (4), but it is lost with the more general approach, based on ideals.

The problem with the analogy extends even more deeply than this with the fact that, in a sense, Proposition 1.2.1 may also fail. To see this, let $\{t_n\}$ be any sequence of positive numbers diverging to infinity. For any $S \subset \mathbb{R}$, define $\Psi_{\mathcal{J}}(S, \{t_n\})$ to be the set of all $a \in S$ such that $\chi_{t_n(S-a)\cap(-1,1)}$ converges ($\mathcal{J}$) to $\chi_{(-1,1)}$. Then, set

$$\Psi_{\mathcal{J}}(S) = \bigcap_{\{t_n\}} \Psi_{\mathcal{J}}(S, \{t_n\}),$$

where the intersection is over all sequences $\{t_n\}$ of positive numbers diverging to infinity. It is not hard to show that

$$\mathcal{T}''_{\mathcal{J}} = \{S \subset \mathbb{R} : S \subset \Psi_{\mathcal{J}}(S)\} \subset \mathcal{T}'_{\mathcal{J}}$$

is a topology on $\mathbb{R}$. This is termed an *abstract $\mathcal{J}$-density topology.*

Example 1.5.3. *There exists a nonmeasurable set $A \subset \mathbb{R}$, which does not have the Baire property,such that $A \subset \Psi_{\mathcal{I}_c}(A)$, where $\mathcal{I}_c$ is the ideal of all sets of cardinality less than c, the cardinality of the continuum.*

Proof. Let $B = \{b^{\zeta+1} \colon \zeta < c\}$ be a transcendental base of $\mathbb{R}$ over $\mathbb{Q}$ such that B intersects every perfect set.[10] For $\xi < c$ let K_ξ be the algebraic closure of $\{b^\zeta \colon \zeta < \xi\}$ in $\mathbb{R}$ and let B^ξ be a linear base of $K_{\xi+1}$ over K_ξ containing 1 and b^ξ. For $a \in K_{\xi+1} \setminus K_\xi$ let $\rho''(a) = |\alpha_0|$, where $a = \alpha_0 b^\xi + \alpha_1 b_1 + \ldots + \alpha_k b_k$ is the representation of a in the base B^ξ. Note that if $x \in K_\xi$ and $y \notin K_\xi$, then $\rho''(x+y) = \rho''(y)$.

Define $A = \{a \in \mathbb{R} \colon \rho''(a) < 1\}$. As in Example 1.5.1,

$$\frac{1}{2}B \subset A \text{ and } B \subset \mathbb{R} \setminus A,$$

so that A is neither measurable, nor does it have the Baire property.

Let $a \in A$ and let $\{t_n\}_{n\in\mathbb{N}}$ be an increasing sequence diverging to infinity. Let $\xi < c$ be such that $a, t_n \in K_\xi$ for every $n \in \mathbb{N}$. We have $\mathrm{card}(K_\xi) < c$. It suffices to show that for every $x \in (-1,1) \setminus K_\xi$

$$\lim_{n\to\infty} \chi_{t_n(A-a)\cap(-1,1)}(x) = \chi_{(-1,1)}(x) = 1.$$

It is enough to show that $x \in t_n(A-a)$ for all but finitely many n. This is equivalent to the fact that

$$\frac{x}{t_n} + a \in A.$$

But

$$\frac{1}{t_n} \in K_\xi,$$

so that

$$\frac{x}{t_n} \notin K_\xi,$$

[10]For the definition of a transcendental base see [**37**]. The additional requirements are obtained in the same way as described in the footnote for Theorem 1.5.1.

and $a \in K_\xi$. Therefore

$$\rho''\left(\frac{x}{t_n} + a\right) = \rho''\left(\frac{x}{t_n}\right) = \frac{\rho''(x)}{|t_n|}.$$

If we choose $n \in \mathbb{N}$ such that

$$\frac{1}{|t_n|} < 1,$$

then $x \in t_n(A - a)$. This finishes the proof. □

In the next chapters we will study these kinds of topologies for the ideal $\mathcal{I}$ of sets of the first category. In particular, the above Example implies

COROLLARY 1.5.4. *If either the Continuum Hypothesis, or Martin's Axiom is assumed, then there is a nonmeasurable set A without the Baire property such that $A \subset \Psi_{\mathcal{I}}(A) \cap \Psi_{\mathcal{N}}(A)$.*

PROOF. If the Continuum Hypothesis holds, then

$$\mathcal{I}_c \equiv \{B : \text{card}(B) < c\} = \{B : \text{card}(B) \leq \aleph_0\} \subset \mathcal{I} \cap \mathcal{N}.$$

Martin's Axiom also implies $\mathcal{I}_c \subset \mathcal{I} \cap \mathcal{N}$. (See [**35**].) □

The above example shows that the topology $\mathcal{T}''_{\mathcal{J}}$ does not behave as well as the ordinary density topology, even when $\mathcal{J} = \mathcal{N}$. Thus, to obtain a good analogy, we must refine the definition somewhat. However, the topologies $\mathcal{T}''_{\mathcal{J}}$ perhaps deserve closer scrutiny. For example the following open problems seem to be interesting.

PROBLEM 1.5.5. *Are the topologies $\mathcal{T}'_{\mathcal{I}}$, $\mathcal{T}'_{\mathcal{N}}$, $\mathcal{T}''_{\mathcal{I}}$ and $\mathcal{T}''_{\mathcal{N}}$ on $\mathbb{R}$ regular? completely regular? normal?*

PROBLEM 1.5.6. *Are the topologies $\mathcal{T}''_{\mathcal{I}}$ and $\mathcal{T}''_{\mathcal{N}}$ disconnected?*

To obtain the desired refinement of $\mathcal{T}''_{\mathcal{J}}$ and $\mathcal{T}'_{\mathcal{J}}$ let us notice that

$$\mathcal{T}_{\mathcal{N}} = \mathcal{T}''_{\mathcal{N}} \cap \mathcal{L} = \mathcal{T}'_{\mathcal{N}} \cap \mathcal{L}.$$

This suggests that the $\mathcal{J}$-topology $\mathcal{T}_{\mathcal{J}}$ should be defined as

$$\mathcal{T}_{\mathcal{J}} = \mathcal{T}''_{\mathcal{J}} \cap \mathcal{S}_{\mathcal{J}}$$

for some family $\mathcal{S}_{\mathcal{J}}$ of subsets of $\mathbb{R}$. The problem with this definition is how to correctly choose the family $\mathcal{S}_{\mathcal{J}}$ for the given ideal $\mathcal{J}$. Our choice should at least guarantee that the family

$$\mathcal{T}_{\mathcal{J}} = \mathcal{T}''_{\mathcal{J}} \cap \mathcal{S}_{\mathcal{J}} \quad \text{is a topology on } \mathbb{R}. \tag{6}$$

It would also be very desirable to have the equation

$$\mathcal{T}_{\mathcal{J}} = \mathcal{T}'_{\mathcal{J}} \cap \mathcal{S}_{\mathcal{J}}. \tag{7}$$

We will see in the next chapters that for the ideal $\mathcal{I}$ of sets of the first category, the family $\mathcal{S}_{\mathcal{I}} = \mathcal{B}$ of sets with the Baire property leads to the satisfaction of both conditions (6) and (7). Moreover, the qualitative topology $\mathcal{T}_Q$ is strictly contained in $\mathcal{T}_{\mathcal{I}}$. However, for a general ideal $\mathcal{J}$, finding the natural and non-trivial family $\mathcal{S}_{\mathcal{J}}$ satisfying conditions (6) and (7) could be very difficult, if not impossible.

The easiest example showing these difficulties comes from the ordinary topology $\mathcal{T}_{\mathcal{O}}$, for which it is easy to see that

$$\mathcal{T}_{\mathcal{O}} = \mathcal{T}''_{\mathcal{O}}.$$

But, there is a decreasing sequence S converging to 0 such that $S^c \in \mathcal{T}'_{\mathcal{O}}$ [**19**]. This implies that

$$\mathcal{T}'_{\mathcal{O}} \cap \mathbf{F}_\sigma \cap \mathbf{G}_\delta \not\subset \mathcal{T}_{\mathcal{O}} = \mathcal{T}''_{\mathcal{O}}.$$

However, to see how bad things can be, let us consider the ideal $\mathcal{I}_\omega$ of countable subsets of $\mathbb{R}$. In Lemma 2.1.8 we will prove the existence of a perfect set C such that for every $x \in C$ the set $\{x\} \cup C^c \in \mathcal{T}''_{\mathcal{I}_\omega}$. Thus, the topology generated by $\mathcal{T}''_{\mathcal{I}_\omega} \cap F_\sigma \cap G_\delta$ contains non-Borel sets. To make things worse we can construct, with a little bit of effort, a perfect set P with $0 \in P \subset [0,1]$, such that $\{0\} \cup P^c \in \mathcal{T}'_{\mathcal{I}_\omega} \setminus \mathcal{T}''_{\mathcal{I}_\omega}$ [**19**, Theorem 3]. Hence,

$$\mathcal{T}'_{\mathcal{I}_\omega} \cap F_\sigma \cap G_\delta \not\subset \mathcal{T}''_{\mathcal{I}_\omega}.$$

PROBLEM 1.5.7. *Is it possible to prove, without additional set theoretical assumptions, that $\mathcal{T}''_{\mathcal{I}} \neq \mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}''_{\mathcal{N}} \neq \mathcal{T}_{\mathcal{N}}$?*

1.6. Historical and Bibliographic Notes

The qualitative limit was defined by Marcus [**47**] in 1953. A deeper study of this notion was done by Evans and Larson [**26**] in 1983. In particular, this last paper contains the proof of Corollary 1.1.8 in the case when $\mathcal{J} = \mathcal{I}$. A form of Theorem 1.1.4 is given by Kuratowski [**36**] in the case when $\mathcal{J} = \mathcal{I}$. The topologies $\mathcal{T}(\mathcal{N})$ and $\mathcal{T}(\mathcal{I})$ were also studied by Hashimoto [**30**] in 1976.

The class of approximately continuous functionswas defined by Denjoy [**24**] as early as 1915. It was also extensively studied by Zahorski [**70**] in 1950 as part of his deep work on the ordinary derivative. In Zahorski's work, the approximately continuous functions correspond to his class $\mathcal{M}_5$.

It was not until 1952 that the density topology was defined by Haupt and Pauc [**31**] in a paper which seems to have had almost no impact. The first real study of the density topology dates from the 1961 paper of Goffman and Waterman [**29**]. Other important contributions can be found in the papers of Goffman, Neugebauer and Nishiura [**28**] and Tall [**61**].

Density continuity is a more recent concept. Bruckner [**6**] and Niewiarowski [**50**] discuss homeomorphic changes of variable which preserve approximate continuity. Such changes of variable must be density continuous, but neither author directly addresses density continuity in general. Bruckner does prove that a homeomorphism h such that both h and h^{-1} satisfy a local Lipschitz condition is density continuous. Density continuity was explicitly introduced in Query 1 of the first issue of the *Real Analysis Exchange*. Later, in response to the query, Maly [**46**] showed that the first coordinate of the classical Peano curve is a density continuous function which does not satisfy Lusin's condition **(N)**. Ostaszewski [**52, 53**] studied the local behavior of density continuous functions and called them *d-continuous*. It seems that the term *density continuity* was first used in a letter from David Preiss to Krzysztof Ostaszewski. More recent deeper investigations of density continuity are also available [**11, 18, 13, 21, 54**].

The equivalence of conditions (A) and (E) from Section 1.2 and the abstract definition of $\mathcal{J}$-density points are due to Wilczyński [**57, 66**].

Essentially all results from Section 1.4 were proved by Ciesielski, Larson and Ostaszewski in papers [**13, 20, 21, 54**] as marked in the text. Some of these results were also proved independently by Burke in [**8**].

Essentially all the results from Section 1.5 were proved by Ciesielski and Larson in [**19**]. An example similar to that from Corollary 1.5.4 could be also found in the 1984 paper [**67**] of Wilczyński. The subject of [**67**] is similar to that of [**19**], however both papers were written independently.

CHAPTER 2

Category Analogues of the Density Topology

In this chapter, the definitions of the $\mathcal{I}$-density and deep-$\mathcal{I}$-density topologies are introduced. Parallels between these topologies and the ordinary density topology are explored, and the topological properties of the $\mathcal{I}$-density topologies are examined.

2.1. $\mathcal{J}$-density and $\mathcal{J}$-dispersion Points.

Let $\mathcal{J}$ be an ideal of subsets of $\mathbb{R}$. As in Section 1.5 we say that a point a is a *$\mathcal{J}$-density point* of a set $A \subset \mathbb{R}$ if $\chi_{n(A-a)\cap(-1,1)}$ converges $(\mathcal{J})$ to $\chi_{(-1,1)}$; i.e., for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ with the property that

$$\lim_{p\to\infty} \chi_{n_{m_p}(A-a)\cap(-1,1)} = \chi_{(-1,1)}, \quad \mathcal{J}\text{-a.e.}$$

We let $\Phi_{\mathcal{J}}(A)$ denote the set of all $\mathcal{J}$-density points of the set A.

Similarly, we say that a point a is a *strong $\mathcal{J}$-density point* of a set $A \subset \mathbb{R}$ if for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity, $\chi_{t_n(A-a)\cap(-1,1)}$ converges $(\mathcal{J})$ to $\chi_{(-1,1)}$; i.e., for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity there exists a subsequence $\{t_{n_p}\}_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{t_{n_p}(A-a)\cap(-1,1)} = \chi_{(-1,1)}, \quad \mathcal{J}\text{-a.e.}$$

A point a is a *(strong) $\mathcal{J}$-dispersion point* of $A \subset \mathbb{R}$ if, and only if, it is a (strong) $\mathcal{J}$-density point of A^c.

For technical reasons it will often be convenient to use unilateral versions of the definitions given above. So, for example, we say that a is a *right $\mathcal{J}$-density point* of $A \subset \mathbb{R}$ if, and only if, for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{n_{m_p}(A-a)\cap[0,1)} = \chi_{[0,1)}, \quad \mathcal{J}\text{-a.e.}$$

The definitions of a point a being a *left $\mathcal{J}$-density point, strong right $\mathcal{J}$-density point* and *strong left $\mathcal{J}$-density point* of A are similar.

Notice that the definition of a non-zero number a being a (strong) $\mathcal{J}$-density or (strong) $\mathcal{J}$-dispersion point of $A \subset \mathbb{R}$ is just a translation of the definition at 0. Thus, in many proofs involving these notions, it suffices to consider only the case when $a = 0$.

The following is an easy, but useful characterization of the fact that 0 is a $\mathcal{J}$-density point.

LEMMA 2.1.1. *Let $\mathcal{J}$ be an ideal of subsets of $\mathbb{R}$ and let $A \subset \mathbb{R}$. Then 0 is a $\mathcal{J}$-density point of A if, and only if, for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there is a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that*

$$(-1,1) \cap (\bigcup_{q\in\mathbb{N}} \bigcap_{p\geq q} n_{m_p} A)^c = (-1,1) \cap (\liminf_{p\to\infty} n_{m_p} A)^c \in \mathcal{J}.$$

The next Lemma is a dual version of Lemma 2.1.1 and the definitions given above.

LEMMA 2.1.2. *Let $\mathcal{J}$ be an ideal of subsets of $\mathbb{R}$ and let $B \subset \mathbb{R}$. The following are equivalent:*

- **(i):** *0 is a $\mathcal{J}$-dispersion point of B;*
- **(ii):** *for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that*

$$\lim_{p\to\infty} \chi_{(n_{m_p}B)\cap(-1,1)} = 0, \quad \mathcal{J}\text{-a.e.};$$

- **(iii):** *for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that*

$$(-1,1) \cap \bigcap_{q\in\mathbb{N}} \bigcup_{p\geq q} n_{m_p} B = (-1,1) \cap \limsup_{p\to\infty} (n_{m_p} B) \in \mathcal{J}.$$

The lemmas given above can be restated substituting strong $\mathcal{J}$-density and strong $\mathcal{J}$-dispersion points in place of the $\mathcal{J}$-density and $\mathcal{J}$-dispersion points used above. In particular, the following lemma, analogous to Lemma 2.1.2(iii) will be needed.

LEMMA 2.1.3. *Let $\mathcal{J}$ be an ideal of subsets of $\mathbb{R}$ and let $B \subset \mathbb{R}$. Then 0 is a strong $\mathcal{J}$-dispersion point of B if, and only if, for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity there exists a subsequence $\{t_{n_p}\}_{p\in\mathbb{N}}$ such that*

$$(-1,1) \cap \bigcap_{q\in\mathbb{N}} \bigcup_{p\geq q} t_{n_p} B = (-1,1) \cap \limsup_{p\to\infty} (t_{n_p} B) \in \mathcal{J}.$$

To find nontrivial (strong) $\mathcal{J}$-density and (strong) $\mathcal{J}$-dispersion points, the following definitions are needed.

Either of the sets $\bigcup_{n\in\mathbb{N}} (a_n, b_n)$ or $\bigcup_{n\in\mathbb{N}} [a_n, b_n]$ is a *right interval set at a point* $a \in \mathbb{R}$ if $b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$. In the case when $a = 0$ it is simply called *a right interval set*. A *left interval set* at a point $a \in \mathbb{R}$

is defined in the same way. The set E is an *interval set* if it is the union of a right interval set and a left interval set at the same point.

Notice that if $E=\bigcup_{n\in\mathbb{N}}[a_n,b_n]$ is an interval set at the point a, then $\operatorname{int}(E)=\bigcup_{n\in\mathbb{N}}(a_n,b_n)$ and

$$E\triangle\operatorname{int}(E)=E\setminus\operatorname{int}(E)=\{a_1,a_2,\dots;b_1,b_2,\dots\}\in\mathcal{I}_\omega.$$

In particular, if $\mathcal{J}$ is an ideal of subsets of $\mathbb{R}$ such that $\mathcal{I}_\omega\subset\mathcal{J}$, then a point a is a $\mathcal{J}$-density ($\mathcal{J}$-dispersion, strong $\mathcal{J}$-density, strong $\mathcal{J}$-dispersion) point of E if, and only if, it is a $\mathcal{J}$-density ($\mathcal{J}$-dispersion, strong $\mathcal{J}$-density, strong $\mathcal{J}$-dispersion, respectively) point of $\operatorname{int}(E)$. Thus, all the results proved below for interval sets consisting of closed intervals, $\bigcup_{n\in\mathbb{N}}[a_n,b_n]$, are also true for those consisting of open intervals, $\bigcup_{n\in\mathbb{N}}(a_n,b_n)$.

Lemma 2.1.4. *Suppose that $\mathcal{J}$ is an ideal of subsets of $\mathbb{R}$ such that $\mathcal{I}_\omega\subset\mathcal{J}$. If*

$$E=\bigcup_{n\in\mathbb{N}}[a_n,b_n]$$

is a right interval set such that

(i): $\lim_{n\to\infty}(b_n-a_n)/a_n=0$, *and*

(ii): $\limsup_{n\to\infty}b_{n+1}/a_n=c\in[0,1)$,

then 0 *is a strong $\mathcal{J}$-dispersion point of E. In particular, $E^c\in\mathcal{T}''_{\mathcal{J}}$.*

Proof. Let $\{t_n\}_{n\in\mathbb{N}}$ be a divergent increasing sequence of positive numbers and define

$$i_n=\min\{k:t_na_k\in(0,1]\}. \tag{8}$$

Since the sequence $\{t_na_{i_n}:n\in\mathbb{N}\}$ is bounded, it must contain a convergent subsequence. There is no generality lost with the assumption that

$$\lim_{n\to\infty}t_na_{i_n}=\alpha_0\in[0,1]. \tag{9}$$

Using (i) and (8) it is apparent that

$$\limsup_{n\to\infty}t_nE\cap(\alpha_0,1)=\emptyset. \tag{10}$$

(If $\alpha_0=1$, then $(\alpha_0,1)=\emptyset$.) If $\alpha_0=0$, combining (10) with Lemma 2.1.3 immediately shows that 0 is a $\mathcal{J}$-dispersion point of E. So, assume that $\alpha_0\neq 0$.

According to (ii) there exists a $\gamma_1\in[0,c]$ such that

$$\limsup_{n\to\infty}b_{i_n+1}/a_{i_n}=\gamma_1.$$

Then

$$\limsup_{n\to\infty}t_nb_{i_n+1}=\limsup_{n\to\infty}t_na_{i_n}\frac{b_{i_n+1}}{a_{i_n}}=\gamma_1\alpha_0<\alpha_0.$$

Therefore, there is an increasing sequence of natural numbers $\{n(1,k) : k \in \mathbb{N}\}$ such that

$$\lim_{k\to\infty} t_{n(1,k)} b_{i_{n(1,k)}+1} = \gamma_1 \alpha_0. \tag{11}$$

Let $\alpha_1 = \gamma_1 \alpha_0$. In light of (i), it is clear that, in addition,

$$\lim_{k\to\infty} t_{n(1,k)} a_{i_{n(1,k)}+1} = \alpha_1.$$

Equations (9) and (11) imply that

$$\limsup_{k\to\infty} t_{n(1,k)} E \cap (\alpha_1, \alpha_0) = \emptyset. \tag{12}$$

Assume that for some natural number m there exists an $\alpha_m \in (0, \alpha_0)$ and an increasing sequence $n(m,k)$ of natural numbers such that

$$\lim_{k\to\infty} t_{n(m,k)} a_{i_{n(m,k)}+m} = \alpha_m. \tag{13}$$

Applying (ii) and (i) precisely as above, yields a subsequence $n(m+1,k)$ of $n(m,k)$ and a constant $\gamma_{m+1} \in [0,c]$ such that

$$\lim_{k\to\infty} t_{n(m+1,k)} a_{i_{n(m+1,k)}+m+1} = \gamma_{m+1} \alpha_m = \alpha_{m+1}.$$

and

$$\limsup_{k\to\infty} t_{n(m+1,k)} E \cap (\alpha_{m+1}, \alpha_m) = \emptyset. \tag{14}$$

In this way we have inductively defined a nonincreasing sequence of numbers $\{\alpha_k : k \in \mathbb{N}\} \subset [0, \alpha_0)$ and a collection of increasing sequences of natural numbers $\{n(j,k) : j,k \in \mathbb{N}\}$ with $\{n(j+1,k) : k \in \mathbb{N}\} \subset \{n(j,k) : k \in \mathbb{N}\}$ for all $j \in \mathbb{N}$. It is clear from the definition that $\alpha_k \le \alpha_0 \Pi_{j=1}^k \gamma_j \le \alpha_0 c^k$. Hence

$$\lim_{k\to\infty} \alpha_k = 0.$$

Let $s_k = t_{n(k,k)}$. From (10) and (14), a more-or-less standard diagonal argument shows that

$$\limsup_{k\to\infty} s_k E \cap (0,1) \subset \{a_k : k \in \mathbb{N}\} \in \mathcal{I}_\omega \subset \mathcal{J}.$$

Now, an application of Lemma 2.1.3 finishes the proof. □

The preceding lemma is used many times in the following chapters, but usually it is used with the condition that the limit supremum in (ii) actually exists as a limit and is 0. It provides a very powerful technique for constructing counterexamples.

COROLLARY 2.1.5. *Let $\mathcal{J}$ be an ideal of subsets of $\mathbb{R}$ such that $\mathcal{I}_\omega \subset \mathcal{J}$. Then*

(i): *there is a right interval set $E = \bigcup_{n\in\mathbb{N}} [a_n, b_n]$ such that 0 is a strong $\mathcal{J}$-dispersion point of E;*

(ii): *there is an open interval set U such that 0 is a strong $\mathcal{J}$-density point of U; in particular, $\{0\} \cup U \in \mathcal{T}''_{\mathcal{J}}$.*

PROOF. (i). By Lemma 2.1.4 it is enough to define

$$b_n = \frac{1}{(n+1)!} \quad \text{and} \quad a_n = \frac{n+1}{n+2} b_n.$$

(ii). Evidently 0 is a strong right $\mathcal{J}$-density point of $D = (0, \infty) \setminus E$. Thus $U = -D \cup D$ has 0 as a strong $\mathcal{J}$-density point. $\square$

The next two lemmas are easy consequences of Lemma 2.1.4.

LEMMA 2.1.6. *Let $\mathcal{J}$ be an ideal of subsets of $\mathbb{R}$ such that $\mathcal{I}_\omega \subset \mathcal{J}$. Moreover, let $\{c_n\}_{n \in \mathbb{N}}$ be a decreasing sequence of positive numbers converging to zero and, for each $n \in \mathbb{N}$, let (a_n, b_n) be an open interval centered at c_n. If*

$$\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = 0 \quad \textit{and} \quad \lim_{n \to \infty} \frac{b_n - a_n}{c_n} = 0,$$

then 0 is a strong $\mathcal{J}$-dispersion point of

$$\bigcup_{n \in \mathbb{N}} [a_n, b_n].$$

PROOF. Since $c_n = (a_n + b_n)/2$ for all n, (i) and (ii) from Lemma 2.1.4 can be established with a short calculation. $\square$

LEMMA 2.1.7. *Suppose that $\mathcal{J}$ is an ideal of subsets of $\mathbb{R}$ such that $\mathcal{I}_\omega \subset \mathcal{J}$. If $\bigcup_{n \in \mathbb{N}} [a_n, b_n]$ is a right interval set with*

$$\lim_{n \to \infty} \frac{(b_n - a_n)}{b_n} = 0,$$

then there exists an increasing sequence $\{n_m\}_{m \in \mathbb{N}}$ of natural numbers such that 0 is a strong $\mathcal{J}$-dispersion point of

$$\bigcup_{m \in \mathbb{N}} [a_{n_m}, b_{n_m}].$$

PROOF. It suffices to choose $\{n_m\}_{m \in \mathbb{N}}$ in such a way that $\{a_{n_m}\}_{m \in \mathbb{N}}$ satisfies (ii) of Lemma 2.1.4; i.e., such that $\lim_{m \to \infty} b_{n_{m+1}} / a_{n_m} = 0$. $\square$

We finish this section with the following

LEMMA 2.1.8. *Let $\mathcal{J}$ be an ideal of subsets of $\mathbb{R}$ such that $\mathcal{I}_\omega \subset \mathcal{J}$. There exists a perfect set C such that for every $x \in C$ there is an interval set E at x with the property that $C \subset E \cup \{x\}$ and x is a strong $\mathcal{J}$-dispersion point of E. In particular, $\{x\} \cup C^c \in \mathcal{T}''_{\mathcal{J}}$ for every $x \in C$. Therefore, C is closed and discrete with respect to $\mathcal{T}''_{\mathcal{J}}$.*

PROOF. We define by induction on n a decreasing sequence of closed sets $\{C_n\}_{n\in\mathbb{N}}$ such that each C_n consists of 2^n pairwise disjoint closed subintervals of $[0,1]$ of equal length. Put $C_0 = [0,1]$. To form C_{n+1} from C_n we remove from every component interval $[a,b]$ of C_n the open interval (c,d) with the same center and such that

$$\frac{d-c}{b-a} = \frac{(n+2)!-1}{(n+2)!}.$$

Put $C = \bigcap_{n\in\mathbb{N}} C_n$.

It is not difficult to check that for every $x \in C$ there is a right interval set $E_r \supset C \cap (x,1]$ at x that satisfies the assumptions of Lemma 2.1.4. Thus, x is a strong $\mathcal{J}$-dispersion point of E_r. Similarly, we can find a left interval set $E_l \supset C \cap [0,x)$ at x for which x is a strong $\mathcal{J}$-dispersion point. The set $E = E_r \cup E_l$ has the desired property. □

2.2. $\mathcal{I}$-density and $\mathcal{I}$-dispersion Points.

In this section, $\mathcal{I}$-density and $\mathcal{I}$-dispersion points for sets with the Baire property are studied. We begin with some well-known properties which connect the collection $\mathcal{B}$ of sets with the Baire property to the ideal $\mathcal{I}$ of first category subsets of $\mathbb{R}$. For their proofs see [**36**] or [**55**].

LEMMA 2.2.1. *For $E \subset \mathbb{R}$, the condition $E \in \mathcal{B}$ is equivalent to any of the following:*

(i): *$E = G \triangle P$, where G is open and $P \in \mathcal{I}$;*
(ii): *$E = F \triangle Q$, where F is closed and $Q \in \mathcal{I}$;*
(iii): *$E = K \cup P$, where K is a G_δ set and $P \in \mathcal{I}$; and,*
(iv): *$E = L \setminus Q$, where L is an F_σ set and $Q \in \mathcal{I}$.*

An open subset G of $\mathbb{R}$ is *regular* if $G = \operatorname{int}(\operatorname{cl}(G))$. If we assume that the open set in the first part of Lemma 2.2.1 is regular, then the decomposition given there is unique ([**55**, Theorem 4.6]). This is, in fact, true in any Baire topological space. For $E \in \mathcal{B}$, let $\tilde{E}$ denote this unique regular open set. Note that $\tilde{\tilde{E}} = \tilde{E}$, and the class of regular open sets is a complete Boolean algebra [**34**].

Now let $\mathcal{I}_0$ stand for the ideal of nowhere dense subsets of $\mathbb{R}$. The following theorem is one of the most important technical tools of this work. In the theorem $\{t_n\}$ always denotes a sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity and $\{n_k\}$ is always an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers.

THEOREM 2.2.2. *Let $B \in \mathcal{B}$. The condition that* 0 *is an $\mathcal{I}$-dispersion point of B is equivalent to any of the following:*

(i): *for every sequence $\{t_n\}$ there exists a subsequence $\{t_{n_k}\}$ such that*

$$\limsup_{k\to\infty} \left(t_{n_k}\tilde{B}\right) \cap (-1,1) \in \mathcal{I}_0;$$

(ii): *for every sequence* $\{t_n\}$ *there exists a subsequence* $\{t_{n_k}\}$ *such that*

$$\limsup_{k\to\infty}\left(t_{n_k}\tilde{B}\right)\cap(-1,1)\in\mathcal{I};$$

(iii): *for every sequence* $\{t_n\}$ *there exists a subsequence* $\{t_{n_k}\}$ *such that*

$$\limsup_{k\to\infty}\left(t_{n_k}B\right)\cap(-1,1)\in\mathcal{I};$$

(iv): *for every sequence* $\{n_k\}$ *there exists a subsequence* $\{n_{k_p}\}$ *such that*

$$\limsup_{p\to\infty}\left(n_{k_p}B\right)\cap(-1,1)\in\mathcal{I};$$

(v): *for every sequence* $\{n_k\}$ *there exists a subsequence* $\{n_{k_p}\}$ *such that*

$$\limsup_{p\to\infty}\left(n_{k_p}\tilde{B}\right)\cap(-1,1)\in\mathcal{I};$$

(vi): *for every sequence* $\{n_k\}$ *there exists a subsequence* $\{n_{k_p}\}$ *such that*

$$\limsup_{p\to\infty}\left(n_{k_p}\tilde{B}\right)\cap(-1,1)\in\mathcal{I}_0;$$

(vii): *for every nonempty interval* $(a,b)\subset(-1,1)$ *there exists a number* $\varepsilon>0$ *and* $n_0\in\mathbb{N}$ *such that for every* $n\geq n_0$ *there is an interval* $(c,d)\subset(a,b)$ *with the properties that*

$$|d-c|>\varepsilon \quad \text{and} \quad (c,d)\cap n\tilde{B}=\emptyset;$$

(viii): *for every sequence* $\{n_k\}$ *and any nonempty open interval* $(a,b)\subset(-1,1)$ *there exists a nonempty subinterval* $(c,d)\subset(a,b)$ *and a subsequence* $\{n_{k_p}\}$ *such that for every* $p\in\mathbb{N}$

$$(c,d)\cap n_{k_p}\tilde{B}=\emptyset;$$

(ix): *for every sequence* $\{t_n\}$ *and every nondegenerate open interval* $(a,b)\subset(-1,1)$ *there exists a nonempty subinterval* $(c,d)\subset(a,b)$ *and a subsequence* $\{t_{n_k}\}$ *such that for every* $k\in\mathbb{N}$

$$(c,d)\cap t_{n_k}\tilde{B}=\emptyset;$$

(x): *for every sequence* $\{t_n\}$ *there is a subsequence* $\{t_{n_k}\}$ *such that for every nonempty interval* $(a,b)\subset(-1,1)$ *there exists a* $k_0\in\mathbb{N}$ *and a nonempty subinterval* $(c,d)\subset(a,b)$ *such that for all* $k\geq k_0$

$$(c,d)\cap t_{n_k}\tilde{B}=\emptyset.$$

Proof. By Lemma 2.1.2(iii) condition (iv) is equivalent to the fact that 0 is an $\mathcal{I}$-dispersion point of B. To finish the proof, it will be shown that (i) through (ix) each imply the next item and that (x) implies (i).

(i)⇒(ii). This is obvious, because $\mathcal{I}_0\subset\mathcal{I}$.

(ii)⇒(iii). It follows immediately from the fact that for any sequence $\{t_n\}$ of positive numbers,

$$\limsup_{n\to\infty}(t_nB)\triangle\limsup_{n\to\infty}(t_n\tilde{B})\in\mathcal{I}. \tag{15}$$

(iii)⇒(iv). This is made obvious by taking $t_m = n_m$.

(iv)⇒(v). This follows immediately from (15).

(v)⇒(vi). Note that $\limsup_{p\to\infty}(n_{k_p}\tilde{B})\cap(-1,1)$ is a $\mathbf{G}_\delta$ set, and a $\mathbf{G}_\delta$ set belongs to $\mathcal{I}$ if, and only if, it belongs to $\mathcal{I}_0$.

(vi)⇒(vii). By way of contradiction, suppose that (vii) is not true. Then there must exist a nonempty open interval $I\subset(-1,1)$ such that for every $k\in\mathbb{N}$, there is an increasing sequence of natural numbers $\{n^k_m\}_{m\in\mathbb{N}}$ such that whenever J is a subinterval of I with length greater then $1/k$, then $n^k_m\tilde{B}\cap J\neq\emptyset$, for all $m\in\mathbb{N}$. Choose a sequence $\{m_k\}$ such that the numbers $n_k = n^k_{m_k}$ form an increasing sequence. Then, for every subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$, the set $\bigcup_{p\geq q} n_{k_p}\tilde{B}\cap I$ must be dense and open in I, for all $q\in\mathbb{N}$. This clearly contradicts (vi).

(vii)⇒(viii). Fix the sequence of natural numbers $\{n_k\}$ and any nonempty interval $(a,b)\subset(-1,1)$. By assumption there is an $\varepsilon>0$ and $n_0\in\mathbb{N}$ such that whenever $n\geq n_0$, then there is a nonempty interval $(c_n,d_n)\subset(a,b)\setminus n\tilde{B}$ such that $d_n-c_n>2\varepsilon$.

Let q be a natural number such that $(b-a)/q<\varepsilon$ and partition (a,b) into q contiguous intervals $J_1,\dots,J_q$ of equal length. From the choice of $0<|J_i|<\varepsilon$ for $1\leq i\leq q$, it is apparent that each interval (c_{n_k},d_{n_k}) must contain at least one of the J_i. The Pigeon Hole Principle now yields the existence of the desired subsequence.

(viii)⇒(ix). There is no generality lost with the assumptions that $t_1\geq 1$, that $t_{n+1}-t_n\geq 1$ for all n and $(a,b)\subset[a,b]\subset(0,1)$. Let $p_n=\lfloor t_n\rfloor$. The sequence of integers $\{p_n\}$ satisfies the conditions of (viii), so there must exist a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ and a nonempty $(c,d)\subset(a,b)$ such that $p_{n_k}\tilde{B}\cap(c,d)=\emptyset$ for all $k\in\mathbb{N}$.

Since $t_{n_k}/p_{n_k}\to 1$ as $k\to\infty$, there exists a k_0 such that

$$1\leq\frac{t_{n_k}}{p_{n_k}}<1+\frac{d-c}{3c},\quad\text{for all}\quad k\geq k_0.$$

Let $J=(c+(d-c)/3,d)$ and $k\geq k_0$. Then

$$\emptyset=\frac{t_{n_k}}{p_{n_k}}\left((c,d)\cap p_{n_k}\tilde{B}\right)=\left(\frac{t_{n_k}c}{p_{n_k}},\frac{t_{n_k}d}{p_{n_k}}\right)\cap t_{n_k}\tilde{B}\supset J\cap t_{n_k}\tilde{B}.$$

Part (ix) follows easily from this.

(ix)⇒(x). Without loss of generality we may assume that a and b are rational. Let $\{(a_s,b_s)\}_{s\in\mathbb{N}}$ be an enumeration of all such intervals, and let us choose a sequence $\{t_n\}$. The idea of the proof is an application of (ix) infinitely many times, and diagonalization.

Let $m_k^0 = k$ for $k \in \mathbb{N}$. We will, by induction on $s \in \mathbb{N}$, construct sequences $\{m_k^s\}_{k\in\mathbb{N}}$ such that, for every $s \in \mathbb{N}$, $\{m_k^s\}_{k\in\mathbb{N}}$ will be a subsequence of $\{m_k^{s-1}\}_{k\in\mathbb{N}}$, and there will be a nonempty interval (c,d) (possibly dependent on s) contained in (a_s, b_s) such that for all $k \in \mathbb{N}$

$$(c,d) \cap (t_{m_k^s}\tilde{B}) = \emptyset.$$

The construction is facilitated by (ix). Now, define $t_{n_k} = t_{m_k^k}$. Then, for every $k_0 \in \mathbb{N}$, there exists $(c,d) \subset (a_{k_0}, b_{k_0})$ such that for every $k \geq k_0$

$$(c,d) \cap (t_{n_k}\tilde{B}) = \emptyset.$$

This proves (x).

(x)⇒(i). Fix $\{t_n\}$ and let $\{t_{n_k}\}$ be a subsequence given by (x). Then, for every nonempty open interval $(a,b) \subset (-1,1)$ we can find a nonempty subinterval $(c,d) \subset (a,b)$ such that

$$(c,d) \cap \limsup_{k\to\infty} t_{n_k}\tilde{B} = \emptyset.$$

This implies (i).

This finishes the proof of Theorem 2.2.2. □

The dual version of condition (iii) implies that for every $B \in \mathcal{B}$

$$0 \in \Phi_{\mathcal{I}}(B) \quad \text{if, and only if,} \quad 0 \in \Psi_{\mathcal{I}}(B).$$

Using the translation version of the above we infer the following corollary.

COROLLARY 2.2.3. *For every* $B \in \mathcal{B}$

$$\Phi_{\mathcal{I}}(B) = \Psi_{\mathcal{I}}(B).$$

For the regular open sets there is one more characterization of $\mathcal{I}$-density points.

LEMMA 2.2.4. *Let A be a regular open set. Then the following are equivalent:*

- **(i):** *0 is an $\mathcal{I}$-density point of A;*
- **(ii):** *there exists an interval set $E \subset A$ consisting of closed intervals such that 0 is an $\mathcal{I}$-density point of E.*

PROOF. 0 is an $\mathcal{I}$-density point of A if, and only if, 0 is an $\mathcal{I}$-dispersion point of A^c. Thus, by Theorem 2.2.2(vii), condition (i) is equivalent to the statement that

> for every nonempty interval $(a,b) \subset (-1,1)$ there exists a number $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there is a nonempty interval $(c,d) \subset (a,b)$ with the properties:
>
> $$|d-c| > \varepsilon \quad \text{and} \quad (c,d) \cap n\widetilde{A^c} = \emptyset.$$

Notice that in the above it is enough to only consider intervals (a,b) with rational endpoints. Moreover, by the regularity of A, $\mathrm{cl}\left(\widetilde{A^c}\right) = A^c$, and so

$$\begin{aligned}(c,d)\cap n\widetilde{A^c}=\emptyset \quad &\iff \quad (c,d)\cap nA^c=\emptyset\\ &\iff \quad \frac{1}{n}(c,d)\cap A^c=\emptyset\\ &\iff \quad \frac{1}{n}(c,d)\subset A.\end{aligned}$$

Hence, we can conclude that (i) is equivalent to the statement

> $(\star)$ for every nonempty interval $I=(a,b)\subset(-1,1)$ with rational endpoints there exists a number $\varepsilon_I>0$ and $n_I\in\mathbb{N}$ such that for every $n\geq n_I$ there is a nonempty interval $(c,d)\subset[c,d]\subset(a,b)$ with the properties:
>
> $$|d-c|>\varepsilon_I \quad \text{and} \quad \frac{1}{n}[c,d]\subset A.$$

Now we can construct E to satisfy (ii).

Replacing A by $A\cap U$, if necessary, where U is from Corollary 2.1.5(ii), we can assume that

(16) A does not contain intervals of the form $(r,0)$ and $(0,r)$.

Let $\{I_k\}_{k\in\mathbb{N}}$ be an enumeration of all nonempty subintervals of $(-1,1)$ with rational endpoints and let us assume that $n_{I_k}\geq k$ for every $k\in\mathbb{N}$.

For $k\in\mathbb{N}$, let E_k be a union of intervals $\frac{1}{n}[c_n,d_n]$ for $n\geq n_{I_k}$, where $[c_n,d_n]$ is a subset of I_k satisfying $(\star)$; i.e., such that

$$|d_n-c_n|>\varepsilon_{I_k} \quad \text{and} \quad \frac{1}{n}[c_n,d_n]\subset A.$$

Let

$$E=\bigcup_{k\in\mathbb{N}}E_k.$$

Notice that

$$E_k\subset\left(-\frac{1}{n_{I_k}},\frac{1}{n_{I_k}}\right)\subset\left(-\frac{1}{k},\frac{1}{k}\right)$$

and that $E_k\setminus(-\frac{1}{n},\frac{1}{n})$ intersects only finitely many closed intervals forming E_k. Then,

$$E\setminus\left(-\frac{1}{k},\frac{1}{k}\right)\subset\bigcup_{i<k}E_i;$$

i.e., it intersects only finitely many closed intervals forming E.

This, together with (16), implies that E is an interval set formed with closed intervals. It is also easy to see that $\mathrm{int}\,(E)$ is an interval set obtained from E by removing the endpoints of the constituent intervals. Hence, $\mathrm{int}\,(E)$ is a regular open set and, by the construction, it is easy to see that it satisfies $(\star)$. Thus it has been proved that 0 is an $\mathcal{I}$-density point of $\mathrm{int}\,(E)$ and so of E as well.

This finishes the proof that (i) implies (ii). The converse implication is obvious. □

The next example will be used to show that in Lemma 2.2.4 the assumption that A is regular open cannot be weakened to the assumption that A is open.

EXAMPLE 2.2.5. *There exists a decreasing sequence $S = \{b_n\}_{n\in\mathbb{N}}$ converging to 0 such that 0 is not a right $\mathcal{I}$-dispersion point of any open set $B \supset S$.*

PROOF. Let

$$D_n = \left\{\frac{i}{2^n} : i = 1, 2, \ldots, 2^n\right\} \subset (0, 1]$$

and let

$$S = \bigcup_{n\in\mathbb{N}} \frac{1}{n} D_n.$$

Notice that $S \subset (0,1]$ and that $S \setminus (0, \frac{1}{n}]$ is finite for every $n \in \mathbb{N}$. Thus, S is a decreasing sequence converging to 0. Moreover, $D_k \subset D_n \subset nS$ for every $k \leq n$. In particular, for every increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers, the set $\bigcup_{k\in\mathbb{N}} n_k S$ contains $\bigcup_{n\in\mathbb{N}} D_n$ and so is dense. Now, if $B \supset S$ is open, then $\limsup_{k\to\infty} n_k B$ is a dense $\mathbf{G}_\delta$ subset of $(0,1)$. Thus, by Lemma 2.1.1(iii), 0 cannot be an $\mathcal{I}$-dispersion point of B. □

Now let S be as above and let $A = \mathbb{R} \setminus \mathrm{cl}\,(S)$. Then A is open and evidently 0 is an $\mathcal{I}$-density point of A. However, condition (ii) of Lemma 2.2.4 fails for A, as no matter how the right interval set $E_r \subset A$ is chosen, 0 cannot be a right $\mathcal{I}$-density point of E_r because the complement of E_r in $(0,1)$ always contains an open set $B \supset S$.

We finish this section with the following technical lemma.

LEMMA 2.2.6. *If $B = \bigcup_{n\in\mathbb{N}} (a_n, b_n) \subset [-1, 1]$ and there exists a positive number c such that*

$$\frac{b_n - a_n}{\max\{|a_n|, |b_n|\}} > c,$$

for every $n \in \mathbb{N}$, then 0 is not an $\mathcal{I}$-dispersion point of B.

PROOF. Without loss of generality we may assume that $B = \bigcup_{n\in\mathbb{N}} (a_n, b_n) \subset [0, 1]$. Put $t_n = 1/b_n$ for $n \in \mathbb{N}$. Then, for every subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ of $\{t_n\}_{n\in\mathbb{N}}$,

$$(1 - c, 1) \subset \limsup_{k\to\infty} (t_{n_k} B).$$

Hence, by Theorem 2.2.2(iii), 0 is not an $\mathcal{I}$-dispersion point of B. □

2.3. The $\mathcal{I}$-density Topology

In Corollary 2.2.3 it was shown that

$$\mathcal{T}''_{\mathcal{I}} \cap \mathcal{B} = \mathcal{T}'_{\mathcal{I}} \cap \mathcal{B}.$$

The purpose of this section is to prove that the family

$$\mathcal{T}_{\mathcal{I}} = \mathcal{T}''_{\mathcal{I}} \cap \mathcal{B} = \mathcal{T}'_{\mathcal{I}} \cap \mathcal{B}$$

forms a topology on $\mathbb{R}$.

We start with a lemma which essentially says that the $\mathcal{I}$-density operator $\Phi_{\mathcal{I}}$, restricted to $\mathcal{B}$, is a *lower density* [**29**]. In particular, condition (iv) is an analogue of the Lebesgue density theorem.

LEMMA 2.3.1. *For $A, B \in \mathcal{B}$*

- **(i):** *if $A \subset B$ then, $\Phi_{\mathcal{I}}(A) \subset \Phi_{\mathcal{I}}(B)$;*
- **(ii):** *$int(A) \subset \Phi_{\mathcal{I}}(A) \subset cl(A)$; in particular, $\Phi_{\mathcal{I}}(\emptyset) = \emptyset$ and $\Phi_{\mathcal{I}}(\mathbb{R}) = \mathbb{R}$;*
- **(iii):** *if $A \triangle B \in \mathcal{I}$, then $\Phi_{\mathcal{I}}(A) = \Phi_{\mathcal{I}}(B)$;*
- **(iv):** *$\Phi_{\mathcal{I}}(A) \triangle A \in \mathcal{I}$;*
- **(v):** *$\Phi_{\mathcal{I}}(A \cap B) = \Phi_{\mathcal{I}}(A) \cap \Phi_{\mathcal{I}}(B)$.*

PROOF. (i) is obvious from the definition.

(ii) and (iii) follow easily from Lemma 2.1.1.

(iv). Let $A \simeq B$ stand for the equivalence relation $A \triangle B \in \mathcal{I}$. Then (iv) is justified by the equivalences $A \simeq \tilde{A} \simeq \Phi_{\mathcal{I}}(\tilde{A}) \simeq \Phi_{\mathcal{I}}(A)$ where the third equivalence follows from (iii) and the second by the inclusions $\tilde{A} \subset \Phi_{\mathcal{I}}(\tilde{A}) \subset \mathrm{cl}(\tilde{A})$.

For (v), first notice that the inclusion $\Phi_{\mathcal{I}}(A \cap B) \subset \Phi_{\mathcal{I}}(A) \cap \Phi_{\mathcal{I}}(B)$ follows from (i). To prove $\Phi_{\mathcal{I}}(A) \cap \Phi_{\mathcal{I}}(B) \subset \Phi_{\mathcal{I}}(A \cap B)$ assume that $a \in \Phi_{\mathcal{I}}(A) \cap \Phi_{\mathcal{I}}(B)$. Without loss of generality we can assume that $a = 0$. Choose an increasing sequence $\{n_m\}_{m \in \mathbb{N}}$ of natural numbers. Then, by Lemma 2.1.1, there exists a subsequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that

$$(-1,1) \cap \left(\liminf_{p \to \infty} n_{m_p} A \right)^c \in \mathcal{I}.$$

We can find a further subsequence $\{n_{m_{p_i}}\}_{i \in \mathbb{N}}$ such that

$$(-1,1) \cap \left(\liminf_{i \to \infty} n_{m_{p_i}} B \right)^c \in \mathcal{I}.$$

Now Lemma 2.1.1 and

$$(-1,1) \cap \left(\liminf_{i \to \infty} n_{m_{p_i}} (A \cap B) \right)^c$$

$$= (-1,1) \cap \left[\left(\liminf_{i \to \infty} n_{m_{p_i}} A \right)^c \cup \left(\liminf_{i \to \infty} n_{m_{p_i}} B \right)^c \right] \in \mathcal{I}$$

imply that $0 \in \Phi_{\mathcal{I}}(A \cap B)$. $\square$

The next theorem follows immediately from the observation that the Boolean algebra of regular open sets is complete [**34**]. However, we would like to provide an elementary proof of it, since this proof gives significant insight to the subject.

THEOREM 2.3.2. *$\mathcal{T}_{\mathcal{I}}$ is a topology on $\mathbb{R}$.*

PROOF. Lemma 2.3.1 implies that the empty set and $\mathbb{R}$ are elements of $\mathcal{T}_{\mathcal{I}}$, and that the class $\mathcal{T}_{\mathcal{I}}$ is closed under finite intersections.

Let us assume that a class $\{A_\gamma : \gamma \in \Gamma\}$ of elements of $\mathcal{T}_{\mathcal{I}}$ is given and write $A = \bigcup_{\gamma\in\Gamma} A_\gamma$. It is obvious that $A \subset \Phi_{\mathcal{I}}(A)$. Thus, it suffices to show that A is a Baire set.

Let $\{\tilde{A}_{\gamma_n} : n \in \mathbb{N}\}$ be a countable subcover from $\{\tilde{A}_\gamma : \gamma \in \Gamma\}$ of the open set $G' = \bigcup_{\gamma\in\Gamma} \tilde{A}_\gamma$. Put $G = \bigcup_{n\in\mathbb{N}} A_{\gamma_n}$. Then $G \triangle G' \in \mathcal{I}$. Moreover, by Lemma 2.3.1(iii) and the fact that $A_\gamma \subset \Phi_{\mathcal{I}}(A_\gamma)$ for every $\gamma \in \Gamma$,

$$\begin{aligned} G \subset A = \bigcup_{\gamma\in\Gamma} A_\gamma \quad &\subset \quad \bigcup_{\gamma\in\Gamma} \Phi_{\mathcal{I}}(A_\gamma) \\ &= \quad \bigcup_{\gamma\in\Gamma} \Phi_{\mathcal{I}}(\tilde{A}_\gamma) \\ &\subset \quad \Phi_{\mathcal{I}}\left(\bigcup_{\gamma\in\Gamma} \tilde{A}_\gamma\right) \\ &= \quad \Phi_{\mathcal{I}}(G') \\ &= \quad \Phi_{\mathcal{I}} G. \end{aligned}$$

But, $G \in \mathcal{B}$ and, by Lemma 2.3.1(iv),

$$A \triangle G = A \setminus G \subset \Phi_{\mathcal{I}}(G) \setminus G \in \mathcal{I}.$$

Thus, A is Baire. This proves that $\mathcal{T}_{\mathcal{I}}$ is a topology. □

The topology $\mathcal{T}_{\mathcal{I}}$ is called the *$\mathcal{I}$-density topology.*

2.4. The $\mathcal{P}^*$-topology.

In this section, an alternative definition of the $\mathcal{I}$-density topology due to L. Zajíček [**72**], called the $\mathcal{P}^*$-topology, is introduced. Although, with only a few exceptions, the $\mathcal{P}^*$-topology is not used directly in later sections, it is useful for developing intuition about the $\mathcal{I}$-density topology.

Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. For $R > 0$ let

$$\begin{aligned} \gamma(x, R, A) = \sup\{\varepsilon > 0 : \exists y \in \mathbb{R} \text{ such that} \\ (y-\varepsilon, y+\varepsilon) \subset (x-R, x+R) \cap A^c \cap \{x\}^c\}. \end{aligned}$$

The *porosity of A at x* is defined to be

$$p(x, A) = \limsup_{R\to 0^+} 2\gamma(x, R, A)/R.$$

The notions of *right* and *left porosity of A at x* can be defined in an obvious way. The set A is *porous* at x if $p(x,A) > 0$. If M is porous at each of its points, then it is called a porous set.

A set which is porous at a point x is, in some sense, "full of holes" nearby x. If two sets are porous at x, this does not guarantee that their union is porous at x. A simple example of this is $A = [-1,0]$ and $B = [0,1]$. Both A and B are porous at 0, but their union is obviously not porous at 0. A more complicated example, which is more instructive for our purposes is as follows. Let

$$A = \bigcup_{n\in\mathbb{N}} (-2^{-2n}, -2^{-2n-1}) \cup \bigcup_{n\in\mathbb{N}} (2^{-2n-1}, 2^{-2n})$$

and

$$B = \bigcup_{n\in\mathbb{N}} (-2^{-2n+1}, -2^{-2n}) \cup \bigcup_{n\in\mathbb{N}} (2^{-2n}, 2^{-2n+1}).$$

It is clear that

$$p(0,A) = p(0,B) = \frac{1}{2} > 0,$$

but

$$p(0, A\cup B) = 0.$$

Examples such as the above motivate the following definition.

The set $A \subset \mathbb{R}$ is *superporous* at x if $A \cup B$ is porous at x whenever B is porous at x. The set A is simply termed to be superporous, if it is superporous at every point. Any such set is obviously porous.

These definitions imply at once that if $\{A_\lambda \colon \lambda \in \Lambda\}$ are each superporous (porous) at x, then $\bigcap_{\lambda\in\Lambda} A_\lambda$ is also superporous (porous) at x and if the sets $\{A_i \colon 1 \le i \le n\}$ for $n \in \mathbb{N}$ are superporous at x, then $\bigcup_{1\le i\le n} A_i$ is also superporous at x. Using this, the $\mathcal{P}$*-topology* is defined as

$$\mathcal{P} = \{G \subset \mathbb{R} : G^c \text{ is superporous at every point of } G\}.$$

Finally, the $\mathcal{P}^*$-topology is defined as

$$\mathcal{P}^* = \{G \setminus N \colon G \in \mathcal{P} \text{ and } N \in \mathcal{I}\}.$$

It will be proved that $\mathcal{P}^* = \mathcal{T}_\mathcal{I}$. To do this, several lemmas are needed.

LEMMA 2.4.1. *$\mathcal{P}^* \subset \mathcal{B}$.*

PROOF. It is easy to see that any $G \in \mathcal{P}$ contains an open set $H \in \mathcal{T}_\mathcal{O}$ such that $H \subset G \subset \operatorname{cl}(H)$. Thus, $G = H \cup (G\setminus H) \in \mathcal{B}$, as $G \setminus H \subset \operatorname{cl}(H) \setminus H \in \mathcal{I}$. □

LEMMA 2.4.2. *The following conditions are equivalent:*

(i): *the set A is superporous at x; and,*

(ii): *given $s \in (0,1)$ there exists $D_s > 0$ and $R_s \in (0,1)$ such that whenever $0 < D < D_s$ and $(y-\delta, y+\delta) \subset (x-D, x+D) \setminus \{x\}$ with $2\delta/D > s$, then there is an interval $J \subset (y-\delta, y+\delta) \cap A^c$ with $m(J)/2\delta > R_s$.*

PROOF. Since porosity is translation invariant, it may be assumed that $x = 0$.

First, assume (i) is true, but (ii) fails for some $s \in (0,1)$. This implies that for each $R \in (0,1)$ and each $D > 0$, there is a closed interval $I(R,D) \subset (-D,D)\backslash\{0\}$ such that

$$\frac{\mathrm{m}(I(R,D))}{D} > s,$$

but every component of $I(R,D) \cap A^c$ can have measure at most

$$R\,\mathrm{m}(I(R,D)).$$

Using this, we will construct a set B, which is porous at 0, such that $A \cup B$ is not porous at 0. This contradiction of (i) will show that (ii) must be true.

To do this, let $R_n = 2^{-n}$ and $I_1 = I(R_1, 1)$. Inductively define

$$I_{n+1} = I(R_{n+1}, \mathrm{dist}(I_n, \{0\})).$$

Let $B = \left(\bigcup_{n\in\mathbb{N}} I_n\right)^c$. Then $p(B,0) \geq s > 0$, but the definition of I_n implies $p(A \cup B, 0) = 0$.

Next, suppose (ii) is true and $B \subset \mathbb{R}$ such that $p(B,0) > s > 0$. There is no generality lost in assuming that the right porosity of B at 0 is greater than s. Then, it is easy to construct a right interval set $E = \bigcup_{n\in\mathbb{N}}(a_n, b_n)$ such that $E \subset B^c$ and $(b_n - a_n)/b_n > s$ for every $n \in \mathbb{N}$. Choose D_s and R_s as in (ii). We may further assume $0 < b_1 < D_s$. Then, for each $n \in \mathbb{N}$, there is an interval $J_n \subset (a_n, b_n) \cap A^c$ such that $\mathrm{m}(J_n)/(b_n - a_n) > R_s$. This easily implies $p(A \cup B, 0) \geq sR_s > 0$, and (i) holds. □

The notion of unilateral right superporosity set can be defined in a very natural way. Using this definition, Lemma 2.4.2 can be reformulated in the following way.

COROLLARY 2.4.3. *The following conditions are equivalent:*

(i): *the set A is unilaterally right superporous at 0; and,*

(ii): *for every $c \in (0,1)$ there exist $\varepsilon > 0$ and $\delta_0 > 0$ such for any $x \in (0,\delta_0)$ there exists a closed interval $J \subset A^c \cap (cx, x)$ such that $m(J) > x\varepsilon$.*

PROOF. It is not difficult to see from Lemma 2.4.2 that A is right superporous at 0 if, and only if,

($\star$) given $s \in (0,1)$ there exists $D_s > 0$ and $R_s \in (0,1)$ such that whenever $0 < D < D_s$ and $(y-\delta, y+\delta) \subset (0,D)$ with $2\delta/D > s$, then there is an interval $J \subset (y-\delta, y+\delta) \cap A^c$ with $\mathrm{m}(J)/2\delta > R_s$.

To prove that ($\star$) implies (ii), it is enough to choose $s < 1-c$, define $\delta_0 = D_s$, $\varepsilon = (1-c)R_s$ and take $D = x < \delta_0 = D_s$ and $(y-\delta, y+\delta) = (cx, x) \subset (-D,D)\setminus\{0\}$. Then, $2\delta/D = x(1-c)/x > s$ and so, there exists $J \subset A^c \cap (y-\delta, y+\delta) = A^c \cap (cx, x)$ such that $\mathrm{m}(J) > 2\delta R_s = x(1-c)\varepsilon/(1-c) = x\varepsilon$.

To prove that (ii) implies ($\star$), choose c such that $1-c < s$ and define $D_s = \delta_0$ and $R_s = \varepsilon/2$. For $D < D_s$ and $(y-\delta, y+\delta) \subset (0,D)$ such that $2\delta > sD$ take

$x = y + \delta \leq D < D_s = \delta_0$. Then, $(1-c)x \leq (1-c)D < sD < 2\delta$ and so, $(cx, x) \subset (x - 2\delta, x) = (y - \delta, y + \delta)$. Thus, there exists $J \subset A^c \cap (cx, x) \subset A^c \cap (y - \delta, y + \delta)$ such that

$$\mathrm{m}(J)/2\delta \geq x\varepsilon/2\delta \geq \delta\varepsilon/2\delta = R_s,$$

since $x \geq y + \delta \geq \delta$. This finishes the proof. □

In the proof of the next theorem, the following easy fact is needed.

PROPOSITION 2.4.4. *Let $\varepsilon, M > 0$ and let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a sequence of subintervals of $[-M, M]$ such that $b_n - a_n > \varepsilon$ for every $n \in \mathbb{N}$. Then there exists a nonempty interval $(a, b) \subset [-M, M]$ and an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive numbers such that $(a, b) \subset (a_{n_k}, b_{n_k})$ for every $k \in \mathbb{N}$.*

PROOF. Let c_n be center of (a_n, b_n) for every $n \in \mathbb{N}$ and let $-M = d_0 < d_1 < \ldots < d_m = M$ be such that $d_i - d_{i-1} < \frac{\varepsilon}{2}$ for every $i = 1, 2, \ldots, m$. Then, by the Pigeon Hole Principle, there exists $i \leq m$ and a sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $d_i \in (a_{n_k}, c_{n_k})$ for every $k \in \mathbb{N}$. Then the sequence $\{n_k\}_{k\in\mathbb{N}}$ and an interval $(a, b) = (d_i, d_i + \frac{\varepsilon}{2})$ suffice. □

The main conclusion of this section is a consequence of the following theorem.

THEOREM 2.4.5. *Let $A \in \mathcal{B}$ and $x \in \mathbb{R}$. Then, $\tilde{A}$ is superporous at x if, and only if, x is an $\mathcal{I}$-dispersion point of A.*

PROOF. Since $A \triangle \tilde{A} \in \mathcal{I}$, x is an $\mathcal{I}$-dispersion point of A if, and only if, x is an $\mathcal{I}$-dispersion point of $\tilde{A}$. Thus, without loss of generality we may assume that $A = \tilde{A}$. We can also assume, without loss of generality that $x = 0$, because the notions of superporosity and $\mathcal{I}$-dispersion are translation invariant.

Before we start the main part of the proof let us rephrase condition (ii) of Lemma 2.4.2 for $x = 0$ in the following way:

> for every $s \in (0, 1)$ there exist $D_s > 0$ and $R_s \in (0, 1)$ such that for every $0 < D < D_s$ and any open interval $I \subset (-D, D) \setminus \{0\}$ with $\mathrm{m}(I)/D > s$, there is an open interval $J \subset I \cap A^c$ such that $\mathrm{m}(J)/\mathrm{m}(I) > R_s$.

This can be rewritten as

> for every $s \in (0, 1)$ there exist $D_s > 0$ and $R_s \in (0, 1)$ such that for every $0 < D < D_s$ and any open interval $\frac{1}{D}I \subset \frac{1}{D}(-D, D) \setminus \{0\}$ with $\mathrm{m}(\frac{1}{D}I) > s$, there is an open interval $\frac{1}{D}J \subset \frac{1}{D}I$ such that $\frac{1}{D}J \cap \frac{1}{D}A = \emptyset$ and $\mathrm{m}(\frac{1}{D}J)/\mathrm{m}(\frac{1}{D}I) > R_s$.

The last version, in turn, is equivalent to

> ($\star$) for every $s \in (0, 1)$ there exist $D_s > 0$ and $R_s \in (0, 1)$ such that for every $0 < D \leq D_s$ and any open interval $(a, b) \subset (-1, 1)\setminus\{0\}$ with $b - a \geq s$, there is an open interval $(c, d) \subset (a, b)$ such that $(c, d) \cap \frac{1}{D}A = \emptyset$ and $(d - c)/(b - a) > R_s$.

Now we are ready to prove the theorem by proving that condition ($\star$) implies Theorem 2.2.2(viii) and that Theorem 2.2.2(ix) implies condition ($\star$).

So, assume that A is superporous at 0. Choose an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$, a nonempty open interval $(a,b) \subset (-1,1) \setminus \{0\}$ and a number $p \in \mathbb{N}$. Put $s = b-a$ and use condition ($\star$) to choose $D_s > 0$ and $R_s \in (0,1)$. We can assume that $D_s^{-1} = n_{k_p} \geq n_p$. Then, since $b-a=s$, there is an interval $(c_p,d_p) \subset (a,b)$ such that $d_p - c_p > s\,R_s$ and $(c_p,d_p) \cap n_{k_p}A = \emptyset$. Now, we can assume that the sequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ is increasing and, using Proposition 2.4.4, if necessary, that there is a nonempty open interval $(c,d) \subset (c_p,d_p) \subset (a,b)$ for every $p \in \mathbb{N}$. This implies $(c,d) \cap n_{k_p}A = \emptyset$ for every $p \in \mathbb{N}$. Theorem 2.2.2(viii) is proved.

To prove the converse, assume that condition ($\star$) is false for some $s > 0$. Then for every $n \in \mathbb{N}$ and $D_s = R_s = \frac{1}{n}$ there are numbers $0 < D \leq D_s$ and $t_n = \frac{1}{D}$ and an interval $(a_n,b_n) \subset (-1,1) \setminus \{0\}$ such that $b_n - a_n \geq s$ and for every interval $(c,d) \subset (a_n,b_n)$ with the property that $(c,d) \cap t_nA = \emptyset$ we have $d - c \leq R_s(b-a) \leq \frac{1}{n}$. Now, by Proposition 2.4.4, we can choose an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers and a nonempty interval (a,b) such that $(a,b) \subset (a_{n_k}, b_{n_k})$ for every $k \in \mathbb{N}$ and the sequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ is increasing and diverging to infinity. In other words, we have found a nonempty interval $(a,b) \subset (-1,1)$ and an increasing sequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ of positive numbers diverging to infinity such that whenever $(c,d) \subset (a,b)$ is such that $(c,d) \cap t_{n_k}A = \emptyset$, then $d - c \leq \frac{1}{n_k}$. This contradicts Theorem 2.2.2(ix). This contradiction forces the conclusion that ($\star$) must be true. □

Combining the previous theorem with Lemma 2.4.1 and the definition of $\mathcal{P}^*$ yields to the folloving corollary.

Corollary 2.4.6. *$\mathcal{P}^* = \mathcal{T}_{\mathcal{I}}$.*

2.5. $\mathcal{I}$-approximate Continuity

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *$\mathcal{I}$-approximately continuous*, if it is continuous with respect to the $\mathcal{I}$-density topology $\mathcal{T}_{\mathcal{I}}$ on the domain, and the natural topology $\mathcal{T}_{\mathcal{O}}$ on the range. The class of all $\mathcal{I}$-approximately continuous functions is denoted by $\mathcal{C}_{\mathcal{IO}}$. This is analogous to the definition of the approximately continuous functions $\mathcal{C}_{\mathcal{NO}}$.

A Baire function $f\colon \mathbb{R} \to \mathbb{R}$ is *$\mathcal{I}$-approximately continuous at a point x* if for any $\varepsilon > 0$, x is an $\mathcal{I}$-density point of $\{t : |f(t) - f(x)| < \varepsilon\} \in \mathcal{B}$. Pointwise *right* and *left $\mathcal{I}$-approximate continuity* are defined in the obvious way.

The following propositions serve to give some insight into the structure of the $\mathcal{I}$-approximately continuous functions.

Lemma 2.5.1. *If f is right $\mathcal{I}$-approximately continuous at each of its points, $a \in \mathbb{R}$ and $A = \{x : f(x) > a\}$, then $int(A)$ is dense in A.*

Proof. It may be supposed without loss of generality that $a = 0$. Let $A_n = \{x : f(x) \geq 1/n\} \in \mathcal{B}$ and let $A = \bigcup_{n\in\mathbb{N}} A_n$. If $x \in \tilde{A}_n$, then, by Lemma 2.1.1, x is an $\mathcal{I}$-density point of A_n, as $A_n \triangle \tilde{A}_n \in \mathcal{I}$. The definition of right $\mathcal{I}$-approximate continuity shows that $f(x) \geq 1/n$. It follows that $\tilde{A}_n \subset A_n$. Therefore, $G = \bigcup_{n\in\mathbb{N}} \tilde{A}_n \subset \bigcup_{n\in\mathbb{N}} A_n = A$. To see that G is dense in A, let $x \in A$ and choose $n \in \mathbb{N}$ such that $1/n < f(x)$. Then x is a right $\mathcal{I}$-density point of $\{w : f(w) > 1/n\}$ and it is apparent that x must be a limit point of $\tilde{A}_n$. From this, we see that G is dense in A. $\square$

Theorem 2.5.2. *Every right $\mathcal{I}$-approximately continuous function is of the first Baire class.*

Proof. Let f be right $\mathcal{I}$-approximately continuous on $\mathbb{R}$. It suffices to show that $\{x : f(x) \geq 0\}$ is a $\mathbf{G}_\delta$ set. To do this, for each $p \in \mathbb{N}$, let $U_p = \{x\colon f(x) > -1/p\}$ and, for $p, q, r, k \in \mathbb{N}$, define

$$A(p,q,r,k) = \left\{x \in \mathbb{R}\colon \left(\frac{k-1}{q}, \frac{k}{q}\right) \cap r\,(U_p - x) \neq \emptyset\right\} \tag{17}$$

and

$$A(p,q,r) = \bigcap_{k=1}^{q} A(p,q,r,k). \tag{18}$$

It is easy to see that each $A(p,q,r)$ is an open set.

Next, define

$$U = \bigcap_{p\in\mathbb{N}} \bigcap_{q\in\mathbb{N}} \bigcup_{r\geq q} A(p,q,r). \tag{19}$$

It is clear that U is a $\mathbf{G}_\delta$ set. We will show that $U = \{x\colon f(x) \geq 0\}$.

To show that $U \subset \{x\colon f(x) \geq 0\}$, fix $p \in \mathbb{N}$ and let

$$V_p = \bigcap_{q\in\mathbb{N}} \bigcup_{r\geq q} A(p,q,r).$$

Suppose that $0 \in V_p$. For each $q \in \mathbb{N}$ there is an $r_q \in \mathbb{N}$, $r_q \geq q$, such that when $1 \leq k \leq q$, then $\left(\frac{k-1}{q}, \frac{k}{q}\right) \cap r_q U_p \neq \emptyset$. From this and Lemma 2.5.1 we see that

$$\left(\frac{k-1}{q}, \frac{k}{q}\right) \cap r_q \mathrm{int}\,(U_p) = int\left(\left(\frac{k-1}{q}, \frac{k}{q}\right) \cap r_q U_p\right) \neq \emptyset \tag{20}$$

for $k = 1, 2, \ldots, q$.

Let $\{r_{q_i}\}_{i\in\mathbb{N}}$ be an increasing subsequence of $\{r_q\}_{q\in\mathbb{N}}$ and put $n_i = r_{q_i}$ for $i \in \mathbb{N}$. From (20) it follows that $\bigcup_{j\in\mathbb{N}} n_{i_j}\,\mathrm{int}\,(U_p)$ is a dense open subset of $(0,1)$ for every subsequence $\{n_{i_j}\}_{j\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$. Therefore,

$$\limsup_{j\to\infty} n_{i_j} U_p \cap (0,1)$$

is a residual subset of $(0,1)$. It follows from Lemma 2.1.2, that 0 is not a right $\mathcal{I}$-dispersion point of U_p and the right $\mathcal{I}$-density continuity of f shows that $0 \in \{x\colon f(x) \geq -1/p\}$. This same argument can be done with any other $x \in V_p$, showing that $V_p \subset \{x\colon f(x) \geq -1/p\}$. Therefore,

$$U = \bigcap_{p\in\mathbb{N}} V_p \subset \{x\colon f(x) \geq 0\}.$$

To show that $\{x\colon f(x) \geq 0\} \subset U$, fix x such that $f(x) \geq 0$. As previously, we consider only the case when $x = 0$. The other cases are similar.

So, suppose $f(0) \geq 0$ and $p, q \in \mathbb{N}$. We must show there is an $r \in \mathbb{N}$, $r \geq q$, such that $0 \in A(p,q,r)$. If not, for every $r \geq q$ there must be an integer k_r, with $1 \leq k_r \leq q$, such that

$$\left(\frac{k_r - 1}{q}, \frac{k_r}{q}\right) \cap rU_p = \emptyset.$$

There must exist an increasing sequence of natural numbers r_i such that $k_{r_i} = k$ for some $1 \leq k \leq q$. This gives

$$\left(\frac{k - 1}{q}, \frac{k}{q}\right) \cap r_i U_p = \emptyset$$

for all i so that for any subsequence $\{r_{i_j}\}_{j\in\mathbb{N}}$ of $\{r_i\}_{i\in\mathbb{N}}$

$$\left(\frac{k - 1}{q}, \frac{k}{q}\right) \cap \liminf_{j\to\infty} r_{i_j} U_p = \emptyset.$$

Therefore, 0 is not a right $\mathcal{I}$-density point of U_p. But, this is impossible because $f(0) \geq 0$ and f is right $\mathcal{I}$-approximately continuous at 0.

We are forced to conclude $U \supset \{x : f(x) \geq 0\}$ and consequently $U = \{x : f(x) \geq 0\}$, which finishes the proof of the theorem. □

Corollary 2.5.3. *If f is $\mathcal{I}$-approximately continuous on $\mathbb{R}$, then f is a Darboux Baire 1 function.*

Proof. Since sets which are open in the $\mathcal{I}$-density topology are bilaterally c-dense in themselves, this is an immediate consequence of the preceding theorem and Young's criterion. (See Bruckner [**7**, p. 9].) □

The properties of the $\mathcal{I}$-approximately continuous functions given so far have concentrated on showing that they behave like the ordinary approximately continuous functions. But, as can be expected, all the widely used properties of the approximately continuous functions do not translate to the case of $\mathcal{I}$-approximate continuity. For example, it was noted in Theorem 1.3.1(iv) that any bounded approximately continuous function is a derivative. The following example shows this is not true of bounded $\mathcal{I}$-approximately continuous functions.

EXAMPLE 2.5.4. *There exists a bounded $\mathcal{I}$-approximately continuous function*[1] *which is not a derivative.*

PROOF. Let $P \subset (0,1]$ be a nowhere dense closed set with positive measure. Choose a sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers satisfying $\lim_{k\to\infty} n_k/n_{k+1} = 0$ and define

$$E = \bigcup_{k\in\mathbb{N}} \frac{1}{n_k} P.$$

It will be shown in Lemma 2.8.1 that there is an open set $G \supset E$ such that 0 is an $\mathcal{I}$-dispersion point of G. Moreover, by Corollary 2.7.4, G can be chosen as an open interval set.

On the other hand, for all $k \in \mathbb{N}$,

$$\frac{\mathrm{m}(G\cap(0,1/n_k))}{1/n_k} \geq \frac{\mathrm{m}(E\cap(0,1/n_k))}{1/n_k} > \mathrm{m}(P) > 0, \tag{21}$$

so 0 is not a dispersion point of G.

Define the function f on $E \cup G^c$ by

$$f(x) = \begin{cases} 1 & x \in E \\ 0 & x \in G^c \end{cases}$$

and extend f on $G \setminus E$ in such a way that it is piecewise linear on $(0,\infty)$ and bounded by 1. Since 0 is an $\mathcal{I}$-dispersion point of G, it is apparent that f is $\mathcal{I}$-approximately continuous.[2] On the other hand, f cannot be a derivative. To see this, suppose F is any primitive function for f and define

$$G(x) = \int_0^x f.$$

Since f is continuous on $\mathbb{R}\setminus\{0\}$, we see that $F - G$ must be constant on both $(-\infty,0)$ and $(0,\infty)$. Since both F and G are continuous, this implies that $F - G$ is constant on $\mathbb{R}$ and therefore G is differentiable on $\mathbb{R}$. But, this is impossible since, by (21),

$$\begin{aligned} D^-G(0) &= 0 < \mathrm{m}(P) \\ &\leq \liminf_{k\to\infty} \frac{\mathrm{m}(E\cap(0,1/n_k))}{1/n_k} \\ &\leq \limsup_{k\to\infty} \frac{G(1/n_k)}{1/n_k} \leq \overline{D}^+ G(0). \quad \square \end{aligned}$$

Now, we are ready to state some of the most important properties of $\mathcal{C}_{\mathcal{IO}}$, similar to those from Theorem 1.3.1.

THEOREM 2.5.5. *The following are some properties of* $\mathcal{C}_{\mathcal{IO}}$.

(i): $\mathcal{C}_{\mathcal{IO}}$ *is closed under pointwise addition and multiplication.*

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[1] In fact, the constructed function is also $\mathcal{I}$-density continuous. $\mathcal{I}$-density continuity is defined in Chapter 3.

[2] In fact, by Corollary 3.4.4, it is also $\mathcal{I}$-density continuous.

(ii): *$\mathcal{C}_{\mathcal{I}\mathcal{O}}$ is closed under uniform convergence; hence, the bounded $\mathcal{I}$-approximately continuous functions form a Banach space.*
(iii): *If $f \in \mathcal{C}_{\mathcal{I}\mathcal{O}}$, then f is a Darboux function of the first Baire class.*
(iv): *There is a bounded $f \in \mathcal{C}_{\mathcal{I}\mathcal{O}}$ which is not a derivative.*

Proof. (i). If $f, g \in \mathcal{C}_{\mathcal{I}\mathcal{O}}$ then $f + g$ is a composition of the continuous function $+\colon (\mathbb{R}^2, \mathcal{T}_{\mathcal{O}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{O}})$, $+(x, y) = x + y$, and the continuous functions $f, g\colon (\mathbb{R}, \mathcal{T}_{\mathcal{I}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{O}})$. Hence, $f + g \in \mathcal{C}_{\mathcal{I}\mathcal{O}}$. Similarly, $fg \in \mathcal{C}_{\mathcal{I}\mathcal{O}}$.

(ii). Assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathcal{I}$-approximately continuous functions converges uniformly to f. We prove that $f \in \mathcal{C}_{\mathcal{I}\mathcal{O}}$. Let $U \in \mathcal{T}_{\mathcal{O}}$, $x \in f^{-1}(U)$ and choose $\varepsilon > 0$ such that $(f(x) - 2\varepsilon, f(x) + 2\varepsilon) \subset U$. Moreover, let $n \in \mathbb{N}$ be such that $|f_n(x) - f(x)| < \varepsilon$ for every $x \in \mathbb{R}$. Then,

$$x \in f_n^{-1}((f(x) - \varepsilon, f(x) + \varepsilon)) \subset f^{-1}((f(x) - 2\varepsilon, f(x) + 2\varepsilon)) \subset f^{-1}(U)$$

and $f_n^{-1}((f(x) - \varepsilon, f(x) + \varepsilon)) \in \mathcal{T}_{\mathcal{I}}$. Thus, $f \in \mathcal{C}_{\mathcal{I}\mathcal{O}}$.

(iii) is a restatement of Corollary 2.5.3.

(iv) is a restatement of Example 2.5.4. □

We finish this section with the following analog of Theorem 1.3.2.

Theorem 2.5.6. *The function $f\colon \mathbb{R} \to \mathbb{R}$ is a Baire function if, and only if, it is $\mathcal{I}$-approximately continuous $\mathcal{I}$-almost everywhere.*

Proof. If f is a Baire function then, by Example 1.1.5, f is $\mathcal{I}$-continuous $\mathcal{I}$-a.e. and so it is $\mathcal{I}$-approximately continuous $\mathcal{I}$-a.e.

If f is $\mathcal{I}$-approximately continuous $\mathcal{I}$-a.e., then there must be a set $K \in \mathcal{I}$ such that $f|_{K^c}$ is Baire. So, f is Baire. □

2.6. Topological Properties of the $\mathcal{I}$-density Topology

This short section summarizes the topological properties of the $\mathcal{I}$-density topology.

We start with the following easy lemma.

Lemma 2.6.1. *If $D \subset \mathbb{R}$ is dense with respect to $\mathcal{T}_{\mathcal{O}}$, then no point $x \in \mathbb{R}$ can be separated in $\mathcal{T}_{\mathcal{I}}$ from D.*

Proof. By Lemma 2.1.2, if $x \in A \in \mathcal{T}_{\mathcal{I}}$, then $(x - \varepsilon, x + \varepsilon) \cap A \notin \mathcal{I}$ for every $\varepsilon > 0$. In particular, by Lemma 2.2.1(iii), any set $G \in \mathcal{T}_{\mathcal{I}}$ containing D must be residual and any set $H \in \mathcal{T}_{\mathcal{I}}$ containing x must be a second category set. Therefore G and H cannot be disjoint. □

Theorem 2.6.2. *The topology $\mathcal{T}_{\mathcal{I}}$ on $\mathbb{R}$ has the following properties:*
(i): *$\mathcal{T}_{\mathcal{O}} \subset \mathcal{T}_{\mathcal{I}}$ and the inclusion is proper; in particular $\mathcal{T}_{\mathcal{I}}$ is Hausdorff;*
(ii): *a subset C of $\mathbb{R}$ is closed and discrete with respect to $\mathcal{T}_{\mathcal{I}}$ if, and only if, $C \in \mathcal{I}$;*
(iii): *$\mathcal{T}_{\mathcal{I}}$ is neither separable nor has the Lindelöf property;*

(iv): *$\mathcal{T}_{\mathcal{I}}$ is not regular;*
(v): *every subinterval of $\mathbb{R}$ is connected in $\mathcal{T}_{\mathcal{I}}$;*
(vi): *a set A is compact with respect to $\mathcal{T}_{\mathcal{I}}$ if, and only if, it is finite;*
(vii): *$\mathcal{T}_{\mathcal{I}}$ is not generated;* [3] *and*
(viii): *if the Continuum Hypothesis holds, then $\mathcal{T}_{\mathcal{I}}$ is not a Blumberg space.* [4]

Proof. (i) follows immediately from Lemma 2.3.1(ii) and Corollary 2.1.5(ii).

(ii). If $A \in \mathcal{I}$, then $A^c \triangle \mathbb{R} \in \mathcal{I}$ and, by Lemma 2.3.1(ii) and (iii), $A^c \subset \mathbb{R} = \Phi_{\mathcal{I}}(\mathbb{R}) = \Phi_{\mathcal{I}}(A^c)$, i.e., A is closed in $\mathcal{T}_{\mathcal{I}}$. So is every subset of A. Thus A closed and discrete.

On the other hand, if A is discrete in $\mathcal{T}_{\mathcal{I}}$, then $\Phi_{\mathcal{I}}(A^c) = \mathbb{R}$. Hence, by Lemma 2.3.1(iv), $A = A^c \triangle \mathbb{R} = A^c \triangle \Phi_{\mathcal{I}}(A^c) \in \mathcal{I}$.

To see first part of (iii) it is enough to notice that, by (ii), any countable set is closed in $\mathcal{T}_{\mathcal{I}}$. To see that $\mathcal{T}_{\mathcal{I}}$ does not have Lindelöf property it is enough to consider the $\mathcal{I}$-density open cover $\mathcal{U} = \{\{x\} \cup C^c\}_{x \in C}$ where C is the Cantor set.

Condition (iv) is established by noticing that the set $\mathbb{Q}$ is closed with respect to $\mathcal{T}_{\mathcal{I}}$ while, by Lemma 2.6.1, it cannot be separated in $\mathcal{T}_{\mathcal{I}}$ from any point $x \in \mathbb{R} \setminus \mathbb{Q}$.

(v) follows immediately from Corollary 2.5.3.

(vi). Evidently any finite set is compact.

To argue the other direction, let A be an infinite set and let $\{a_n\}_{n \in \mathbb{N}}$ be any sequence of distinct points from A. The set $\{a_n\}_{n \in \mathbb{N}}$ is closed and discrete with respect to $\mathcal{T}_{\mathcal{I}}$. Thus, the sets $G_n = \mathbb{R} \setminus \{a_k : k \geq n\}$ form a $\mathcal{T}_{\mathcal{I}}$-open cover of A without a finite subcover.

(vii). We have to show that the family

$$\mathcal{F} = \{\left(f^{-1}(\{x\})\right)^c : x \in \mathbb{R} \text{ and } f\colon (\mathbb{R}, \mathcal{T}_{\mathcal{I}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{I}}) \text{ is continuous}\}$$

is not a subbase for $\mathcal{T}_{\mathcal{I}}$. But $\mathcal{F} \subset \{\left(f^{-1}(\{x\})\right)^c : x \in \mathbb{R} \text{ and } f \in \mathcal{C}_{\mathcal{I}\mathcal{O}}\}$ and the second family obviously is not a subbase for $\mathcal{T}_{\mathcal{I}}$, as $\mathcal{T}_{\mathcal{I}}$ is not regular.

(viii) is obvious by the above properties and the following theorem of White [**65**]:

Let X be a Baire space with cardinality of continuum such that
: (a) X satisfies the countable chain condition,
: (b) the weight and the character of X is that of the continuum,[5]

[3] Compare to the footnote for Theorem 1.2.3 and Chapter 4.

[4] Compare to the footnote for Theorem 1.2.3 or [**63, 65**].

[5] For a topological space X it's *weight* is defined by

$$w(X) = \min\{\text{card}\,(\mathcal{U})\colon \mathcal{U} \text{ is a base of } X\}$$

and it's *character* by $\chi(X) = \sup\{\chi(X,x)\colon x \in X\}$, where

$$\chi(X,x) = \min\{\text{card}\,(\mathcal{U})\colon \mathcal{U} \text{ is a base of } X \text{ at } x\}.$$

: (c) every set of first category in X is nowhere dense in X.

Then X is not Blumberg. □

2.7. The Deep-$\mathcal{I}$-density Topology

As seen in Theorem 2.6.2(iv) the topology $\mathcal{T}_I$ on $\mathbb{R}$ is not regular, so, the weak topology generated by $\mathcal{C}_{\mathcal{I}\mathcal{O}}$ is strictly smaller than $\mathcal{T}_I$. In this section this weak topology is described and some of its properties are examined.

A point $a \in \mathbb{R}$ is said to be a *deep-$\mathcal{I}$-density point* of a set $A \in \mathcal{B}$ if, and only if, there exists a closed set $F \subset A \cup \{a\}$ such that a is an $\mathcal{I}$-density point of F. A point $a \in \mathbb{R}$ is said to be a *deep-$\mathcal{I}$-dispersion point* of an $A \in \mathcal{B}$ if, and only if, it is a deep-$\mathcal{I}$-density point of A^c. Similarly defined are *left* and *right deep-$\mathcal{I}$-density* and *deep-$\mathcal{I}$-dispersion points*.

Notice that for a closed set the notions of an $\mathcal{I}$-density point and a deep-$\mathcal{I}$-density point coincide. Similarly, the notions of an $\mathcal{I}$-dispersion point and a deep-$\mathcal{I}$-dispersion point coincide for open sets.

We first present the following easy equivalences.

Lemma 2.7.1. *Let $A \subset \mathbb{R}$. The following are equivalent:*

- **(i):** 0 *is a deep-$\mathcal{I}$-density point of A;*
- **(ii):** *there is a regular open set $V \subset A$ such that* 0 *is an $\mathcal{I}$-density point of V;*
- **(iii):** *there exists an interval set $E \subset A$ composed of closed intervals such that* 0 *is an $\mathcal{I}$-density point of E.*

Proof. (i) implies (ii). If $F \subset A \cup \{0\}$ is a closed set such that 0 is an $\mathcal{I}$-density point of F, then $W = \operatorname{int}(F)$ is a regular open and $W \triangle F \in \mathcal{I}$, so, by Lemma 2.3.1(iii), 0 is an $\mathcal{I}$-density point of W.

If $0 \notin W$, then $W \subset A$ and $V = W$ satisfies (ii).

If $0 \in W$, then $V = W \cap U$ works, where U is from Corollary 2.1.5(ii).

(ii) implies (iii). $E \subset V$ from Lemma 2.2.4(ii) works.

(iii) implies (i). $F = E \cup \{0\} \subset A \cup \{0\}$ has the desired property. □

Corollary 2.7.2. *The notions of $\mathcal{I}$-density point and deep-$\mathcal{I}$-density point coincide on regular open sets. In particular, they coincide on open interval sets.*

Define

$$\Phi_{\mathcal{D}}(A) = \{x \in \mathbb{R} \colon x \text{ is a deep-}\mathcal{I}\text{-density point of } A\},$$

and let

$$\mathcal{T}_D = \{A \in \mathcal{B} \colon A \subset \Phi_{\mathcal{D}}(A)\} \subset \mathcal{T}_{\mathcal{I}}.$$

Notice that $\Phi_{\mathcal{D}}(A) \subset \Phi_{\mathcal{I}}(A)$ and that properties (i), (ii) and (v) of Lemma 2.3.1 remain true if we replace $\Phi_{\mathcal{I}}$ by $\Phi_{\mathcal{D}}$. Thus, the following Theorem is an easy consequence of Theorem 2.3.2.

Theorem 2.7.3. *$\mathcal{T}_D$ is a topology on $\mathbb{R}$.*

The topology $\mathcal{T}_D$ on $\mathbb{R}$ is called the *deep-$\mathcal{I}$-density topology.*

The following theorem is an immediate consequence of Lemma 2.7.1.

COROLLARY 2.7.4. *For every $x \in \mathbb{R}$ the family*

$$\mathcal{U}(x) = \{U \cup \{x\} : U \text{ is an open interval set at } x \text{ and } x \in \Phi_{\mathcal{D}}(U)\}$$

forms a base of $\mathcal{T}_D$ at the point x.

A function $f : \mathbb{R} \to \mathbb{R}$ is *deep-$\mathcal{I}$-approximately continuous*, if it is continuous with respect to the natural topology $\mathcal{T}_{\mathcal{O}}$ on the range, and the deep-$\mathcal{I}$-density topology $\mathcal{T}_D$ on the domain. The class of all deep-$\mathcal{I}$-approximately continuous functions is denoted by $\mathcal{C}_{\mathcal{DO}}$.

As the next step the following theorem is needed.

THEOREM 2.7.5. *$\mathcal{T}_D$ is completely regular.*

PROOF. Let $x \in A \in \mathcal{T}_D$. Assume for simplicity that $x = 0$. We must define a deep-$\mathcal{I}$-approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$ and $A^c \subset f^{-1}(0)$. By Corollary 2.7.4 and Lemma 2.7.1 we can find an open interval set $V \subset A$ and an interval set $E \subset V$ composed of closed intervals such that 0 is an $\mathcal{I}$-density point of E. Define

$$f(x) = \begin{cases} 1 & x = 0 \\ \dfrac{\operatorname{dist}(x, V^c)}{\operatorname{dist}(x, V^c) + \operatorname{dist}(x, E)} & x \neq 0 \end{cases}$$

It is easy to see that $f(0) = 1$, $A^c \subset V^c \subset f^{-1}(0)$ and that f is continuous on $\mathbb{R} \setminus \{0\}$. Moreover, f is deep-$\mathcal{I}$-approximately continuous at 0 as $E \subset f^{-1}(1)$. □

We are ready to prove the main theorem of this section.

THEOREM 2.7.6. *$\mathcal{T}_D$ is the weak topology generated by $\mathcal{C}_{\mathcal{IO}}$; i.e., $\mathcal{T}_D$ is equal to the topology $\mathcal{T}$ generated by the family*

$$\{f^{-1}((a, b)) : f \in \mathcal{C}_{\mathcal{IO}} \text{ and } a, b \in \mathbb{R}\}.$$

PROOF. Evidently $\mathcal{T}_D \subset \mathcal{T}$ as $\mathcal{T}_D$ is completely regular. To prove the reverse inclusion, fix $a < b$, $f \in \mathcal{C}_{\mathcal{IO}}$ and $x \in f^{-1}((a, b))$. It must be proved that x is a deep-$\mathcal{I}$-density point of $f^{-1}((a, b))$.

Let $c < d$ be such that $f(x) \in (c, d) \subset [c, d] \subset (a, b)$ and define $E = f^{-1}([c, d])$. Thus, $E^c = f^{-1}([c, d]^c) \in \mathcal{T}_{\mathcal{I}}$ while

$$x \in f^{-1}((c, d)) \subset E \subset f^{-1}((a, b)).$$

So, x is an $\mathcal{I}$-density point of $\tilde{E}$ and, by Corollary 2.7.2, x is also a deep-$\mathcal{I}$-density point of $\tilde{E}$. Moreover, $\tilde{E} \cap E^c \in \mathcal{I} \cap \mathcal{T}_{\mathcal{I}}$. Thus, $\tilde{E} \cap E^c = \emptyset$; i.e., $\tilde{E} \subset E$. □

Theorem 2.7.6 immediately implies

COROLLARY 2.7.7. *A function f is $\mathcal{I}$-approximately continuous if, and only if, it is deep-$\mathcal{I}$-approximately continuous; i.e.,*

$$\mathcal{C}_{\mathcal{IO}} = \mathcal{C}_{\mathcal{DO}}.$$

The next theorem summarizes the topological properties of $\mathcal{T}_D$.

THEOREM 2.7.8. *The topology $\mathcal{T}_D$ on $\mathbb{R}$ has the following properties:*

(i): *$\mathcal{T}_{\mathcal{O}} \subset \mathcal{T}_D \subset \mathcal{T}_{\mathcal{I}}$ and the inclusions are proper;*

(ii): *there exists a set $S \in \mathcal{I}_0$ which is not closed with respect to $\mathcal{T}_D$; however, there also exists a perfect set C which is discrete with respect to $\mathcal{T}_D$;*

(iii): *$\mathcal{T}_D$ is separable; moreover, every set D which is dense with respect to $\mathcal{T}_{\mathcal{O}}$ is also dense with respect to $\mathcal{T}_D$;*

(iv): *$\mathcal{T}_D$ is completely regular but not normal;*

(v): *every subinterval of $\mathbb{R}$ is connected in $\mathcal{T}_D$;*

(vi): *a set A is compact with respect to $\mathcal{T}_D$ if, and only if, it is finite;*

(vii): *$\mathcal{T}_D$ is generated;*[6]

(viii): *$\mathcal{T}_D$ is a Blumberg space.*[7]

PROOF. (i). The inclusions are obvious. The first inclusion is proper by Corollary 2.1.5(ii). The second inclusion is proper because $\mathcal{T}_D$ is regular, but $\mathcal{T}_{\mathcal{I}}$ is not.

(ii). For the first part it is enough to take the set S from Example 2.2.5. For the second part, the set C from Lemma 2.1.8 works.

(iii). By Lemma 2.6.1, a set D which is dense with respect to $\mathcal{T}_{\mathcal{O}}$ cannot be separated in $\mathcal{T}_{\mathcal{I}}$, and so in $\mathcal{T}_D$, from any point $x \in \mathbb{R} \setminus D$. Thus, by the regularity of $\mathcal{T}_D$, the set D is dense with respect to $\mathcal{T}_D$.

(iv). The first part is a restatement of Theorem 2.7.5.

To prove the second part let us notice that the set C from Lemma 2.1.8 is closed and discrete with respect to $\mathcal{T}_D$ and has cardinality equal to the continuum $\mathfrak{c}$. Thus, by the following theorem (see [**23**, Theorem 8.10])

> If X is a separable normal space and E is a subset of X with cardinality at least equal to the continuum, then E has a limit point in X.

$\mathcal{T}_D$ cannot be normal.

(v) follows immediately from (i) and Theorem 2.6.2(v).

(vi). Evidently, any finite set is compact. To argue the other direction, let A be an infinite set. It must be shown that it is not compact with respect to $\mathcal{T}_D$. As in Theorem 2.6.2(vi) it is enough to prove that A has an infinite subset D which is closed and discrete with respect to $\mathcal{T}_D$.

If A is closed and discrete with respect to $\mathcal{T}_{\mathcal{O}}$ then $D = A$ works.

[6] See the footnote for Theorem 1.2.3 and Chapter 4.

[7] See the footnote for Theorem 1.2.3 or [**63, 65**].

Otherwise, there is a monotone sequence $\{c_n\}_{n\in\mathbb{N}} \subset A$ converging to a point $x \in \mathbb{R}$. Without loss of generality, assume that $x = 0$ and that the sequence $\{c_n\}_{n\in\mathbb{N}}$ is decreasing. Choosing a subsequence, if necessary, we can also assume that $\lim_{n\to\infty} c_{n+1}/c_n = 0$. Now, using Lemma 2.1.6, it is not difficult to argue that the set $D = \{c_n\}_{n\in\mathbb{N}}$ is closed and discrete with respect to $\mathcal{T}_D$.

(vii). Let $x \in A \in \mathcal{T}_D$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be as in the proof of Theorem 2.7.5. Then $x \in \left(f^{-1}(0)\right)^c \subset A$. So, it is enough to argue that f is continuous with respect to $\mathcal{T}_D$ both on the domain and the range.[8] But, f is continuous at 0 in this sense by the same argument as in Theorem 2.7.5. Moreover, at points $\neq 0$, f is unilaterally linear and so obviously satisfies the desired continuity. (Compare Proposition 3.1.7.)

(viii). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Then, by Blumberg's theorem [**4, 5**], there exists a set $D \subset \mathbb{R}$ dense with respect to $\mathcal{T}_\mathcal{O}$ such that $f|_D\colon (D, \mathcal{T}_\mathcal{O}) \to (\mathbb{R}, \mathcal{T}_\mathcal{O})$ is continuous. But, by (iii), D is dense with respect to $\mathcal{T}_D$ and, by (i), $f|_D\colon (D, \mathcal{T}_D) \to (\mathbb{R}, \mathcal{T}_\mathcal{O})$ is continuous. □

There is yet one more advantage that $\mathcal{T}_D$ has over $\mathcal{T}_\mathcal{I}$. The examples from Section 1.5 do not apply to $\mathcal{T}_D$, as can be seen from the following theorem.

THEOREM 2.7.9. $\mathcal{T}_D = \{A \subset \mathbb{R}\colon A \subset \Phi_\mathcal{D}(A)\}$.

PROOF. If $A \subset \Phi_\mathcal{D}(A)$ then, by Lemma 2.7.1(ii), $\operatorname{int}(A) \subset A \subset \operatorname{cl}(\operatorname{int}(A))$. Hence $A \in \mathcal{B}$. □

As a final remark in this section, notice that it follows easily from Theorem 2.4.5 that for a regular open set A and $x \in \mathbb{R}$, $A^c = \operatorname{cl}\left(\widetilde{A^c}\right)$ is superporous at x if, and only if, x is an $\mathcal{I}$-dispersion point of A^c. This means that the sets of the form $\{x\} \cup A$ where A is regular open and x is an $\mathcal{I}$-density point of A form a base for the topology $\mathcal{P}$ as well as for $\mathcal{T}_D$. This immediately implies

COROLLARY 2.7.10. $\mathcal{P} = \mathcal{T}_D$.

and

COROLLARY 2.7.11. *A is superporous at the point x if, and only if, x is a deep-$\mathcal{I}$-dispersion point of A.*

2.8. $\mathcal{I}$-density Topologies Versus the Density Topology

The purpose of this section is to discuss the relations between the $\mathcal{I}$-density topologies and the density topology. For this, we need the following lemma.

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[8]Such functions will be termed *deep-$\mathcal{I}$-density continuous*. Compare also Chapter 3.

LEMMA 2.8.1. *Let $P \subset (0,1]$ be closed and nowhere dense and let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty} b_{n+1}/b_n = 0$. Then 0 is a deep-$\mathcal{I}$-dispersion point of the set*

$$Q = \bigcup_{n\in\mathbb{N}} b_n P.$$

In particular, $Q^c \in \mathcal{T}_D$.

PROOF. By Corollaries 2.4.3 and 2.7.11 suffices to prove that for every $c \in (0,1)$ there exist $\varepsilon > 0$ and $\delta > 0$ such for any $x \in (0,\delta)$ there exists a closed interval $I \subset Q^c \cap (cx, x)$ such that $\mathrm{m}(I) \geq x\varepsilon$.

Let $c \in (0,1)$, $p = \min P$ and let $\delta > 0$ be such that

$$\frac{b_{n+1}}{b_n} < pc \ \text{ for every } \ n \in \mathbb{N} \ \text{ for which } \ pb_n \leq \delta. \tag{22}$$

Moreover, let $\varepsilon_0 > 0$ be a number such that for every interval $K \subset (0,1)$ of length $\geq p(1-c)/2$ there exists a closed interval $J \subset K \setminus P$ of length $\geq \varepsilon_0$. Such a number can be found, by partitioning $(0,1)$ into intervals $J_1, \ldots, J_k$, of length $< p(1-c)/4$ and defining

$$\varepsilon_0 < \min\{\sup\{b-a\colon [a,b] \subset J_i \setminus P\}\colon 1 \leq i \leq k\}.$$

Put $\varepsilon = \min\{\varepsilon_0/2, (1-c)/2\}$.

Now, let $x \in (0,\delta)$ and define

$$m = \min\{n\colon pb_n \leq x\}.$$

Then, by (22), $b_{m+1} < pb_m c \leq xc$. In particular,

$$Q^c \cap (cx, x) = (cx, x) \setminus b_m P.$$

Let a be the middle point of (cx, x). If $pb_m > a$, then $I = [xc, a]$ works, as $\mathrm{m}(I) = x(1-c)/2 \geq x\varepsilon$. Similarly, if $b_m < a$, then $I = [a, x]$ works.

So, let us assume that $pb_m \leq a \leq b_m$. Then $x \leq 2a \leq 2b_m$ and $pb_m < x = \frac{2}{1-c}(x-a)$, as $2(x-a) = (x - xc)$. In particular, $(x-a)/b_m > p(1-c)/2$. Thus, by the definition of ε_0, there exists a closed interval $J \subset \frac{1}{b_m}(a,x) \setminus P$ of the length $\geq \varepsilon_0$. Hence,

$$I = b_m J \subset (a,x) \setminus b_m P \subset (cx, x) \setminus b_m P = Q^c \cap (cx, x)$$

has length $\geq \varepsilon_0 b_m = (2\varepsilon_0)(b_m/2) \geq \varepsilon x = x\varepsilon$. This finishes the proof of Lemma 2.8.1. □

Now we are ready to prove

THEOREM 2.8.2. *If $\mathcal{P}(\mathbb{R})$ stands for the discrete topology on $\mathbb{R}$, then*

$$\begin{array}{ccccccc}
\mathcal{T}_{\mathcal{O}} \cap \mathcal{T}_{\mathcal{N}} & \subset & \mathcal{T}_D \cap \mathcal{T}_{\mathcal{N}} & \subset & \mathcal{T}_{\mathcal{I}} \cap \mathcal{T}_{\mathcal{N}} & \subset & \mathcal{T}_{\mathcal{N}} \\
\| & & \cap & & \cap & & \cap \\
\mathcal{T}_{\mathcal{O}} & \subset & \mathcal{T}_D & \subset & \mathcal{T}_{\mathcal{I}} & \subset & \mathcal{P}(\mathbb{R})
\end{array}$$

Moreover, all the inclusions are proper.

PROOF. All the inclusions follow immediately from Theorems 2.7.8(i) and 1.2.3(i).

To show that the horizontal inclusions are proper, it is enough to argue for the inclusions in the first row. Thus, it is enough to prove that $\mathcal{T}_D \cap \mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{O}$, $\mathcal{T}_\mathcal{I} \cap \mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_D$ and $\mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{I}$. To show that the vertical inclusions are proper we will show that $\mathcal{T}_D \not\subset \mathcal{T}_\mathcal{N}$.

$\mathcal{T}_D \cap \mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{O}$. If U is from Corollary 2.1.5(ii), then obviously $\{0\} \cup U \in \mathcal{T}_\mathcal{I} \cap \mathcal{T}_\mathcal{N}$, as $\mathcal{I}_\omega \subset \mathcal{I} \cap \mathcal{N}$. Moreover, U is an interval set. So, by Corollary 2.7.2, $\{0\} \cup U \in \mathcal{T}_D$. Evidently, $\{0\} \cup U \notin \mathcal{T}_\mathcal{O}$.

$\mathcal{T}_\mathcal{I} \cap \mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_D$. Let $E = \mathbb{R} \setminus \mathbb{Q}$. By Theorems 2.6.2(ii) and 1.2.3(ii), $\mathbb{Q}$ is nowhere dense in $\mathcal{T}_\mathcal{I}$ and $\mathcal{T}_\mathcal{N}$ and so $E \in \mathcal{T}_\mathcal{I} \cap \mathcal{T}_\mathcal{N}$. But, by Theorem 2.7.8(iii), $\mathbb{Q}$ is dense in $\mathcal{T}_D$. Hence, $E \notin \mathcal{T}_D$.

$\mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{I}$. Let C be a nowhere dense set of positive Lebesgue measure. By the Lebesgue Density Theorem, $D = C \cap \Phi_\mathcal{N}(C) \in \mathcal{T}_\mathcal{N}$ and $D \neq \emptyset$. Moreover, by Theorem 2.6.2(ii), D is nowhere dense in $\mathcal{T}_\mathcal{I}$, so $D \notin \mathcal{T}_\mathcal{I}$.

$\mathcal{T}_D \not\subset \mathcal{T}_\mathcal{N}$. Let $C \subset [\frac{1}{2}, 1]$ be a closed nowhere dense set with positive Lebesgue measure and let $\{b_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that

$$\lim_{n\to\infty} b_{n+1}/b_n = 0.$$

Then, by Lemma 2.8.1, 0 is a deep-$\mathcal{I}$-dispersion point of $E = \bigcup_{n\in\mathbb{N}} b_n C$ and $E^c \in \mathcal{T}_D$. On the other hand, for every $n \in \mathbb{N}$,

$$\frac{m\left(E \cap (0, b_n)\right)}{b_n} = m\left(b_n^{-1} E \cap (0,1)\right) \geq m(C) > 0.$$

Thus, 0 is not a dispersion point of E and $E^c \notin \mathcal{T}_\mathcal{N}$. □

2.9. Historical and Bibliographic Notes

Lemma 2.1.8 is new, although the set exhibited there, in a similar setting, was used earlier by Natkaniec [**48**]. Other results from Section 2.1 can be found in Wilczyński's survey paper [**68**] in the case of $\mathcal{J} = \mathcal{I}$, although Lemma 2.1.4 was proved there only in the case when $\limsup_{n\to\infty} b_{n+1}/a_n = 0$. In a slightly more general setting some of those results were proved in the earlier papers from Wilczyński [**66**], Poreda, Wagner-Bojakowska, Wilczyński [**58, 57**] and Aversa, Wilczyński [**1**].

Theorem 2.2.2 generalizes the results obtained by Poreda, Wagner-Bojakowska and Wilczyński [**57**, Theorem 1]. Condition (vii) of Theorem 2.2.2 is due to Łazarow [**38**]. The notion of regular open sets in this setting was first used by Łazarow, Johnson and Wilczyński [**40**] and, independently, by Ciesielski and Larson [**17, 15**].

Lemma 2.2.4 and Example 2.2.5 has been proved by Ciesielski and Larson in [**14**] and [**15**], respectively. Lemma 2.2.6 was originally proved by Aversa and Wilczyński [**1**, Lemma 4].

Lemma 2.3.1 and Theorem 2.3.2 were originally proved by Poreda, Wagner-Bojakowska and Wilczyński [**57**], although Theorem 2.3.2 was first stated, without any proof, by Wilczyński [**66**].

Essentially all the results from Section 2.4 are interpretations of results due to Zajíček [**72**]. Related results are contained in [**71**].

The class of $\mathcal{I}$-approximately continuous functions was defined by Poreda, Wagner-Bojakowska and Wilczyński [**57**]. In the same paper they also prove the two-sided version of Theorem 2.5.2. The version of this theorem presented here has been proved by Ciesielski and Larson [**17**]. The same authors presented Example 2.5.4 [**14**]. Properties from Theorem 2.6.2 can be found in [**57**] in the cases of (i), (iii) (for separability), (iv) and (v), in [**58**] in the case of (ii), in [**15**] in the case of (vi) and in [**22**] in the case of (vii). Part of (iii), concerning the Lindelöf property, and (viii) have never been published before.

The deep-$\mathcal{I}$-density topology was first introduced by Łazarow [**38**] and, independently, by Poreda and Wagner-Bojakowska [**56**]. Both these papers essentially contain Theorems 2.7.3, 2.7.5, 2.7.6, Corollaries 2.7.4, 2.7.7 and the equivalence of (i) and (iii) from Lemma 2.7.1. The equivalence of condition (ii) of this lemma with the remaining condition seems not to have been previously published.

Properties from Theorem 2.7.8 can be found in [**38**] and [**56**] in the cases of (i), (iii) and (iv) (for regularity), in [**56**] in the case of (iv) (for normality), in [**15**] in the cases of (ii) and (vi) and in [**22**] in the case of (vii). Condition (viii) has never been published before.

Lemma 2.8.1 and Theorem 2.8.2 have been proved by Ciesielski and Larson [**15**]. The examples showing that $\mathcal{T}_{\mathcal{N}} \not\subset \mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}_{\mathcal{I}} \not\subset \mathcal{T}_{\mathcal{N}}$ can be also found in [**1**].

CHAPTER 3

$\mathcal{I}$-density Continuous Functions

In this chapter, the $\mathcal{I}$-density continuous and deep-$\mathcal{I}$-density continuous functions are defined and some of their properties are explored. In particular, the relationships between these classes and the classes of continuous, $\mathcal{C}^\infty$ and analytic functions are presented.

3.1. $\mathcal{I}$-density and Deep-$\mathcal{I}$-density Continuous Functions

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *$\mathcal{I}$-density continuous* if it is continuous with respect to the $\mathcal{I}$-density topology $\mathcal{T}_\mathcal{I}$ on both the domain and the range. The class of all $\mathcal{I}$-density continuous functions is denoted by $\mathcal{C}_{\mathcal{I}\mathcal{I}}$. Similarly, a function $f\colon \mathbb{R} \to \mathbb{R}$ is *deep-$\mathcal{I}$-density continuous*, if it is continuous with respect to the deep-$\mathcal{I}$-density topology $\mathcal{T}_\mathcal{D}$ on both the domain and the range. The class of all deep-$\mathcal{I}$-density continuous functions is denoted by $\mathcal{C}_{\mathcal{D}\mathcal{D}}$.

The following definitions will be needed for technical reasons. The function $f\colon \mathbb{R} \to \mathbb{R}$ is *$\mathcal{I}$-density continuous at the point x* if whenever $f(x) \in B \in \mathcal{B}$ such that $f(x)$ is an $\mathcal{I}$-density point of B, then x is an $\mathcal{I}$-density point of $f^{-1}(B)$. By only requiring x to be a left or right $\mathcal{I}$-density point of $f^{-1}(B)$, the definitions of pointwise left and right $\mathcal{I}$-density continuity are obtained. *Right, left* and *bilateral deep-$\mathcal{I}$-density continuity* of f *at the point x* are defined similarly.

The following propositions are clear from the definitions.

Proposition 3.1.1. *The function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{I}$-density (deep-$\mathcal{I}$-density) continuous at a point $x \in \mathbb{R}$ if, and only if, f is simultaneously right and left $\mathcal{I}$-density (deep-$\mathcal{I}$-density) continuous at x.*

Proposition 3.1.2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. The following conditions are equivalent.*

(i): *f is $\mathcal{I}$-density (deep-$\mathcal{I}$-density) continuous at x.*

(ii): *For every $B \in \mathcal{B}$, if $f(x) \in B$ and $f(x)$ is an $\mathcal{I}$-density (deep-$\mathcal{I}$-density) point of B, then x is an $\mathcal{I}$-density (deep-$\mathcal{I}$-density) point of $f^{-1}(B)$.*

(iii): *For every $B \in \mathcal{B}$, if $f(x) \notin B$ and $f(x)$ is an $\mathcal{I}$-dispersion (deep-$\mathcal{I}$-dispersion) point of B, then x is an $\mathcal{I}$-dispersion (deep-$\mathcal{I}$-dispersion)* *point of $f^{-1}(B)$.*

To find relationships between the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$, the following lemma is needed.

LEMMA 3.1.3. *Let X be a completely regular topological space and Y any topological space. The following are equivalent:*

- **(i):** *the function $f\colon Y \to X$ is continuous;*
- **(ii):** *the function $g \circ f\colon Y \to [0,1]$ is continuous for every continuous $g\colon X \to [0,1]$.*

PROOF. The implication (i)⇒(ii) is obvious. (ii)⇒(i) follows immediately from the fact that, for a completely regular space X, the family

$$\{g^{-1}([0,\frac{1}{2})) : g\colon X \to [0,1] \text{ is continuous}\}$$

is a base for X. □

In particular, because $\mathcal{T}_D$ is completely regular (Theorem 2.7.5) and $\mathcal{C}_{\mathcal{I}\mathcal{O}} = \mathcal{C}_{\mathcal{D}\mathcal{O}}$ (Corollary 2.7.7), we obtain

COROLLARY 3.1.4. *For every function $f\colon \mathbb{R} \to \mathbb{R}$*

$$f \in \mathcal{C}_{\mathcal{D}\mathcal{D}} \quad \textit{if, and only if,} \quad g \circ f \in \mathcal{C}_{\mathcal{I}\mathcal{O}} \quad \textit{for every} \quad g \in \mathcal{C}_{\mathcal{I}\mathcal{O}}.$$

Now we are ready to prove

THEOREM 3.1.5. *Every $\mathcal{I}$-density continuous function is deep-$\mathcal{I}$-density continuous; i.e.,*

$$\mathcal{C}_{\mathcal{I}\mathcal{I}} \subset \mathcal{C}_{\mathcal{D}\mathcal{D}} \subset \mathcal{C}_{\mathcal{D}\mathcal{O}} = \mathcal{C}_{\mathcal{I}\mathcal{O}}.$$

PROOF. To prove the first inclusion let $f \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$. Then, evidently, $g \circ f \in \mathcal{C}_{\mathcal{I}\mathcal{O}}$ for every $g \in \mathcal{C}_{\mathcal{I}\mathcal{O}}$. Hence, by Corollary 3.1.4, $f \in \mathcal{C}_{\mathcal{D}\mathcal{D}}$. The second inclusion is an immediate consequence of the definitions. The equation is a restatement of Corollary 2.7.7. □

In particular, as a consequence of Corollary 2.5.3 we obtain

COROLLARY 3.1.6. *If $f \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$ or $f \in \mathcal{C}_{\mathcal{D}\mathcal{D}}$, then f is a Darboux Baire 1 function.*

Easy examples of deep-$\mathcal{I}$-density continuous functions are obtained from the following propositions. Others will follow in the next sections.

PROPOSITION 3.1.7. *Linear functions, $f(x) = ax + b$, are deep-$\mathcal{I}$-density continuous.*

Proof. Translations of x such as $f(x) = x + b$ are evidently deep-$\mathcal{I}$-density continuous, as the deep-$\mathcal{I}$-density topology is translation invariant. Thus, since the class $\mathcal{C}_{\mathcal{DD}}$ is closed under composition, it is enough to argue for functions like $f(x) = ax$. But the proposition now follows from known properties of $\mathcal{I}$-dispersion points, such as those contained in Theorem 2.2.2(iii). □

We say that a function f is *piecewise linear*, if every point of its domain has right and left neighborhoods on each of which f is linear. More generally, a function f satisfies a condition P *piecewise* if every point of its domain has right and left neighborhoods on each of which f satisfies condition P.

Notice that Propositions 3.1.1 and 3.1.7 immediately imply the following corollary.

Corollary 3.1.8. *Every piecewise linear function is $\mathcal{I}$-density continuous.*

3.2. Homeomorphisms and $\mathcal{I}$-density

In this section $\mathcal{I}$-density continuous homeomorphisms are examined. We also introduce and investigate the functions which preserve $\mathcal{I}$-density and deep-$\mathcal{I}$-density points.

Let $\mathcal{H}$ be the class of all homeomorphisms from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$ (or any subinterval). It will be shown that the inclusion $\mathcal{C}_{\mathcal{II}} \subset \mathcal{C}_{\mathcal{DD}}$ is proper (even in the class of continuous functions) and that $\mathcal{H} \not\subset \mathcal{C}_{\mathcal{DD}}$. However, first we present the following theorem.

Theorem 3.2.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f^{-1}(E) \in \mathcal{I}$ for every $E \in \mathcal{I}$. Then f is deep-$\mathcal{I}$-density continuous if, and only if, f is $\mathcal{I}$-density continuous. In particular,*

$$\mathcal{H} \cap \mathcal{C}_{\mathcal{II}} = \mathcal{H} \cap \mathcal{C}_{\mathcal{DD}}.$$

Proof. The inclusion $\mathcal{C}_{\mathcal{II}} \subset \mathcal{C}_{\mathcal{DD}}$ follows from Theorem 3.1.5.

To prove the converse, choose $f \in \mathcal{C}_{\mathcal{DD}}$ satisfying the assumption and let $f(x)$ be an $\mathcal{I}$-density point of $E \in \mathcal{B}$ with $f(x) \in E$. Then, $f(x)$ is an $\mathcal{I}$-density point of $\tilde{E}$ and, by the regularity of $\tilde{E}$, $f(x)$ is also a deep-$\mathcal{I}$-density point of $\tilde{E}$. (See Corollary 2.7.2.) Thus, x is a deep-$\mathcal{I}$-density point of $f^{-1}(\tilde{E})$. Moreover, by the assumption,

$$f^{-1}(\tilde{E}) \triangle f^{-1}(E) = f^{-1}(\tilde{E} \triangle E) \in \mathcal{I}.$$

So, by Lemma 2.3.1(iii), x is an $\mathcal{I}$-density point of $f^{-1}(E)$.

The additional part follows easily from the fact that homeomorphisms satisfy the assumption. □

Let $\mathcal{A}$ denote the class of all analytic functions from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$. It is easy to see that Theorem 3.2.1 implies that $\mathcal{A} \cap \mathcal{C}_{\mathcal{II}} = \mathcal{A} \cap \mathcal{C}_{\mathcal{DD}}$. However, it will be shown in the following sections that $\mathcal{A} \subset \mathcal{C}_{\mathcal{II}}$, which makes this observation trivial.

Now consider the following notions dual to those of $\mathcal{I}$-density and deep-$\mathcal{I}$-density continuity at a point. A function $h\colon \mathbb{R} \to \mathbb{R}$ *preserves $\mathcal{I}$-density points*, if for every $B \in \mathcal{B}$, $h(x)$ is an $\mathcal{I}$-density point of $h(B)$ whenever x is an $\mathcal{I}$-density point of B. Similarly defined are functions that *preserve deep-$\mathcal{I}$-density points*. The notions of preserving $\mathcal{I}$-dispersion points and deep-$\mathcal{I}$-dispersion points are similar to the above. The left-hand and right-hand versions of the above definitions are defined in the obvious way.

The following lemma is an easy consequence of Theorem 3.2.1 and Proposition 3.1.2.

LEMMA 3.2.2. *If $h \in \mathcal{H}$ is a homeomorphism, then the following are equivalent:*

- **(i):** *h^{-1} is $\mathcal{I}$-density continuous;*
- **(ii):** *h^{-1} is deep-$\mathcal{I}$-density continuous;*
- **(iii):** *h preserves $\mathcal{I}$-density points;*
- **(iv):** *h preserves deep-$\mathcal{I}$-density points;*
- **(v):** *h preserves $\mathcal{I}$-dispersion points;*
- **(vi):** *h preserves deep-$\mathcal{I}$-dispersion points.*

COROLLARY 3.2.3. *If f is piecewise homeomorphic, then f preserves $\mathcal{I}$-density points if, and only if, f preserves deep-$\mathcal{I}$-density points.*

The following lemma shows that piecewise homeomorphisms which preserve $\mathcal{I}$-density points form a lattice. Similar results for $\mathcal{I}$-density and deep-$\mathcal{I}$-density continuous functions will be proved in Proposition 3.2.15.

LEMMA 3.2.4. *If f and g are piecewise preserving $\mathcal{I}$-density points, then both $\min\{f,g\}$ and $\max\{f,g\}$ also preserve $\mathcal{I}$-density points.*

PROOF. Let f and g be piecewise homeomorphic functions that preserve $\mathcal{I}$-density points. It is clear that a function h preserves $\mathcal{I}$-density points if, and only if, $-h$ preserves $\mathcal{I}$-density points. Thus, by Lemma 3.2.2 and the fact that $\min\{f,g\} = -\max\{-f,-g\}$, it is enough to prove that $h = \max\{f,g\}$ preserves deep-$\mathcal{I}$-dispersion points.

So, let $B \in \mathcal{B}$ and let $p \notin B$ be a deep-$\mathcal{I}$-dispersion point of B. It is enough to prove that $h(p)$ is an $\mathcal{I}$-dispersion point of $h(B)$.

Without any loss of generality it may be assumed that $p = f(p) = g(p) = h(p) = 0$ and that $B \subset (0,\infty)$, as the left hand case is similar. By Lemma 2.7.1 we may also assume that $B = \bigcup_{n\in\mathbb{N}}(a_n,b_n)$ is a right interval set such that f and g are strictly monotone on $(0,b_1)$.

We use Theorem 2.2.2(viii) for this proof. Fix an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers and a nonempty interval $(a,b) \subset (-1,1)$. We must find a nonempty subinterval $(c,d) \subset (a,b)$ and a subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$ such that for every $p \in \mathbb{N}$

$$(c,d) \cap n_{k_p} \max\{f,g\}(B) = \emptyset;$$

i.e., such that

$$\frac{1}{n_{k_p}}(c,d) \cap \max\{f,g\}(B) = \emptyset. \tag{23}$$

Using Theorem 2.2.2(viii) twice, once for each of the functions f and g, we can find a nonempty subinterval $(c,d) \subset (a,b)$ and a subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$ such that for every $p \in \mathbb{N}$

$$\frac{1}{n_{k_p}}(c,d) \cap f(B) = \emptyset \quad \text{and} \quad \frac{1}{n_{k_p}}(c,d) \cap g(B) = \emptyset.$$

But it is easy to see that

$$\frac{1}{n_{k_p}}(c,d) \cap f((a_n,b_n)) = \emptyset \quad \text{and} \quad , \frac{1}{n_{k_p}}(c,d) \cap g((a_n,b_n)) = \emptyset$$

implies

$$\frac{1}{n_{k_p}}(c,d) \cap \max\{f,g\}((a_n,b_n)) = \emptyset$$

for every $p, n \in \mathbb{N}$. Condition (23) and Lemma 3.2.4 follow. □

The next theorem is used to establish the existence of several interesting examples.

THEOREM 3.2.5. *Let $f \in \mathcal{C}^\infty$ be such that for every $n \geq 0$*

$$f^{(n)}(0) = 0 \quad \textit{and} \quad , f^{(n)}((0,\varepsilon_n)) \subset (0,\infty) \quad \textit{for some} \quad \varepsilon_n > 0.$$

Then f is not deep-$\mathcal{I}$-density continuous and does not preserve deep-$\mathcal{I}$-density points.

Before proving this theorem we state three immediate consequences.

COROLLARY 3.2.6. *There exists a $\mathcal{C}^\infty$ homeomorphism which is not $\mathcal{I}$-density continuous and does not preserve $\mathcal{I}$-density points. In particular,*

$$\mathcal{H} \cap \mathcal{C}^\infty \not\subset \mathcal{C}_{\mathcal{D}\mathcal{D}}.$$

PROOF. Define

$$f(x) = \begin{cases} e^{-x^{-2}} & x > 0 \\ 0 & x = 0 \\ -e^{-x^{-2}} & x < 0 \end{cases}$$

It is known that $f \in \mathcal{C}^\infty$ and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. It is also easy to see that for every $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that $f^{(n)}(x) > 0$ for every $x \in (0, \varepsilon_n)$. Use of Theorem 3.2.5 finishes the proof. □

The next corollary shows that the full analogue of Theorem 1.4.3 for $\mathcal{I}$-density continuous functions cannot be proved.

COROLLARY 3.2.7. *There exists a convex $\mathcal{C}^\infty$ function which is not deep-$\mathcal{I}$-density continuous and does not preserve $\mathcal{I}$-density and deep-$\mathcal{I}$-density points.*

Proof. Define $g\colon (-\infty, 0.5) \to \mathbb{R}$ by

$$g(x) = \begin{cases} e^{-x^{-2}} & x \in (0, 0.5) \\ 0 & x \in (-\infty, 0] \end{cases}$$

Examining the second derivative of g it is easy to see that g is convex on $(-\infty, 0.5)$. The other properties follow as in Corollary 3.2.6. □

Corollary 3.1.8 and the existence of a continuous function which is not $\mathcal{I}$-density continuous (Corollary 3.2.6) imply immediately

Corollary 3.2.8. *The classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ are not closed under uniform convergence.*

The following lemma is needed for the proof of Theorem 3.2.5.

Lemma 3.2.9. *Let $f \in \mathcal{C}^\infty$ be such that for every $n \geq 0$*

$$f^{(n)}(0) = 0 \quad \textit{and} \quad f^{(n)}((0, \varepsilon_n)) \subset (0, \infty), \quad \textit{for some} \quad \varepsilon_n > 0.$$

Then

$$\lim_{x \to 0^+} \frac{f(ax)}{f(x)} = 0,$$

for every $a \in (0, 1)$.

Proof. Let $a \in (0, 1)$ and $n \in \mathbb{N}$. Moreover, choose $\varepsilon > 0$ such that $0 < \varepsilon < \varepsilon_k$ for every $k \leq n + 1$. In particular, $f^{(n)}$ is increasing on $(0, \varepsilon)$, and so

$$\left| \frac{f^{(n)}(a\xi)}{f^{(n)}(\xi)} \right| < 1 \quad \text{for every} \quad \xi \in (0, \varepsilon).$$

Now let $x \in (0, \varepsilon)$ and let $g(x) = f(ax)$. Using Cauchy's Generalized Mean Value Theorem n-times we can find $\xi \in (0, x)$ such that

$$\left| \frac{f(ax)}{f(x)} \right| = \left| \frac{g(x)}{f(x)} \right| = \left| \frac{g^{(n)}(\xi)}{f^{(n)}(\xi)} \right| = |a^n| \left| \frac{f^{(n)}(a\xi)}{f^{(n)}(\xi)} \right| < a^n.$$

Thus,

$$\lim_{x \to 0^+} \frac{f(ax)}{f(x)} = 0. \quad \square$$

Proof of Theorem 3.2.5. Any function f with the described properties is a homeomorphism in a right neighborhood of 0. So, by Theorem 3.2.1, it is enough to prove that f is not right $\mathcal{I}$-density continuous at 0 and does not preserve 0 as a right $\mathcal{I}$-density point.

We start with a proof that f is not right $\mathcal{I}$-density continuous at 0. Let $D_n = \{\frac{i}{2^n} : i = 1, 2, \ldots, 2^n\}$ for $n \in \mathbb{N}$. First notice that if a sequence $\{n_k\}_{k \in \mathbb{N}}$ satisfies

$$n_{k+1} > 2^k n_k \quad \text{for every } k \in \mathbb{N}, \tag{24}$$

then
$$\min \frac{1}{n_k} D_k = \frac{1}{n_k}\frac{1}{2^k} > \frac{1}{n_{k+1}} = \max \frac{1}{n_{k+1}} D_{k+1}.$$
This means that if $\{s_i\}_{i>1}$ is a decreasing ordering of $D = \bigcup_{k\in\mathbb{N}} \frac{1}{n_k} D_k$, then
$$\frac{1}{n_k} D_k = \{s_i \colon 2^k \leq i < 2^{k+1}\}.$$
A sequence $\{n_k\}_{k\in\mathbb{N}}$ will be defined by induction on k such that it will satisfy condition (24) and for every $k > 0$

$$\frac{f(s_i)}{f(s_{i-1})} \leq \frac{1}{k} \quad \text{for} \quad 2^k \leq i < 2^{k+1}. \tag{25}$$

Put $n_1 = 1$ and assume that n_{k-1} has already been chosen for some $k > 1$. Choose $n_k > 2^{k-1} n_{k-1}$ such that
$$\frac{f(\frac{2^k-1}{2^k}x)}{f(x)} < \frac{1}{k}, \quad \text{for all} \quad x \in (0, \frac{1}{n_k}).$$
Such a choice is possible by Lemma 3.2.9. Then, the above condition obviously implies condition (25) for $2^k < i < 2^{k+1}$. Increasing n_k, if necessary, we can also obtain condition (25) for $i = 2^k$. This finishes the construction of D.

Now let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint intervals such that every interval (a_n, b_n) is centered at $c_n = f(s_n)$ and that
$$\lim_{n\to\infty} \frac{b_n - a_n}{c_n} = 0.$$
By (25),
$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$$
so, by Lemma 2.1.6, 0 is an $\mathcal{I}$-dispersion point of the interval set
$$E = \bigcup_{n\in\mathbb{N}} (a_n, b_n).$$
On the other hand, as in Example 2.2.5, we notice that for every subsequence $\{n_{k_i}\}_{i\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$, the set
$$\bigcup_{i\in\mathbb{N}} n_{k_i} f^{-1}(E) \supset \bigcup_{i\in\mathbb{N}} D_{k_i}$$
is dense and open in $[0,1]$. So, 0 is not a right $\mathcal{I}$-dispersion point of $f^{-1}(E)$ and f is not $\mathcal{I}$-density continuous at 0.

To prove the second part of the theorem, define a right interval set $E = \bigcup_{n\in\mathbb{N}} (a_n, b_n)$ such that
$$\lim_{n\to\infty} (b_n - a_n)/a_n = 0$$
and

$$f(b_n) = \frac{1}{2} f(a_n), \quad \text{for every } n \in \mathbb{N}. \tag{26}$$

Such a choice can be made by Lemma 3.2.9. Then, by Lemma 2.1.7, there exists an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers such that 0 is an $\mathcal{I}$-dispersion point of

$$\bigcup_{m\in\mathbb{N}} (a_{n_m}, b_{n_m}),$$

while, by Lemma 2.2.6 and (26), 0 is not an $\mathcal{I}$-dispersion point of

$$f\left(\bigcup_{m\in\mathbb{N}} (a_{n_m}, b_{n_m})\right).$$

This finishes the proof of Theorem 3.2.5. □

To prove the next theorem and find more $\mathcal{I}$-density continuous homeomorphisms we need the following

LEMMA 3.2.10. *Let $f, h\colon [0,+\infty) \to [0,+\infty)$ be homeomorphisms such that*

$$\lim_{x\to 0^+} \frac{h^{-1}(x)}{f^{-1}(x)} = 1.$$

Then for every $0 < c < c' < d' < d$ there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$,

$$f\left((\varepsilon c', \varepsilon d')\right) \subset h\left((\varepsilon c, \varepsilon d)\right).$$

PROOF. Since $c/c' < 1$ and $d/d' > 1$ there exist a $\delta_0 > 0$ such that for every $x \in (0, \delta_0)$

$$\frac{c}{c'} < \frac{h^{-1}(x)}{f^{-1}(x)} < \frac{d}{d'}. \tag{27}$$

Using the continuity of f^{-1} at 0, we can find $\varepsilon_0 > 0$ such that

$$f((0, \varepsilon_0 d)) \subset (0, \delta_0).$$

Now let $\varepsilon \in (0, \varepsilon_0)$ and

$$x \in f((\varepsilon c', \varepsilon d')) \subset f((0, \varepsilon_0 d)) \subset (0, \delta_0).$$

So, (27) holds and $f^{-1}(x) \in (\varepsilon c', \varepsilon d')$; i.e.,

$$\varepsilon c' < f^{-1}(x) < \varepsilon d'.$$

Multiplying the above inequality by (27), we obtain

$$\varepsilon c < h^{-1}(x) < \varepsilon d,$$

which implies $x \in h\left((\varepsilon c, \varepsilon d)\right)$. □

THEOREM 3.2.11. *Let* $f, h\colon [0,+\infty) \to [0,+\infty)$ *be homeomorphisms such that*

$$\lim_{x\to 0^+} \frac{h^{-1}(x)}{f^{-1}(x)} = 1.$$

Then h *is right* $\mathcal{I}$*-density continuous at* 0 *if, and only if,* f *is right* $\mathcal{I}$*-density continuous at* 0.

PROOF. Without loss of generality we may assume that both functions are increasing, as the decreasing case is essentially the same.

So assume that h is right $\mathcal{I}$-density continuous at 0. It will be shown that f is right $\mathcal{I}$-density continuous at 0. This will finish the proof, as the converse implication follows by exchanging f with h.

Choose $B \in \mathcal{B}$, $0 \notin B$, which has 0 as an $\mathcal{I}$-dispersion point. By the right hand side versions of Theorem 3.2.1 and Corollary 2.7.4 we may assume that B is an open right interval set. We will use condition (viii) of Theorem 2.2.2 to prove that 0 is a right $\mathcal{I}$-dispersion point of $f^{-1}(B)$. So, choose an increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ and a nonempty interval $(a,b) \subset (0,1)$. Because 0 is a right $\mathcal{I}$-dispersion point of $h^{-1}(B)$, there exists a nonempty interval $(c,d) \subset (a,b)$ and a subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$ such that for every $p \in \mathbb{N}$

$$(c,d) \cap n_{k_p} h^{-1}(B) = \emptyset.$$

But this last condition is equivalent to

$$h\left(\left(\frac{1}{n_{k_p}}c, \frac{1}{n_{k_p}}d\right)\right) \cap B = \emptyset.$$

Now let $0 < c < c' < d' < d$. Then, by Lemma 3.2.10,

$$f\left(\frac{1}{n_{k_p}}c', \frac{1}{n_{k_p}}d'\right) \subset h\left(\frac{1}{n_{k_p}}c, \frac{1}{n_{k_p}}d\right)$$

for almost all $p \in \mathbb{N}$. This implies that for almost all $p \in \mathbb{N}$

$$f\left(\left(\frac{1}{n_{k_p}}c', \frac{1}{n_{k_p}}d'\right)\right) \cap B = \emptyset,$$

or

$$(c', d') \cap n_{k_p} f^{-1}(B) = \emptyset.$$

Therefore, Theorem 2.2.2(viii) is satisfied. This finishes the proof of Theorem 3.2.11. □

COROLLARY 3.2.12. *If* h *is a differentiable function on* $\mathbb{R}$*, or on any subinterval of* $\mathbb{R}$*, with nonzero derivative everywhere, then* h *is* $\mathcal{I}$*-density continuous.*

PROOF. It is enough to prove that h is right $\mathcal{I}$-density continuous at 0, while $h(0) = 0$. For this case, let $a = h'(0)$ and let $f(x) = ax$. An elementary calculation shows that the assumption of Theorem 3.2.11 is satisfied. □

We can also conclude

Corollary 3.2.13. *The classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\mathcal{C}_{\mathcal{I}\mathcal{I}} \cap \mathcal{C}^\infty$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}} \cap \mathcal{C}^\infty$ are not closed under addition.*

Proof. Let f be as in Corollary 3.2.6. Define $g(x) = f(x)+x$ and $h(x) = -x$. Then, by Corollary 3.2.12, the functions g and h are $\mathcal{C}^\infty$ $\mathcal{I}$-density continuous homeomorphisms, while $f = g + h$ is not. □

The same example as in Corollary 3.2.13 also gives the following corollary, which contradicts a theorem of Aversa and Wilczyński [**1**, Theorem 4]. Thus, the oversight in their proof cannot be repaired.

Corollary 3.2.14. *The class of all $\mathcal{C}^\infty$ homeomorphisms preserving $\mathcal{I}$-density points is not closed under addition.*

We will finish this section with the following proposition, similar to Lemma 3.2.4, which will be used in the next section.

Proposition 3.2.15. *The classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ of $\mathcal{I}$-density continuous functions and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ of deep-$\mathcal{I}$-density continuous functions are closed under the supremum and infimum operation applied to a finite number of functions.*

Proof. We prove this only for $\mathcal{C}_{\mathcal{I}\mathcal{I}}$. For $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ the same proof works. Only two functions need be considered in the proof.

Let $f, g \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $h = \max\{f, g\}$ and $x_0 \in \mathbb{R}$. Assume first that $h(x_0) = f(x_0) > g(x_0)$ and that $m \in (g(x_0), f(x_0))$. If G is an $\mathcal{I}$-density neighborhood of $f(x_0)$ contained in (m, ∞), then $H = f^{-1}(G) \cap g^{-1}((-\infty, m))$ is an $\mathcal{I}$-density neighborhood of x_0 with the property that $h|H = f|H$. This implies that h is $\mathcal{I}$-density continuous at x_0. A symmetrical argument handles the case when $h(x_0) = g(x_0) > f(x_0)$.

Now, assume $f(x_0) = g(x_0) = h(x_0)$ and let G be an $\mathcal{I}$-density neighborhood of $h(x_0)$. Both $f^{-1}(G)$ and $g^{-1}(G)$ are $\mathcal{I}$-density neighborhoods of x_0, so $H = f^{-1}(G) \cap g^{-1}(G)$ is also an $\mathcal{I}$-density neighborhood of x_0. If $x \in H$, then $f(x) \in G$ and $g(x) \in G$, so $h(x) = \max(f(x), g(x)) \in G$. From this, it follows that $h^{-1}(G) \supset H$ and h is $\mathcal{I}$-density continuous at x_0.

Therefore $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ is closed under the operation of taking the maximum of two functions. Since $\min(f(x), g(x)) = -\max(-f(x), -g(x))$, we see $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ is also closed under the minimization operation. □

3.3. Addition within $\mathcal{H} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$

In the previous section we concluded that the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ and the class of functions preserving $\mathcal{I}$-density points are not closed under addition. In fact, we found an increasing homeomorphism $g \in \mathcal{C}_{\mathcal{I}\mathcal{I}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$ such that $h(x) = g(x) - x$ is a homeomorphism which does not preserve $\mathcal{I}$-density points and $h \notin \mathcal{C}_{\mathcal{I}\mathcal{I}} \cup \mathcal{C}_{\mathcal{N}\mathcal{N}}$. However, there are some positive things which can be proved in this direction.

The purpose of this section is to prove that the sum of two increasing $\mathcal{I}$-density continuous homeomorphisms is $\mathcal{I}$-density continuous and that the sum of two increasing homeomorphisms preserving $\mathcal{I}$-density points is a homeomorphism that preserves $\mathcal{I}$-density points.

To prove the first of these theorems we need the following lemmas.

LEMMA 3.3.1. *Let $D \in \mathcal{B}$ be such that 0 is not a right $\mathcal{I}$-dispersion point of D. Then there exists an increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers diverging to infinity and a nonempty interval $(a,b) \subset (0,1)$ such that*

$$\left(\liminf_{k\to\infty} t_k D\right) \text{ is dense in } (a,b).$$

PROOF. Since 0 is not a right $\mathcal{I}$-dispersion point of D then, by Theorem 2.2.2(iii), there exists an increasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity such that for each of its subsequences $\{s_{n_k}\}_{k\in\mathbb{N}}$

$$\left(\limsup_{k\to\infty} s_{n_k} D\right) \cap (0,1) \notin \mathcal{I}. \tag{28}$$

Let $(p_k,q_k) \subset (0,1)$ be a sequence containing all nonempty intervals with rational endpoints. Let us construct, by induction on k, sequences $\{s_n^k\}_{n\in\mathbb{N}}$ such that $\{s_n^0\}_{n\in\mathbb{N}} = \{s_n\}_{n\in\mathbb{N}}$ and $\{s_n^k\}_{n\in\mathbb{N}}$ is a subsequence of $\{s_n^{k-1}\}_{n\in\mathbb{N}}$ such that

$$\left(\limsup_{n\to\infty} s_n^k D\right) \cap (p_k,q_k) = \emptyset \quad \text{or} \quad \left(\liminf_{n\to\infty} s_n^k D\right) \cap (p_k,q_k) \neq \emptyset. \tag{29}$$

To see that this is possible, suppose that the left-hand equation from (29) cannot be satisfied with any subsequence of $\{s_n^{k-1}\}_{n\in\mathbb{N}}$. Then,

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} s_m^{k-1} D \cap (p_k,q_k) \neq \emptyset.$$

This implies there is an $x \in (p_k,q_k)$ such that for each $n \in \mathbb{N}$ there exists an $m_n > n$ with $x \in s_{m_n}^{k-1} D$. There is no generality lost with the assumption that m_n is an increasing sequence. Define $s_n^k = s_{m_n}^{k-1}$. Then

$$x \in \bigcap_{n=1}^{\infty} s_n^k D \subset \left(\liminf_{n\to\infty} s_n^k D\right) \cap (p_k,q_k),$$

and the right-hand expression from (29) is satisfied.

Put $t_k = s_k^k$. Then, by (28), $(\limsup_{k\to\infty} t_k D) \cap (0,1) \notin \mathcal{I}$; i.e., there exists a nonempty interval $(a,b) \subset (0,1)$ such that

$$\left(\limsup_{k\to\infty} t_k D\right) \text{ is dense in } (a,b).$$

But this, together with (29), guarantees also

$$\left(\liminf_{k\to\infty} t_k D\right) \text{ is dense in } (a,b).$$

This finishes the proof of Lemma 3.3.1. □

LEMMA 3.3.2. *Suppose $h\colon \mathbb{R} \to \mathbb{R}$ is an increasing $\mathcal{I}$-density continuous homeomorphism such that $h(0) = 0$ and let $\{t_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers diverging to infinity. Then for every nontrivial interval $[a, b] \subset (0, 1)$ there exists a nonempty interval $(c, d) \subset (a, b)$ and a subsequence $\{t_{k_i}\}_{i\in\mathbb{N}}$ of $\{t_k\}_{k\in\mathbb{N}}$ such that the limit*

$$\lim_{i\to\infty} \frac{h(c/t_{k_i})}{h(d/t_{k_i})}$$

exists and is positive.

PROOF. By way of contradiction assume that it cannot be done; i.e., that

$$\limsup_{k\to\infty} \frac{h(c/t_k)}{h(d/t_k)} = 0 \quad \text{for every} \quad a \le c < d \le b. \tag{30}$$

We will show that this contradicts the $\mathcal{I}$-density continuity of h.

So, let $\{q_k\colon k \in \mathbb{N}\}$ be an enumeration of $Q = [a, b] \cap \mathbb{Q}$ and for each $i \in \mathbb{N}$ let $d_1, \ldots, d_i$ be a decreasing enumeration of $q_1, \ldots, q_i$. Choose $\{t_{k_i}\}_{i\in\mathbb{N}}$ such that

$$\frac{h(b/t_{k_{i+1}})}{h(a/t_{k_i})} \le \frac{h(d_{j+1}/t_{k_i})}{h(d_j/t_{k_i})} \le \frac{1}{i} \quad \text{for every} \quad j < i, \; i \in \mathbb{N}. \tag{31}$$

This can be done by (30). Let

$$U_i = \bigcup_{j\le i} h(d_j/t_{k_i}) \left(1 - \frac{1}{i}, 1 + \frac{1}{i}\right)$$

and put $U = \bigcup_{i\in\mathbb{N}} U_i$. Then, by (31) and Lemma 2.1.6, 0 is an $\mathcal{I}$-dispersion point of U. But 0 is not an $\mathcal{I}$-dispersion point of $h^{-1}(U)$, since for any subsequence $\{t_m\}_{m\in\mathbb{N}}$ of $\{t_{k_i}\}_{i\in\mathbb{N}}$ the open set

$$\bigcup_{m\ge m_0} t_m h^{-1}(U) \supset Q$$

is dense in (a, b) for every $m_0 \in \mathbb{N}$, and so,

$$(-1, 1) \cap \limsup_{m\to\infty} (t_m h^{-1}(U)) \notin \mathcal{I}.$$

This finishes the proof of Lemma 3.3.2. □

LEMMA 3.3.3. *Let $a < b$, $H_k\colon [a, b] \to \mathbb{R}$ be a sequence of increasing homeomorphisms such that there exists a dense subset Q of $[a, b]$ containing a and b such that the limit $H(q) = \lim_{k\to\infty} H_k(q)$ exists for every $q \in Q$. If $H(Q)$ is dense in $[H(a), H(b)]$ and $H(x) = \inf H(Q \cap [x, \infty))$ for every $x \in [a, b]$, then H_k converges uniformly to H.*

PROOF. It can be proved that the functions H_k are equicontinuous. Then an appropriate version of the Ascoli-Arzelá Theorem implies the lemma. However, the proof showing the equicontinuity of H_k is essentially as complicated as the elementary proof presented below.

First notice that the function $H(q) = \lim_{k\to\infty} H_k(q)$ on Q is nondecreasing, so $H(q) = \inf H(Q \cap [q, \infty))$ for every $q \in Q$.

Let us fix $\varepsilon > 0$. For $x \in [a, b]$ choose distinct $q_1, q_2 \in Q$, $q_1 \le x \le q_2$, such that $q_1 < x < q_2$ for $x \in (a, b)$ and

$$|H(q_2) - H(q_1)| < \varepsilon/5.$$

Let $N_x \in \mathbb{N}$ be such that

$$|H(q_i) - H_n(q_i)| < \varepsilon/5$$

for every $n > N_x$ and $i = 1, 2$. Put $U_x = (q_1, q_2)$ for $x \in (a, b)$, $U_x = [q_1, q_2)$ for $x = a$ and $U_x = (q_1, q_2]$ for $x = b$. Thus, U_x is an open neighborhood of x in $[a, b]$ and, for every $y \in U_x$ and $n > N_x$,

$$H(q_1) \le H(y) \le H(q_2) \quad \text{and} \quad H_n(q_1) \le H_n(y) \le H_n(q_2),$$

so that

$$\begin{aligned}|H(y) - H_n(y)| &\le |H(y) - H(q_2)| + |H(q_2) - H_n(q_2)| + |H_n(q_2) - H_n(y)| \\ &< |H(q_1) - H(q_2)| + \varepsilon/5 + |H_n(q_2) - H_n(q_1)| \\ &< \varepsilon/5 + \varepsilon/5 + |H_n(q_2) - H(q_2)| + |H(q_2) - H(q_1)| \\ &\quad + |H(q_1) - H_n(q_1)| \\ &< \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon.\end{aligned}$$

Choose a finite subcover $\{U_{x_1}, \dots, U_{x_k}\}$ of the open cover $\{U_x\}_{x\in[a,b]}$ of $[a, b]$ and put $N = \sup\{N_{x_1}, \dots, N_{x_k}\}$. Then we obtain

$$|H(y) - H_n(y)| < \varepsilon$$

for every $y \in [a, b]$ and $n > N$. The proof of Lemma 3.3.3 is finished. □

LEMMA 3.3.4. *Suppose* $h\colon \mathbb{R} \to \mathbb{R}$ *is an increasing* $\mathcal{I}$*-density continuous homeomorphism such that* $h(0) = 0$ *and let* $[a, b] \subset (0, 1)$ *be a nontrivial interval. If* $\{s_k\}_{k\in\mathbb{N}}$ *and* $\{t_k\}_{k\in\mathbb{N}}$ *are increasing sequences of positive numbers diverging to infinity such that* $H_k(x) = s_k h(x/t_k) \in [0, 1]$ *for every* $x \in [a, b]$*, then there exists a nonempty interval* $(c, d) \subset (a, b)$ *and a subsequence* $\{H_{k_i}\}_{i\in\mathbb{N}}$ *of* $\{H_k\}_{k\in\mathbb{N}}$ *such that the sequence* $H_{k_i}|_{[c,d]}$ *converges uniformly to a function* $H\colon [c, d] \to [0, 1]$.

Moreover, if $\liminf_k H_k(a) > 0$*, then the function* H *is one-to-one.*

PROOF. First notice that the functions H_k are increasing.

Let $Q = \{q_i : i \in \mathbb{N}\}$ be a dense subset of $[a, b]$ containing a and b. The functions $H_k|_Q$ are elements of the compact metric space $[0, 1]^Q$. So, there exists a subsequence $\{H_{k_i}\}_{i\in\mathbb{N}}$ of $\{H_k\}_{k\in\mathbb{N}}$ that converges in $[0,1]^Q$; i.e., such that for every $j\in\mathbb{N}$ there exists $H(q_j)\in[0,1]$ with the property that

$$\lim_{i\to\infty} H_{k_i}(q_j)=H(q_j).$$

If $H(q_r)=0$ for some $q_r\in(a,b)$ then, by Lemma 3.3.3, the interval $[c,d]=[a,q_r]$ and a function $H=0\,\chi_{[c,d]}$ suffice. So, decreasing $[a,b]$, if necessary, we can assume that $H(a)>0$. The same is true when $\liminf_k H_k(a)>0$.

We prove that

$$P=\mathrm{cl}\,(H(Q\cap[a',b']))\subset(0,1] \tag{32}$$

is not nowhere dense for every nonempty interval $(a',b')\subset(a,b)$ such that $a',b'\in Q$. Notice that this will finish the proof, because it implies the existence of a nontrivial interval $[c,d]\subset[a,b]$, $c,d\in Q$, such that $H(Q\cap[c,d])$ is dense in $[H(c),H(d)]$. So, Lemma 3.3.3 gives the desired uniform convergence. Moreover, condition (32) guarantees that H will be one-to-one on $[c,d]$.

By way of contradiction assume condition (32) fails; i.e., that P is nowhere dense for some nonempty interval $(a',b')\subset(a,b)$ such that $a',b'\in Q$. Choosing a subsequence, if necessary, we can assume that

$$\lim_{i\to\infty} s_{k_i}/s_{k_{i+1}}=\lim_{i\to\infty} s^{-1}_{k_{i+1}}/s^{-1}_{k_i}=0.$$

Then, by Lemma 2.8.1, there exists an open set $W\supset\bigcup_{i\in\mathbb{N}} s^{-1}_{k_i}P$ such that 0 is an $\mathcal{I}$-dispersion point of W. We will construct a set V such that 0 is not an $\mathcal{I}$-dispersion point of V, while $h(V)\subset W$; i.e., $h(0)=0$ is an $\mathcal{I}$-dispersion point of $h(V)$. This contradicts the assumption that h is $\mathcal{I}$-density continuous.

So, choose a countable base $\{I_i\}_{i\in\mathbb{N}}$ of $[a',b']$ and for every $i,j\in\mathbb{N}$, $j\le i$, choose $q_{i,j},q'_{i,j}\in Q$ such that $q_{i,j}<q'_{i,j}$, $[q_{i,j},q'_{i,j}]\subset I_j$, and

$$H([q_{i,j},q'_{i,j}])\subset s_{k_i}W. \tag{33}$$

This can be done, since $P\subset s_{k_i}W$, so the distance d_i between P and the complement of $s_{k_i}W$ is positive and any interval $[q_{i,j},q'_{i,j}]$ for which $H(q'_{i,j})-H(q_{i,j})<d_i$ satisfies condition (33). Moreover, choosing a subsequence of $\{k_i\}_{i\in\mathbb{N}}$, if necessary, we can also assume that for every $i,j\in\mathbb{N}$, $j\le i$, $H_{k_i}(q_{i,j})$ and $H_{k_i}(q'_{i,j})$ are closer to $H(q_{i,j})$ than d_i. This means that

$$H_{k_i}((q_{i,j},q'_{i,j}))\subset s_{k_i}W\quad\text{for every } i,j\in\mathbb{N},\ j\le i. \tag{34}$$

Let $V_i=\bigcup_{j\le i}(q_{i,j},q'_{i,j})$ and

$$V=\bigcup_{i\in\mathbb{N}}\frac{1}{t_{k_i}}V_i.$$

Then, by (34),

$$h\left(\frac{1}{t_{k_i}}V_i\right)=\frac{1}{s_{k_i}}\left[s_{k_i}h\left(\frac{1}{t_{k_i}}V_i\right)\right]=\frac{1}{s_{k_i}}H_{k_i}(V_i)\subset\frac{1}{s_{k_i}}\left[s_{k_i}W\right]=W$$

for every $i \in \mathbb{N}$ and, indeed, $h(V) \subset W$.

On the other hand, 0 is not an $\mathcal{I}$-dispersion point of V, since for any subsequence $\{t_{k_{i_p}}\}_{p\in\mathbb{N}}$ of $\{t_{k_i}\}_{i\in\mathbb{N}}$ the set $\bigcup_{p\geq p_0} t_{k_{i_p}} V \supset \bigcup_{p\geq p_0} V_{i_p}$ is open and dense in (a', b') for every $p_0 \in \mathbb{N}$, and so,

$$(-1,1) \cap \limsup_{p\to\infty} (t_{k_{i_p}} V) \notin \mathcal{I}.$$

This finishes the proof of Lemma 3.3.4. $\square$

It is necessary in Lemma 3.3.4 to impose some condition such as $\mathcal{I}$-density continuity on the function h. To see this, let f_n be a sequence of increasing homeomorphisms from $[1/2, 1]$ onto itself which converge pointwise to a function f, which is discontinuous on a dense subset of $[1/2, 1]$. (Such a sequence is a bit tedious to construct and will be left as an exercise for the reader.) For $n \in \mathbb{N}$ and $x \in [2^{1-n}, 2^{-n}]$ let

$$h(x) = \frac{f_n(2^{n-1}x)}{2^{n-1}}.$$

Apparently, if $s_k = t_k = 2^{k-1}$ and $(a, b) = (1/2, 1)$, then the conclusion of Lemma 3.3.4 cannot be satisfied.

LEMMA 3.3.5. *Let f and g be increasing homeomorphisms such that $f(a) = g(a)$ for some $a \in \mathbb{R}$. If $g(x) \leq f(x)$ for every $x \geq a$ and f is right $\mathcal{I}$-density continuous at a then $f + g$ is also right $\mathcal{I}$-density continuous at a.*

PROOF. Without loss of generality we may assume that $a = f(a) = g(a) = 0$. Let D be an open interval set for which 0 is not an $\mathcal{I}$-dispersion point. By Lemma 2.7.1(iii) (or Corollary 2.7.4) it is enough to prove that 0 is not an $\mathcal{I}$-dispersion point of $(f + g)(D)$.

By Lemma 3.3.1, there is an increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers diverging to infinity and a nontrivial interval $[a, b] \subset (0, 1)$ such that

$$Q = \liminf_{k\to\infty} t_k D \cap (a, b) \text{ is dense in } (a, b). \tag{35}$$

Now, by Lemma 3.3.2 used for the function f, the sequence $\{t_k\}_{k\in\mathbb{N}}$ and the interval $[a, b]$, we may find a subsequence $\{t_{k_i}\}_{i\in\mathbb{N}}$ of $\{t_k\}_{k\in\mathbb{N}}$, and a nonempty interval $(c, d) \subset (a, b)$ such that

$$\lim_{i\to\infty} f(c/t_{k_i})/f(d/t_{k_i}) > 0.$$

Without loss of generality we may assume that $\{t_{k_i}\}_{i\in\mathbb{N}} = \{t_k\}_{k\in\mathbb{N}}$ and $[c, d] = [a, b]$; i.e., that

$$\lim_{k\to\infty} \frac{f(a/t_k)}{f(b/t_k)} > 0. \tag{36}$$

Let $s_k = 1/(f+g)(b/t_k)$, $F_k(x) = s_k f(x/t_k)$ and $G_k(x) = s_k g(x/t_k)$ for $x \in [0,1]$. Then,

$$(F_k + G_k)(x) = s_k(f+g)(x/t_k) \in [0,1] \quad \text{for} \quad x \in [a,b],\ k \in \mathbb{N}, \tag{37}$$

$$s_k(f+g)(b/t_k) = (F_k + G_k)(b) = 1$$

and, by condition (36) and the inequality $g(x) \le f(x)$ for $x \ge 0$,

$$\begin{aligned} \liminf_{k\to\infty} F_k(a) &= \liminf_{k\to\infty} s_k f(a/t_k) \\ &= \liminf_{k\to\infty} \frac{f(a/t_k)}{(f+g)(b/t_k)} \\ &\ge \liminf_{k\to\infty} \frac{f(a/t_k)}{2f(b/t_k)} \\ &> 0. \end{aligned} \tag{38}$$

Using Lemma 3.3.4 twice, we can find a nonempty interval $(c,d) \subset (a,b)$ and a sequence $\{k_i\}_{i\in\mathbb{N}}$ of natural numbers such that $\{F_{k_i}|_{[c,d]}\}_{i\in\mathbb{N}}$ converges uniformly to some function F and $\{G_{k_i}|_{[c,d]}\}_{i\in\mathbb{N}}$ converges uniformly to a function G. Moreover, by (38), we can also assume that F and $F+G$ are increasing homeomorphisms on $[c,d]$. Without loss of generality we may assume that $[c,d] = [a,b]$.

Let $(A,B) = ((F+G)(a),(F+G)(b)) \subset (0,1]$. By (35), the set $(F+G)(Q)$ is dense in (A,B). But if $q \in Q$ then, by (35), $q/t_k \in D$ for almost all $k \in \mathbb{N}$. So, for every sequence $\{k_i\}_{i\in\mathbb{N}}$ of natural numbers and every $j \in \mathbb{N}$,

$$(F+G)(q) = \lim_{i\to\infty} s_{k_i}(f+g)(q/t_{k_i}) \in cl\left(\bigcup_{i\ge j} s_{k_i}(f+g)(D)\right),$$

which implies that the set $\bigcup_{i\ge j} s_{k_i}(f+g)(D)$ is dense in (A,B). Thus, the G_δ set

$$(-1,1) \cap \limsup_{i\to\infty} s_{k_i}(f+g)(D) = (-1,1) \cap \bigcap_{j\in\mathbb{N}} \bigcup_{i\ge j} s_{k_i}(f+g)(D) \notin \mathcal{I}.$$

So, 0 is not an $\mathcal{I}$-dispersion point of $(f+g)(D)$. This finishes the proof of Lemma 3.3.5. □

THEOREM 3.3.6. *If f and g are increasing $\mathcal{I}$-density continuous homeomorphisms then $f+g$ is $\mathcal{I}$-density continuous.*

PROOF. Let f and g be the increasing $\mathcal{I}$-density continuous homeomorphisms and let $a \in \mathbb{R}$. It is enough to prove that $f+g$ is right $\mathcal{I}$-density continuous at a. Without loss of generality we may assume that $f(a) = g(a)$. But, by Proposition 3.2.15, $\max\{f,g\}$ is an increasing $\mathcal{I}$-density continuous homeomorphism and $\min\{f,g\} \le \max\{f,g\}$. Hence, by Lemma 3.3.5,

$$f + g = \min\{f,g\} + \max\{f,g\}$$

is right $\mathcal{I}$-density continuous at a. Theorem 3.3.6 is proved. $\square$

COROLLARY 3.3.7. *If f and g are positive and increasing $\mathcal{I}$-density continuous homeomorphisms, then their product fg is also $\mathcal{I}$-density continuous.*

PROOF. By Corollary 3.2.12, functions exp and ln are $\mathcal{I}$-density continuous. So, $\ln f$ and $\ln g$ are $\mathcal{I}$-density continuous, as composition of $\mathcal{I}$-density continuous increasing homeomorphisms is an $\mathcal{I}$-density continuous increasing homeomorphism. Thus, by Theorem 3.3.6,

$$fg = \exp(\ln f + \ln g)$$

is also $\mathcal{I}$-density continuous. $\square$

It might seem that in the above corollary the assumption that functions f and g are positive is unnecessary, since it was not used in Theorem 3.3.6. However, in general, the product of two increasing homeomorphisms does not to be a homeomorphism. This can be used to construct the following example.

EXAMPLE 3.3.8. *There are two $\mathcal{I}$-density continuous and increasing homeomorphisms f and g such that their product is not $\mathcal{I}$-density continuous. In fact, functions f and g also preserve $\mathcal{I}$-density points, while their product does not.*

PROOF. Let $g : (0,\infty) \to \mathbb{R}$ be defined by $g(x) = -1/x$. Then, by Corollary 3.2.12, g is $\mathcal{I}$-density continuous. To construct $f : (0,\infty) \to \mathbb{R}$ let $\bigcup_{k\in\mathbb{N}}(a_k, b_k)$ be a right interval set at 1 and put $c_k = 1 + 1/(k!)$. Define f on $[a_k, b_k]$ by $f(x) = c_k x$ and extend it on the remaining set in linear way. It is easy to see that f is increasing. Also, $(fg)(x) = -c_k$ for all $x \in [a_k, b_k]$. So,

$$(fg)^{-1}(\{-c_k : k \in \mathbb{N}\}) \supset \bigcup_{k\in\mathbb{N}}(a_k, b_k)$$

what easily implies that fg is not $\mathcal{I}$-density continuous. The fact that f is $\mathcal{I}$-density continuous follows immediately from Theorem 3.4.3 (used with $u_k = 1$ and $K = L = 2$.) $\square$

To prove the theorem that $f + g$ preserves $\mathcal{I}$-density points, provided f and g are increasing homeomorphisms preserving $\mathcal{I}$-density points, we need the following lemma, analogous to Lemma 3.3.4.

LEMMA 3.3.9. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism which preserves $\mathcal{I}$-density points such that $h(0) = 0$ and let $\{s_k\}_{k\in\mathbb{N}}$ and $\{t_k\}_{k\in\mathbb{N}}$ be increasing sequences of positive numbers diverging to infinity such that $H_k(x) = s_k h(x/t_k) \in [0, 1]$ for every $x \in [0, 1]$. Then for every nontrivial interval $[a, b] \subset (0, 1)$, there exists a nonempty interval $(c, d) \subset (a, b)$ and a subsequence $\{H_{k_i}\}_{i\in\mathbb{N}}$ of $\{H_k\}_{k\in\mathbb{N}}$ such that the sequence $H_{k_i}|_{[c,d]}$ converges uniformly to a function $H\colon [c, d] \to [0, 1]$.*

Moreover, if $\limsup_k(H_k(b) - H_k(a)) > 0$, then we can assume that the function H is one-to-one.

Proof. Let $Q = \{q_i : i \in \mathbb{N}\}$ be a dense subset of $[a, b]$ containing a and b. The functions $H_k|_Q$ are elements of the compact metric space $[0, 1]^Q$. So, there exists an increasing sequence $\{k_i\}_{i\in\mathbb{N}}$ of natural numbers such that $H_{k_i}|_Q$ converges in $[0, 1]^Q$; i.e., for every $j \in \mathbb{N}$ there exists $H(q_j) \in [0, 1]$ such that

$$\lim_{i\to\infty} H_{k_i}(q_j) = H(q_j).$$

Moreover, if $\limsup_k (H_k(b) - H_k(a)) > 0$, then we can also assume that

$$H(a) < H(b).$$

If $H(a) = H(b)$ then, by Lemma 3.3.3, the interval $[c, d] = [a, b]$ and the function $H = H(a)\chi_{[c,d]}$ work. So, we can assume that $H(a) < H(b)$.

By Lemma 3.3.3 in order to prove the first part of Lemma 3.3.9 it is enough to show that $H(Q)$ is dense in $[H(a), H(b)] \subset [0, 1]$. So, by way of contradiction, assume that $H(Q)$ is not dense in $[H(a), H(b)]$. Then, there exists a nonempty interval $(A, B) \subset [H(a), H(b)]$ such that $H(Q) \cap [A, B] = \emptyset$, and we can find $a_i, b_i \in Q$, $0 < b_i - a_i < 1/i$, such that $H(a_i) < A < B < H(b_i)$ for every $i \in \mathbb{N}$. Now, taking a subsequence of $\{k_i\}_{i\in\mathbb{N}}$, if necessary, we can conclude that

$$s_{k_i} h(a_i/t_{k_i}) = H_{k_i}(a_i) < A < B < H_{k_i}(b_i) = s_{k_i} h(b_i/t_{k_i})$$

for every $i \in \mathbb{N}$.

Let $U = \bigcup_{i\in\mathbb{N}} t_{k_i}^{-1}(a_i, b_i)$. Then, by Lemma 2.1.4, 0 is an $\mathcal{I}$-dispersion point of U. But,

$$[A, B] \subset (H_{k_i}(a_i), H_{k_i}(b_i)) = s_{k_i} h\left(t_{k_i}^{-1}(a_i, b_i)\right) \subset s_{k_i} h(U)$$

for every $i \in \mathbb{N}$. So, by Theorem 2.2.2(viii), 0 is not $\mathcal{I}$-dispersion point of $h(U)$. This contradicts the assumption that h preserves $\mathcal{I}$-density points.

To prove the additional condition let us assume, by way of contradiction, that H is not one-to-one on any nonempty interval $(c, d) \subset (a, b)$. Then, the set

$$U = \bigcup\{(c, d) \subset (a, b) : H(c) = H(d)\}$$

is dense in (a, b) and the set $H(U)$ is countable. In particular, the set $P = [a, b]\setminus U$ is nowhere dense in $[a, b]$, while $H(P)$ is dense in $[H(a), H(b)]$. We will show that this implies h does not preserve right $\mathcal{I}$-density at 0.

Choosing a subsequence of $\{k_i\}_{i\in\mathbb{N}}$, if necessary, we may assume that

$$\lim_{i\to\infty} t_{k_i}/t_{k_{i+1}} = \lim_{i\to\infty} t_{k_{i+1}}^{-1}/t_{k_i}^{-1} = 0.$$

Then, by Lemma 2.8.1, there exists an open set $V \supset \bigcup_{i\in\mathbb{N}} t_{k_i}^{-1} P$ such that 0 is an $\mathcal{I}$-dispersion point of V. We will show that 0 is not an $\mathcal{I}$-dispersion point of the open set $h(V)$.

So, let $\{k_p\}_{p\in\mathbb{N}}$ be an arbitrary subsequence of $\{k_i\}_{i\in\mathbb{N}}$. Then, for every $x \in P$,

$$H(x) = \lim_{p\to\infty} H_{k_p}(x) = \lim_{p\to\infty} s_{k_p} h(x/t_{k_p}) \in \mathrm{cl}\left(\bigcup_{r\geq p} s_{k_r} h(V)\right),$$

which implies that the set $\bigcup_{r\geq p} s_{k_r} h(V)$ is dense in $[H(a), H(b)]$ for every $p \in \mathbb{N}$. Thus, the G_δ set

$$(0,1) \cap \limsup_{p\to\infty} s_{k_p} h(V) = (0,1) \cap \bigcap_{r\in\mathbb{N}} \bigcup_{p\geq r} s_{k_p} h(V) \notin \mathcal{I},$$

because it is dense in $(H(a), H(b)) \neq \emptyset$. Now, by Theorem 2.2.2(iii), 0 is not an $\mathcal{I}$-dispersion point of $h(V)$. This finishes the proof of Lemma 3.3.9. □

THEOREM 3.3.10. *If f and g are increasing homeomorphisms which preserve $\mathcal{I}$-density points, then $f+g$ also preserves $\mathcal{I}$-density points.*

PROOF. Let f and g be increasing homeomorphisms which preserve $\mathcal{I}$-density points and let $a \in \mathbb{R}$. It is enough to prove that $f+g$ preserves right $\mathcal{I}$-density at a. Without loss of generality we may assume that $a = f(a) = g(a) = 0$. Suppose E is a right interval set such that 0 is a right $\mathcal{I}$-density point of E. Let $\{s_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers diverging to infinity and let $0 < A < B < 1$. By Theorems 2.2.2(viii) and 3.2.1 and Corollary 2.7.4, it suffices to prove that there exists a subsequence $\{s_{k_i}\}_{i\in\mathbb{N}}$ of $\{s_k\}_{k\in\mathbb{N}}$ and a nonempty open interval $J \subset (A, B)$ such that

$$J \subset s_{k_i}(f+g)(E)$$

for every $i \in \mathbb{N}$.

Define

$$t_k = 1/(f+g)^{-1}(B/s_k), \quad a_k = 1/(f+g)^{-1}(A/s_k),$$
$$F_k(x) = s_k f(x/t_k) \quad \text{and} \quad G_k(x) = s_k g(x/t_k)$$

and

$$H_k(x) = (F_k + G_k)(x) = s_k(f+g)(x/t_k).$$

Then, $t_k < a_k$, $A = s_k(f+g)(1/a_k)$ and $B = s_k(f+g)(1/t_k)$. In particular,

$$H_k\left(\left[\frac{t_k}{a_k}, 1\right]\right) = s_k(f+g)\left(\frac{1}{t_k}\left[\frac{t_k}{a_k}, 1\right]\right) = [A, B].$$

Let $\{k_i\}_{i\in\mathbb{N}}$ be a sequence of natural numbers such that the following limits exist

$$a = \lim_{i\to\infty} \frac{t_{k_i}}{a_{k_i}} \in [0,1],$$
$$F(a) = \lim_{i\to\infty} F_{k_i}(a) \quad \text{and} \quad G(a) = \lim_{i\to\infty} G_{k_i}(a).$$

We will show that

$$(F+G)(a) = A. \tag{39}$$

By way of contradiction, assume that this is not the case. We will assume that

$$s_{k_i}(f+g)(1/a_{k_i}) = A < (F+G)(a) = \lim_{i\to\infty} s_{k_i}(f+g)(a/t_{k_i}).$$

The other inequality is similar. Let $A < C < (F+G)(a)$. Then,

$$s_{k_i}(f+g)(1/a_{k_i}) = A < C < s_{k_i}(f+g)(a/t_{k_i}) \tag{40}$$

for almost all $i \in \mathbb{N}$. Assume that (40) is true for all $i \in \mathbb{N}$. Then,

$$\frac{f(1/a_{k_i}) + g(1/a_{k_i})}{f(a/t_{k_i}) + g(a/t_{k_i})} = \frac{s_{k_i}(f+g)(1/a_{k_i})}{s_{k_i}(f+g)(a/t_{k_i})} < \frac{A}{C} < 1.$$

Hence, for every $i \in \mathbb{N}$, either

$$\frac{f(1/a_{k_i})}{f(a/t_{k_i}))} \le \frac{A}{C} \quad \text{or} \quad \frac{g(1/a_{k_i})}{g(a/t_{k_i}))} \le \frac{A}{C}.$$

Without loss of generality, passing to a subsequence, if necessary, we can assume that for all $n \in \mathbb{N}$

$$\frac{f\left(\frac{1}{t_{k_i}}\frac{t_{k_i}}{a_{k_i}}\right)}{f\left(\frac{1}{t_{k_i}}a\right)} = \frac{f(1/a_{k_i})}{f(a/t_{k_i})} \le \frac{A}{C} < 1.$$

Let $u_{k_i} = f\left(\frac{1}{t_{k_i}}a\right)$. Then

$$u_{k_i}^{-1} f\left(\frac{1}{t_{k_i}}\frac{t_{k_i}}{a_{k_i}}\right) \le \frac{A}{C} < 1 = u_{k_i}^{-1} f\left(\frac{1}{t_{k_i}}a\right);$$

i.e.,

$$\left(\frac{A}{C}, 1\right) \subset u_{k_i}^{-1} f\left(\frac{1}{t_{k_i}}\left(\frac{t_{k_i}}{a_{k_i}}, a\right)\right) \tag{41}$$

for every $i \in \mathbb{N}$. But, choosing a subsequence, if necessary, we can assume that

$$\lim_{i\to\infty} t_{k_{i+1}}^{-1}/t_{k_i}^{-1} = 0$$

and hence, by Lemma 2.1.4, 0 is an $\mathcal{I}$-dispersion point of

$$D = \bigcup_{i\in\mathbb{N}} \frac{1}{t_{k_i}}\left(\frac{t_{k_i}}{a_{k_i}}, a\right).$$

On the other hand, by (41),

$$\left(\frac{A}{C}, 1\right) \subset u_{k_i}^{-1} f(D)$$

for every $i \in \mathbb{N}$; i.e., 0 is not an $\mathcal{I}$-dispersion point of $f(D)$. This contradicts the assumption that f preserves $\mathcal{I}$-density points. Condition (39) is proved.

Notice that condition (39) implies, in particular, that $a < 1$, since

$$\lim_{i\to\infty}(F_{k_i} + G_{k_i})(1) = B > A = \lim_{i\to\infty}(F_{k_i} + G_{k_i})(a).$$

Using Lemma 3.3.9 twice for the functions F_{k_i} and G_{k_i}, and passing to a subsequence, if necessary, we can find a nontrivial interval $[c,d] \subset (a,1)$ such that $\{F_{k_i}|_{[c,d]}\}_{i\in\mathbb{N}}$ converges uniformly to some F and $\{G_{k_i}|_{[c,d]}\}_{i\in\mathbb{N}}$ converges uniformly to a function G. Notice also that condition (39) implies that either $\limsup_{k\to\infty}(F_k(1) - F_k(a)) > 0$ or $\limsup_{k\to\infty}(G_k(1) - G_k(a)) > 0$, since

$$\begin{aligned} \limsup_{k\to\infty}(F_k(1) - F_k(a)) + (G_k(1) - G_k(a)) &= H(1) - H(a) \\ &= B - A > 0. \end{aligned}$$

Thus, we can also assume that the function $H = F + G$ is a homeomorphism on $[c,d]$.

By Theorem 2.2.2(viii), choosing a subsequence of $\{k_i\}_{i\in\mathbb{N}}$ and a subinterval of (c,d), if necessary, we may also assume that

$$(c,d) \subset t_{k_i}E \quad \text{for every } i \in \mathbb{N}.$$

This implies

$$\begin{aligned} ((F_{k_i} + G_{k_i})(c), (F_{k_i} + G_{k_i})(d)) &= (F_{k_i} + G_{k_i})((c,d)) \\ &\subset (F_{k_i} + G_{k_i})(t_{k_i}E) \\ &= s_{k_i}(f+g)\left(\frac{1}{t_{k_i}}(t_{k_i}E)\right) \\ &= s_{k_i}(f+g)(E). \end{aligned}$$

Now, if $c < c' < d' < d$, then

$$A \le (F+G)(c) < (F+G)(c') < (F+G)(d') < (F+G)(d) \le B$$

so,

$$J = ((F+G)(c'), (F+G)(d')) \subset (A,B)$$

and

$$J \subset ((F_{k_i} + G_{k_i})(c), (F_{k_i} + G_{k_i})(d)) \subset s_{k_i}(f+g)(E)$$

for $i's$ large enough, since $\{F_{k_i} + G_{k_i}\}$ converges to $F+G$. Thus, we may assume that

$$J \subset s_{k_i}(f+g)(E) \cap (A,B)$$

for every $i \in \mathbb{N}$. This finishes the proof of Theorem 3.3.10. □

Corollary 3.3.11. *If f and g are positive and increasing homeomorphisms preserving $\mathcal{I}$-density points then their product fg also preserves $\mathcal{I}$-density points.*

Proof. See the proof of Corollary 3.3.7. □

3.4. More $\mathcal{I}$-density Continuous Functions

In this section we show that functions from several classes are $\mathcal{I}$-density continuous.

The next theorem is the main step in the proof that the analytic functions are $\mathcal{I}$-density continuous. However, it is of independent interest.

THEOREM 3.4.1. *The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^\alpha$, is $\mathcal{I}$-density continuous for any $\alpha \in \mathbb{R}$.*

PROOF. By Corollary 3.2.12 it suffices to prove that, for $\alpha > 0$, f is $\mathcal{I}$-density continuous from the right at 0. For those values of α for which x^α is defined on $(-\infty, 0)$, the left-hand argument is similar.

Let $B \in \mathcal{B}$ and let 0 be an $\mathcal{I}$-dispersion point of B. By Corollary 2.7.4 and Theorem 3.2.1 we may assume that B is a regular open interval set. We will show that $f^{-1}(B)$ satisfies condition (viii) of Theorem 2.2.2. Let $\{n_k\}_{k\in\mathbb{N}}$ be an arbitrary increasing sequence of natural numbers and let $(a, b) \subset (0, 1)$ be a nonempty interval. We will find a nonempty interval $(c, d) \subset (a, b)$ and a subsequence $\{n_{k_m}\}_{m\in\mathbb{N}}$ such that

$$(c,d) \cap \left(n_{k_m} f^{-1}(B)\right) = \emptyset. \tag{42}$$

For every $t > 0$,

$$tf^{-1}(x) = tx^{\frac{1}{\alpha}} = \left(\left(\frac{1}{\frac{1}{t}}\right)^{\alpha} x\right)^{\frac{1}{\alpha}} = f^{-1}\left(\frac{1}{f\left(\frac{1}{t}\right)}x\right).$$

This means that (42) is equivalent to

$$(c,d) \cap f^{-1}\left(\frac{1}{f\left(\frac{1}{n_k}\right)}B\right) = \emptyset,$$

which in turn is equivalent to

$$f(c,d) \cap \left(\frac{1}{f\left(\frac{1}{n_k}\right)}B\right) = \emptyset. \tag{43}$$

Let $(a', b') = f((a, b))$ and

$$t_{n_k} = \frac{1}{f\left(\frac{1}{n_k}\right)}.$$

Then $\{t_{n_k}\}_{k\in\mathbb{N}}$ is an increasing sequence diverging to infinity. Since 0 is an $\mathcal{I}$-dispersion point of B, we can use condition (ix) of Theorem 2.2.2 with (a', b') and $\{t_{n_k}\}$ to find a nonempty subinterval (c', d') of (a', b') and a subsequence $\{t_{n_{k_i}}\}$ of $\{t_{n_k}\}$ such that

$$(c', d') \cap \left(t_{n_{k_i}} B\right) = \emptyset.$$

Let

$$(c,d) = f^{-1}(c',d') \subset f^{-1}(a',b') = (a,b).$$

Then (43) is satisfied. This finishes the proof of Theorem 3.4.1. $\square$

THEOREM 3.4.2. *Analytic functions are $\mathcal{I}$-density continuous; i.e.,*

$$\mathcal{A} \subset \mathcal{C}_{\mathcal{I}\mathcal{I}}.$$

PROOF. Let $h \in \mathcal{A}$. It is enough to prove that h is $\mathcal{I}$-density continuous at 0. We prove that h is right $\mathcal{I}$-density continuous at 0. As usual, the left-hand argument is similar.

Let $h(x) = \sum_{n=0}^{\infty} a_n x^n$. We can assume that $a_0 = 0$. Moreover, by Proposition 3.1.7 and the fact that the class $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ is closed under composition, we can also assume that for $i = \min\{n\colon a_n \neq 0\}$ we have $a_i = 1$. Now let $f(x) = x^i$. Because h is analytic, h^{-1} exists on some right neighborhood of 0. Assume that h^{-1} is positive on this neighborhood, the other case being similar. Then

$$\begin{aligned} 1 = \lim_{x\to 0^+} \frac{h(x)}{x^i} &= \lim_{x\to 0^+} \frac{h(h^{-1}(x))}{(h^{-1}(x))^i} \\ &= \lim_{x\to 0^+} \left(\frac{x^{1/i}}{h^{-1}(x)}\right)^i \\ &= \left(\lim_{x\to 0^+} \frac{f^{-1}(x)}{h^{-1}(x)}\right)^i. \end{aligned}$$

Hence,

$$\lim_{x\to 0^+} \frac{h^{-1}(x)}{f^{-1}(x)} = 1$$

and, by Theorems 3.4.1 and 3.2.11, h is $\mathcal{I}$-density continuous at 0. $\square$

The last theorem of this section is the following.

THEOREM 3.4.3. *Let 0 be a right $\mathcal{I}$-density point of a right interval set $E = \bigcup_{k\in\mathbb{N}}(a_k,b_k) \subset [0,1]$ and let $h\colon [-1,1] \to [-1,1]$ be such that the restricted function $h|_{(a_k,b_k)}$ is a homeomorphism for every $k \in \mathbb{N}$. Moreover, let us assume that there exists a nondecreasing sequence $\{u_k\}_{k\in\mathbb{N}}$ of positive numbers and constants $K, L > 0$ such that for every $k \in \mathbb{N}$ the functions*

$$h_k = u_k h|_{\{0\}\cup(a_k,b_k)} : \{0\} \cup (a_k,b_k) \to [-K,K]$$

and $\left[h_k|_{(a_k,b_k)}\right]^{-1}$ satisfy the Lipschitz condition with constant L. Then h is right $\mathcal{I}$-density continuous at 0.

Before proving the theorem we state some immediate corollaries. First, recall that a real function f satisfies a *local Lipschitz condition*, if for every point a from the domain of f there is an open neighborhood U of a and a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for every $x, y \in U$. The next corollary is a generalization of Corollary 3.2.12.

Corollary 3.4.4. *If h is a homeomorphism such that h and h^{-1} fulfill a local Lipschitz condition, then h and h^{-1} are $\mathcal{I}$-density continuous.*

Proof. To see that h is right $\mathcal{I}$-density continuous at 0, use Theorem 3.4.3 with an arbitrary right interval set E having 0 as a right $\mathcal{I}$-density point and with $u_k = 1$ for all k. The other cases can easily be reduced to this one. □

The following corollary is also needed. It can be considered an $\mathcal{I}$-density analogue of Theorem 1.4.3. (Also compare it with Corollary 3.2.7.)

Corollary 3.4.5. *Suppose 0 is a right $\mathcal{I}$-density point of the right interval set $\bigcup_{k\in\mathbb{N}}(a_k, b_k)$ and let $h\colon [-1,1] \to [-1,1]$ be a convex function such that the restricted functions $h|_{(a_k,b_k)}$ are linear homeomorphisms for every $k \in \mathbb{N}$. Then h is right $\mathcal{I}$-density continuous at 0.*

Proof. Let $c_k = h'((a_k + b_k)/2)$ for $k \in \mathbb{N}$. Then, by the convexity of h, the sequence $\{c_k\}$ is nonincreasing. Let $c = \inf_{k\in\mathbb{N}} c_k$. Notice that $c \neq -\infty$, because this would contradict the convexity of h on $(-1, 1)$. If $c \neq 0$, then h and h^{-1} satisfy a Lipschitz condition on a right neighborhood of 0 and, by Corollary 3.4.4, this implies that h is $\mathcal{I}$-density continuous at 0. If $c = 0$, we may also assume that $h(0) = 0$. Then, evidently, the sequence $u_k = c_k^{-1}$ is nondecreasing and the derivative of $u_k h$ on (a_k, b_k) equals 1. Moreover, the convexity of h implies that $h_k = u_k h|_{\{0\}\cup(a_k,b_k)}\colon \{0\} \cup (a_k, b_k) \to [0, 1]$. An application of Theorem 3.4.3 finishes the proof. □

Notice that the assumption that h is convex in Corollary 3.4.5 is essential. To see this, let 0 be an $\mathcal{I}$-dispersion point of the right interval set $E = \bigcup_{k\in\mathbb{N}}[a_k, b_k] \subset (0, 1)$ and let $(0, 1] \setminus E = \bigcup_{k\in\mathbb{N}}(c_k, d_k)$. The homeomorphism $h\colon [0, 1] \to [0, 1]$ transforming $[c_k, d_k]$ onto $[a_k, b_k]$ in a linear way is not right $\mathcal{I}$-density continuous, even though it satisfies all the assumptions of Corollary 3.4.5, except convexity.

Proof of Theorem 3.4.3. Let h be as in the assumptions. Replacing $h(x)$ by $(h(x) - h(0))/K$, if necessary, it may be assumed that $h(0) = 0$ and $K = 1$.

For $k, n \in \mathbb{N}$, let

$$h_{k,n}(x) = n\, h_k\left(\frac{1}{n}x\right) = n\, u_k\, h\left(\frac{1}{n}x\right) \text{ with domain } \{0\} \cup (na_k, nb_k).$$

Then, the functions $h_{k,n}$ and $\left[h_{k,n}|_{(na_k,nb_k)}\right]^{-1}$ also satisfy a Lipschitz condition with L as the constant.

To prove that h is right $\mathcal{I}$-density continuous at 0, let $B \in \mathcal{B}$, $0 \notin B$, be such that 0 is an $\mathcal{I}$-dispersion point of B. It suffices to prove that 0 is a right $\mathcal{I}$-dispersion point of $D = h^{-1}(B) \cap E$. Notice that $\tilde{D} = h^{-1}(\tilde{B}) \cap E$. Thus, it may as well be assumed from the beginning that B is regular open and so is D.

It will be shown that 0 is an $\mathcal{I}$-dispersion point of $h^{-1}(B) \cap E$ by using Theorem 2.2.2(viii).

So, let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of integers and let $(a,b)\subset[-1,1]$ be a nonempty interval. We must find a nonempty interval $(c,d)\subset(a,b)$ and a subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$ such that for every $p\in\mathbb{N}$

$$(c,d)\cap n_{k_p}\left(h^{-1}(B)\cap E\right)=\emptyset. \tag{44}$$

Evidently, we may now assume that $(a,b)\subset(0,1)$, since otherwise condition (44) is obviously satisfied. Moreover, by the fact that 0 is a right $\mathcal{I}$-density point of E, we may also assume, choosing a subsequence of $\{n_k\}_{k\in\mathbb{N}}$ and a subinterval of (a,b), if necessary, that $(a,b)\subset n_kE$; i.e., that for every $k\in\mathbb{N}$ there exists an m_k such that $(a,b)\subset n_k(a_{m_k},b_{m_k})$. In particular, (a,b) is a subset of the domain of every h_{m_k,n_k} and $m_k\le m_{k+1}$ for every $k\in\mathbb{N}$.

But the functions h_{m_k,n_k} and $\left[h_{m_k,n_k}|_{(n_ka_{m_k},n_kb_{m_k})}\right]^{-1}$ satisfy a Lipschitz condition with constant L. So, the intervals $h_{m_k,n_k}((a,b))\subset h_{m_k,n_k}(\{0\}\cup(a,b))\subset[-L,L]$ must have length at least $(b-a)/L$ and, using Proposition 2.4.4, we may also assume, passing to a subsequence, if necessary, that for some nonempty interval $(a',b')\subset[-L,L]$ and every $k\in\mathbb{N}$

$$(a',b')\subset h_{m_k,n_k}((a,b)).$$

Now, according to Theorem 2.2.2(ix) used with B, (a',b') and the divergent sequence $\{u_{m_k}n_k\}_{k\in\mathbb{N}}$, it may be assumed, by passing to a subsequence, if necessary, that for some nonempty interval $(c',d')\subset(a',b')$ and every $k\in\mathbb{N}$

$$(c',d')\cap u_{m_k}n_kB=\emptyset.$$

This is equivalent to

$$n_kh^{-1}\left(\frac{1}{u_{m_k}n_k}(c',d')\right)\cap n_kh^{-1}(B)=\emptyset.$$

But, for x in the domain of $h^{-1}_{m_k,n_k}$, we have

$$n_kh^{-1}\left(\frac{1}{u_{m_k}n_k}x\right)=h^{-1}_{m_k,n_k}(x).$$

Thus, the above is equivalent to

$$h^{-1}_{m_k,n_k}((c',d'))\cap n_kh^{-1}(B)=\emptyset.$$

But,

$$h^{-1}_{m_k,n_k}((c',d'))\subset h^{-1}_{m_k,n_k}((a',b'))\subset(a,b)$$

for every $k\in\mathbb{N}$. Thus, using once again Proposition 2.4.4 and the fact that the functions h_{m_k,n_k} satisfy a Lipschitz condition with the same constant L, we may choose an increasing sequence $\{k_p\}_{p\in\mathbb{N}}$ of natural numbers and a nonempty interval (c,d) such that for every $p\in\mathbb{N}$

$$(c,d)\subset h^{-1}_{m_{k_p},n_{k_p}}((c',d'))\subset(a,b)\subset[-1,1].$$

The last three conditions easily imply (44). The proof of Theorem 3.4.3 is finished. □

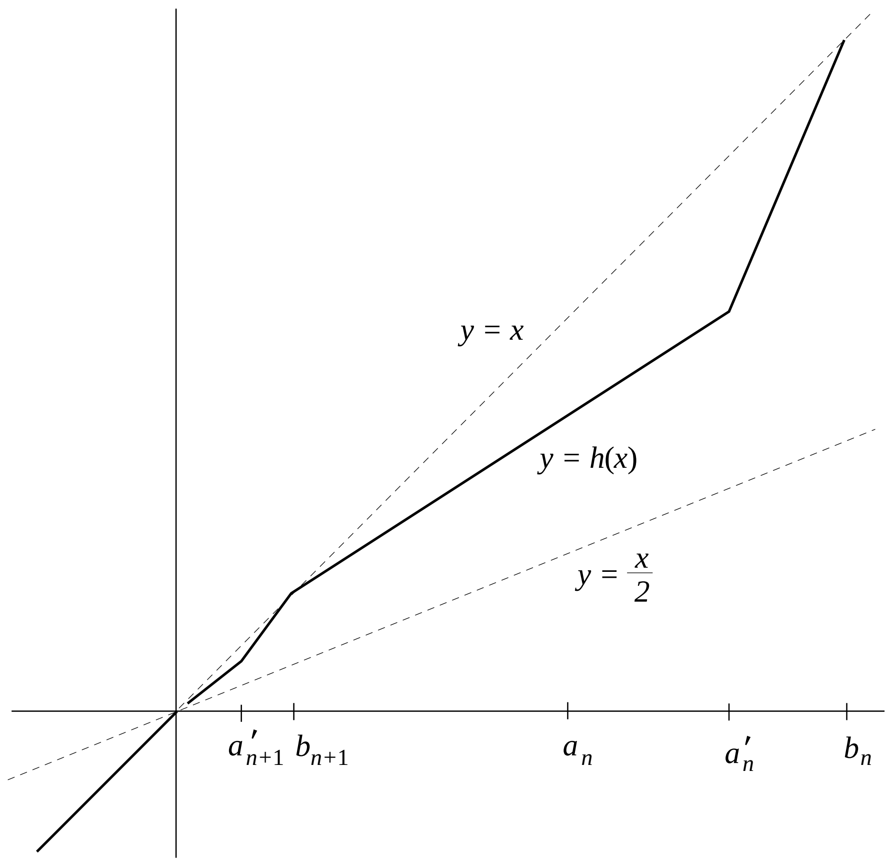

FIGURE 3.1. Homeomorphism h from Example 3.4.6.

The last example of this section shows that there are $\mathcal{I}$-density continuous homeomorphisms which do not preserve $\mathcal{I}$-density points. It also shows that in Corollary 3.4.4 the local Lipschitz condition cannot be replaced by the analogous property at each point.

EXAMPLE 3.4.6. *There is a homeomorphism* $h \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$ *which does not preserve* $\mathcal{I}$*-density points at* 0. *Moreover, for every* $x \in \mathbb{R}$

$$\frac{1}{2}|x| \leq |h(x) - h(0)| \leq |x|.$$

PROOF. For $n \in \mathbb{N}$, $n \geq 2$, let $b_n = \frac{1}{n!}$, $a_n = \frac{1}{2}(b_{n+1} + b_n)$, and $a'_n = b_n - \frac{1}{(n+1)!}$. Define the function $h\colon \mathbb{R} \to \mathbb{R}$ by putting $h(x) = x$ for $x \leq 0$ and $x \geq \frac{1}{2}$, $h(b_n) = b_n$, $h(a'_n) = a_n$ and let h be linear on every interval $[a'_n, b_n]$ and $[b_{n+1}, a'_n]$. (See Figure 3.1.)

It is easy to see that h satisfies the desired inequality. Moreover, by Lemma 2.1.4, 0 is an $\mathcal{I}$-dispersion point of $E = \bigcup_{n\in\mathbb{N}}(a'_n, b_n)$ while, by Lemma 2.2.6, 0 is not an $\mathcal{I}$-dispersion point of $h(E) = \bigcup_{n\in\mathbb{N}}(a_n, b_n)$. Thus, h does not preserve $\mathcal{I}$-density points at 0.

The fact that h is $\mathcal{I}$-density continuous follows immediately from Corollary 3.1.8 and Theorem 3.4.3 used with $K = u_k = 1$ and $L = 2$. □

3.5. $\mathcal{I}$-density Continuous Functions are Baire*1

To prove the theorem of the title we need the following definition and lemma [**33**, Lemma 29.1].

A *partition* of a set E is a pairwise disjoint family $\Pi = \{E_i \colon i \in \Lambda\}$ such that $\bigcup_{i\in\Lambda} E_i = E$. Note that with any partition Π we can associate a function $F\colon E \to \Lambda$ such that $F(x) = F(y)$ if, and only if, x and y belong to the same $E_i \in \Pi$. Conversely, any function $F\colon E \to \Lambda$ determines a partition of E.

For a set A and $n \in \mathbb{N}$ define

$$[A]^n = \{B \subset A\colon\ \mathrm{card}(B) = n\}\,.$$

If $\Pi = \{E_i \colon i \in \Lambda\}$ is a partition of $[A]^n$, then a set $H \subset A$ is *homogeneous* for the partition Π if, for some $i \in \Lambda$, $[H]^n \subset E_i$. That is, all n-element subsets of H are in the same piece of the partition Π.

LEMMA 3.5.1. ***(Ramsey's Theorem)*** *If $n, k \in \mathbb{N}$, then every finite partition $\Pi = \{E_1, E_2, \ldots, E_k\}$ of $[\mathbb{N}]^n$ has an infinite homogeneous set. In other words, for every $F\colon [\mathbb{N}]^n \to \{1, 2, \ldots, k\}$ there exists an infinite $H \subset \mathbb{N}$ such that F is constant on $[H]^n$.*

The following theorem is an analogue of Theorem 1.4.2.

THEOREM 3.5.2. *If $f\colon \mathbb{R} \to \mathbb{R}$ is deep-$\mathcal{I}$-density continuous, then f belongs to Baire*1.*

It will be convenient to extract the main step of the proof of the previous theorem in the form of the following lemma.

LEMMA 3.5.3. *Let $z \in \mathbb{R}$ and let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of open intervals and $\{J_n\}_{n\in\mathbb{N}}$ a sequence of compact intervals with $I_n \subset J_n$ and I_n centered in J_n such that for all $i > 1$*

- **(A):** *$J_{i-1} \cap J_i = \emptyset$;*
- **(B):** *$m(J_i) \leq \frac{1}{3}\min\{dist(J_k, J_{k+1})\colon k \in \mathbb{N}, k < i-1\}$; and,*
- **(C):** *$0 < m(I_i) < 2^{-i} m(J_i)$.*

Then, there exists a sequence $\{n_i\}_{i\in\mathbb{N}}$ *such that* z *is a deep-*$\mathcal{I}$*-dispersion point of* $\bigcup_{i\in\mathbb{N}} I_{n_i}$.

Proof. Without loss of generality we may assume that $z = 0$.

We will consider two cases.

Case 1^o. There exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that the sets $\{J_{n_i}\}$ are pairwise disjoint.

If $\sup_{i\in\mathbb{N}}\{\max J_{n_i}\} = \infty$ or $\inf_{i\in\mathbb{N}}\{\max J_{n_i}\} = -\infty$, then we can choose a subsequence of $\{J_{n_i}\}$ tending monotonically to ∞ or $-\infty$. In this case it is clear that this subsequence satisfies the conclusion of the lemma. Otherwise, by taking a subsequence of $\{n_i\}_{i\in\mathbb{N}}$, if necessary, we may assume that

$$\bigcup_{i\in\mathbb{N}} J_{n_i}$$

is either a right or left interval set at some point $a \in \mathbb{R}$. For simplicity, we assume it is a right interval set.

Let $J_{n_i} = [c_i, d_i]$ and $I_{n_i} = (\alpha_i, \beta_i)$. We have

$$a < d_{i-1} < c_i < \alpha_i < \beta_i < d_i$$

for all i. If $a \neq 0 = z$ then the choice of $\bigcup_{i\in\mathbb{N}} I_{n_i}$ is obvious. So, assume that $a = 0 = z$. Condition (C) states that

$$\frac{\beta_i - \alpha_i}{d_i - c_i} = \frac{\mathrm{m}(I_{n_i})}{\mathrm{m}(J_{n_i})} < \frac{1}{2^{n_i}}.$$

Let z_n be the common center of I_n and J_n, for $n \geq 0$. We have

$$\lim_{i\to\infty} \frac{\beta_i - \alpha_i}{\beta_i} \leq \lim_{i\to\infty} \frac{\beta_i - \alpha_i}{z_{n_i}} \leq \lim_{i\to\infty} \frac{\beta_i - \alpha_i}{z_{n_i} - c_i} = 2 \lim_{i\to\infty} \frac{\beta_i - \alpha_i}{d_i - c_i} = 0.$$

By Lemma 2.1.7 we can choose a subsequence of $\{n_i\}_{i\in\mathbb{N}}$ with the property that 0 is an $\mathcal{I}$-dispersion point of the right interval set $\bigcup_{i\in\mathbb{N}} I_{n_i}$. Therefore, by Corollary 2.7.2, z is a deep-$\mathcal{I}$-dispersion point of $\bigcup_{i\in\mathbb{N}} I_{n_i}$.

Case 2^o. Assume there is no pairwise disjoint subsequence $\{J_{n_i}\}_{i\in\mathbb{N}}$ of the sequence $\{J_n\}_{n\in\mathbb{N}}$.

First consider the subsequence $\{J_{2n+1}\}_{n\in\mathbb{N}}$, indexed by the odd numbers, of the sequence $\{J_n\}_{n\in\mathbb{N}}$. Define a partition function $F\colon [\mathbb{N}]^2 \to \{0,1\}$ by

$$F(\{n,m\}) = 1 \quad \text{if, and only if,} \quad J_{2n+1} \cap J_{2m+1} \neq \emptyset.$$

By Lemma 3.5.1 (Ramsey's Theorem) there exists an infinite homogeneous subset $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$; i.e., a sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that there exists a $k \in \{0,1\}$ with $F(\{n_i, n_j\}) = k$ for all positive integers $i \neq j$. But $k = 0$ would contradict the definition of case 2^o, which is currently being considered. Thus $k = 1$. So,

$$J_{2n_i+1} \cap J_{2n_j+1} \neq \emptyset \tag{45}$$

for all nonnegative integers $i \neq j$.

Now we repeat the Ramsey-type argument given above for the even-numbered counterparts of $\{J_{2n_i+1}\}_{i\in\mathbb{N}}$. Define $G\colon [\mathbb{N}]^2 \to \{0,1\}$ by

$$G(\{i,j\}) = 1 \quad \text{if, and only if,} \quad J_{2n_i} \cap J_{2n_j} \neq \emptyset.$$

Lemma 3.5.1 (Ramsey's Theorem) gives the existence of a subsequence $\{n_{i_s}\}_{s\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$ such that

$$J_{2n_{i_s}} \cap J_{2n_{i_t}} \neq \emptyset \tag{46}$$

for all nonnegative integers $s \neq t$, while the condition (45) is still preserved, or more precisely

$$J_{2n_{i_s}+1} \cap J_{2n_{i_t}+1} \neq \emptyset \tag{47}$$

for $s \neq t$. Define $\varepsilon = \operatorname{dist}\left(J_{2n_{i_0}}, J_{2n_{i_0}+1}\right)$. By (A), $\varepsilon > 0$. Moreover, by (B), (46) and (47)

$$S_0 = \bigcup_{s\geq 0} I_{2n_{i_s}} \subset \bigcup_{s\in\mathbb{N}} J_{2n_{i_s}} \subset \left\{x\colon \operatorname{dist}\left(x, J_{2n_{i_0}}\right) < \frac{\varepsilon}{3}\right\}$$

and

$$S_1 = \bigcup_{s\geq 0} I_{2n_{i_s}+1} \subset \bigcup_{s\in\mathbb{N}} J_{2n_{i_s}+1} \subset \left\{x\colon \operatorname{dist}\left(x, J_{2n_{i_0}+1}\right) < \frac{\varepsilon}{3}\right\}.$$

Hence

$$\operatorname{dist}(S_0, S_1) \geq \frac{\varepsilon}{3} > 0,$$

which implies that either $\operatorname{dist}(z, S_0) > 0$ or $\operatorname{dist}(z, S_1) > 0$. This clearly means that z is a deep-$\mathcal{I}$-dispersion point of either S_0 or S_1.

This finishes the proof of Lemma 3.5.3. □

We now begin the proof of Theorem 3.5.2. Assume to the contrary that for some perfect set P the set

$$Z = \{x \in P\colon f|_P \text{ is not continuous at } x\}$$

is dense in P.

We will construct sequences: $\{x_n\}_{n\in\mathbb{N}}$ of points of P, $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ of open intervals, $\{J_n\}_{n\in\mathbb{N}}$ of compact intervals, and $\{I_n\}_{n\in\mathbb{N}}$ of open intervals having the same midpoint as the corresponding J_n, and contained in that corresponding J_n. The construction is inductive, and aimed at having all the objects obtained satisfy the conditions (a) through (f) listed below.

For the reminder of this proof let $\widetilde{f^{-1}}(A)$ stand for $\tilde{B}$, where $B = f^{-1}(A)$.

We start by choosing $x_0 \in Z$, $(a_0, b_0) = (x_0 - 1, x_0 + 1)$ and $I_0 = J_0 = \emptyset$. Assume that for all $n \in \mathbb{N}$ and all $i \in \mathbb{N}$, $i \leq n$, we have:

(a): $f(x_i) \in I_i$;

(b): $J_{i-1} \cap J_i = \emptyset$ and, for $i > 2$,

$$\mathrm{m}(J_i) \leq \frac{1}{3} \min\{\mathrm{dist}(J_k, J_{k+1})\colon k \in \mathbb{N}, k < i-1\};$$

(c): $\mathrm{m}(J_i) < \omega(f|_P, x_i)$ and $0 < \mathrm{m}(I_i) < 2^{-i}\mathrm{m}(J_i)$;
(d): $x_i \in (a_i, b_i) \cap Z \subset [a_i, b_i] \subset (a_{i-1}, b_{i-1})$ and $|b_i - a_i| < 2^{-i}$;
(e): for every $k \in \mathbb{N}$, $2^i \leq k < 4^i$,

$$\left(\frac{1}{b_i - x_i}\left(\widetilde{f^{-1}}(I_i) - x_i\right)\right) \cap \left(\frac{k}{4^i}, \frac{k+1}{4^i}\right) \neq \emptyset;$$

(f): for every $x \in [a_i, b_i]$ and every $k \in \mathbb{N}$, $2^{i-1} \leq k < 4^{i-1}$,

$$\left(\frac{1}{b_{i-1} - x}\left(\widetilde{f^{-1}}(I_{i-1}) - x\right)\right) \cap \left(\frac{k}{4^{i-1}}, \frac{k+1}{4^{i-1}}\right) \neq \emptyset;$$

Let us present the inductive construction. Assume it is done for some $n \geq 0$. We will show the next step. Start with the condition (f). If $n+1 > 1$, then, by (e), the set

$$U_k = \left\{x\colon \left(\frac{1}{b_n - x}\left(\left(\widetilde{f^{-1}}(I_n)\right) - x\right)\right) \cap \left(\frac{k}{4^n}, \frac{k+1}{4^n}\right) \neq \emptyset\right\}$$

contains x_n for every $k \in \mathbb{N}$, $2^n \leq k < 4^n$. It is also not difficult to see that the sets U_k are open. Therefore

$$U = \bigcap_{2^n \leq k < 4^n} U_k$$

is also open and contains x_n. It is easy to see that condition (f) is satisfied for $x \in U$. For $n+1 = 1$, (f) is void and we ignore it by defining $U = \mathbb{R}$.

Now, find

$$y \in P \cap f^{-1}\left(J_n^c\right) \cap \left((a_n, b_n) \cap U\right).$$

The existence of such a y is guaranteed because U is open, $x_n \in (a_n, b_n) \cap U$ and (c). If $y \in Z$, let $x_{n+1} = y$. Otherwise $f|_P$ is continuous at y. In this case, the fact that Z is dense in P and U is open guarantees the existence of

$$x_{n+1} \in P \cap f^{-1}\left(J_n^c\right) \cap \left((a_n, b_n) \cap U\right) \cap Z.$$

Since $f(x_{n+1}) \notin J_n$ and $x_{n+1} \in Z$, we can easily find a small interval J_{n+1} centered at $f(x_{n+1})$ satisfying conditions (b) and (c). Choosing I_{n+1} centered at $f(x_{n+1})$ of length

$$\frac{\mathrm{m}(J_{n+1})}{2^{n+2}}$$

guarantees (a), (b), and (c).

Defining $\left(a'_{n+1}, b'_{n+1}\right)$ to be centered at x_{n+1} and such that

$$\left[a'_{n+1}, b'_{n+1}\right] \subset (a_n, b_n) \cap U \quad \text{and} \quad b'_{n+1} - a'_{n+1} < \frac{1}{2^{n+1}}$$

guarantees (d) and (f) for the interval $\left[a'_{n+1}, b'_{n+1}\right]$. However, we still need to make certain that the condition (e) is satisfied. We will do it by choosing $(a_{n+1}, b_{n+1}) \subset (a'_{n+1}, b'_{n+1})$.

Note that x_{n+1} is an $\mathcal{I}$-density point of $f^{-1}(I_{n+1})$, so that 0 is an $\mathcal{I}$-density point of $\widetilde{f^{-1}}(I_{n+1}) - x_{n+1}$. Therefore, by Lemma 2.1.1, there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that the set

$$S = \liminf_{i\to\infty} \left(n_i \left(\widetilde{f^{-1}}(I_{n+1}) - x_{n+1}\right)\right) \cap (-1,1)$$

is residual in $(-1,1)$. Define

$$W_i = n_i \left(\widetilde{f^{-1}}(I_{n+1}) - x_{n+1}\right).$$

The set

$$\bigcup_{r=1}^{\infty} \bigcap_{i\geq r} W_i$$

is residual in $(-1,1)$. In particular, for every $k \in \mathbb{N}$, $2^{n+1} \leq k < 4^{n+1}$,

$$\left(\bigcup_{r=1}^{\infty} \bigcap_{i\geq r} W_i\right) \cap \left(\frac{k}{4^{n+1}}, \frac{k+1}{4^{n+1}}\right) \neq \emptyset.$$

The sequence $\left\{\bigcap_{i\geq r} W_i\right\}_{r\in\mathbb{N}}$ is increasing. Thus, there is an $r_0 \in \mathbb{N}$ such that

$$W_i \cap \left(\frac{k}{4^{n+1}}, \frac{k+1}{4^{n+1}}\right) \neq \emptyset$$

for every $i \geq r_0$ and $k \in \mathbb{N}$, $2^{n+1} \leq k < 4^{n+1}$. But

$$W_i = n_i \left(\widetilde{f^{-1}}(I_{n+1}) - x_{n+1}\right) = \frac{1}{x_{n+1} + \frac{1}{n_i} - x_{n+1}} \left(\widetilde{f^{-1}}(I_{n+1}) - x_{n+1}\right).$$

Define (a_{n+1}, b_{n+1}) as

$$\left(x_{n+1} - \frac{1}{n_i}, x_{n+1} + \frac{1}{n_i}\right),$$

where $i \geq r_0$ and $[a_{n+1}, b_{n+1}] \subset [a'_{n+1}, b'_{n+1}]$. The desired condition (e) is satisfied. This ends the inductive construction.

We will show now how the conclusion of the theorem follows from the construction.

Let

$$\{x\} = \bigcap_{n\in\mathbb{N}} [a_n, b_n] = \bigcap_{n\in\mathbb{N}} \left([a_n, b_n] \cap Z\right).$$

We will show that f is not $\mathcal{I}$-density continuous at x. To be more specific, we will find a sequence $\{n_i\}_{i\in\mathbb{N}}$ such that

(1): $f(x)$ is a deep-$\mathcal{I}$-dispersion point of $\bigcup_{i\in\mathbb{N}} I_{n_i}$, and
(2): x is not a deep-$\mathcal{I}$-dispersion point of $f^{-1}\left(\bigcup_{i\in\mathbb{N}} I_{n_i}\right)$.

We will first show x is not an $\mathcal{I}$-dispersion point of $f^{-1}\left(\bigcup_{i\in\mathbb{N}} I_{n_i}\right)$ for every sequence $\{n_i\}_{i\in\mathbb{N}}$.

Let $\{n_i\}_{i\in\mathbb{N}}$ be any increasing sequence of natural numbers. By the definition of x, the condition (f) implies

$$\left(t_n\left(\widetilde{f^{-1}}(I_n)-x\right)\right)\cap\left(\frac{k}{4^n},\frac{k+1}{4^n}\right)\neq\emptyset$$

for every $k\in\mathbb{N}$, $2^n\leq k<4^n$, where t_n is defined as $1/(b_n-x)$. Note that sequence $\{t_n\}_{n\in\mathbb{N}}$ is increasing and diverging to infinity. Thus, the open set $U_n=t_n\left(\widetilde{f^{-1}}(I_n)-x\right)$ intersects every interval $\left(\frac{k}{4^n},\frac{k+1}{4^n}\right)\subset\left[\frac{1}{2},1\right]$. This implies that for every increasing sequence $\{n_j\}_{j\in\mathbb{N}}$ of natural numbers and for every $s\in\mathbb{N}$ the set $\bigcup_{j\geq s}U_{n_j}$ is dense in $\left[\frac{1}{2},1\right]$. Hence,

$$\limsup_{j\to\infty} t_{n_{i_j}}\left(\widetilde{f^{-1}}\left(\bigcup_{i\in\mathbb{N}} I_{n_i}\right)-x\right)\supset\limsup_{j\to\infty} U_{n_{i_j}}\notin\mathcal{I}$$

for every subsequence $\{n_{i_j}\}_{j\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$. Thus, by Theorem 2.2.2(ii), x is not an $\mathcal{I}$-dispersion point of $f^{-1}\left(\bigcup_{i\in\mathbb{N}} I_{n_i}\right)$.

Condition **(1)** follows immediately from Lemma 3.5.3 for $z=f(x)$ since (A), (B) and (C) from the lemma follows from (b) and (c).

This finishes the proof of Theorem 3.5.2. □

As a corollary we obtain the following theorem. (We also refer the reader to the comments following Theorem 1.4.2.)

THEOREM 3.5.4. *The spaces $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ and $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, equipped with the topology of uniform convergence, are of the first category in themselves.*

PROOF. We prove this only for the class $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ as the other case is essentially the same.

Let $\{I_n\}_{n\in\mathbb{N}}$ be the sequence of all open intervals with rational endpoints and let C_n be the family of all deep-$\mathcal{I}$-density continuous functions that are continuous on I_n in the ordinary sense. By Theorem 3.5.2, $\mathcal{C}_{\mathcal{D}\mathcal{D}}=\bigcup_{n\in\mathbb{N}}C_n$. Also, it is evident that the sets C_n are closed in the uniform convergence topology. Finally, for any function $f\in C_n$ and any of its neighborhoods U, it is easy to slightly modify the function g from Corollary 3.2.7 in such a way that $g\in U\setminus\mathcal{C}_{\mathcal{D}\mathcal{D}}$. Thus, the sets C_n are nowhere dense. □

3.6. Inclusions and Examples

Before stating the main theorem of this section, several examples are constructed. The first two are technical and are used to construct other examples.

EXAMPLE 3.6.1. *Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers. For every right interval set $E=\bigcup_{n\in\mathbb{N}}[a_n,b_n]\subset[0,1]$ for which 0 is a right $\mathcal{I}$-density point there exists an $\mathcal{I}$-density continuous, $\mathcal{C}^\infty$, convex increasing homeomorphism $h\colon[0,b_1]\to[0,\infty)$ such that*

(1): $h^{(n)}(0) = 0$ *for every* $n \geq 0$, *and*

(2): *for every* $n \in \mathbb{N}$ *there exists a positive number* $d_n \leq c_n$ *such that* $h'(x) = d_n$ *for every* $x \in (a_n, b_n)$.

PROOF. Let $D = \bigcup_{n\in\mathbb{N}}(l_n, r_n) = (0, b_1] \setminus E$ and let f be a nonnegative $\mathcal{C}^\infty$ function such that $f(x) = 0$ for $x \in D^c$ while $f(x) > 0$ for $x \in D$. To choose such a function, it is enough to define f on $[l_n, r_n]$ by

$$f(x) = u_n e^{-(x-l_n)^{-2}-(x-r_n)^{-2}},$$

which is known to be $\mathcal{C}^\infty$ with poles at l_n and r_n, and in which the constants $u_n \leq c_n$ are chosen in such a way that $f^{(i)}(x) \leq 1/n$ for every $i \leq n$ and $x \in [l_n, r_n]$.

Let $g(x) = \int_0^x f(y)\,dy$ and $h(x) = \int_0^x g(y)\,dy$.

Evidently h is a $\mathcal{C}^\infty$, convex homeomorphism satisfying **(1)**. It is also easy to see **(2)** holds because $f(x) \leq u_n \leq c_n$ for all $x \in [0, b_n] \subset [0, 1]$. Also, by Corollary 3.2.12, h is $\mathcal{I}$-density continuous at every point $\neq 0$ and, by Corollary 3.4.5, h is right $\mathcal{I}$-density continuous at 0. □

EXAMPLE 3.6.2. *Let* $k < l < m < a < b < c$. *There exists an* $\mathcal{I}$*-density continuous,* $\mathcal{C}^\infty$ *function* $g\colon [k, c] \to \mathbb{R}$ *constant on* $[m, a]$ *and such that* $g(x) = x$ *for* $x \in [k, l] \cup [b, c]$. *(See Figure 3.2.)*

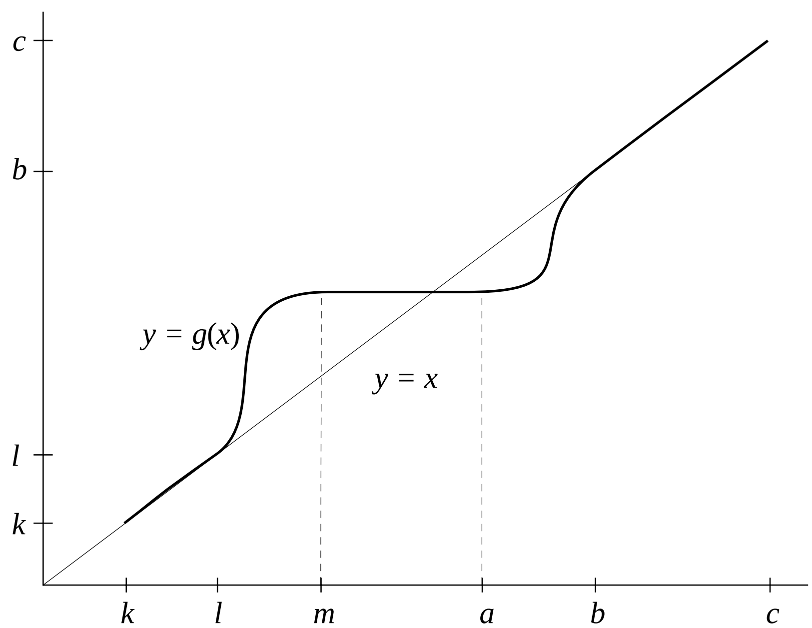

FIGURE 3.2. The function g from Example 3.6.2.

PROOF. Without loss of generality we may assume that $m < 0 < a$. It suffices to define g on $[0, c]$, because defining g on $[k, 0]$ merely involves homothetically altering the odd extension of g.

Let a_0, b_0 be such that $a < a_0 < b_0 < b$ and choose $h\colon [0, a_0 - a] \to [0, \infty)$ as in Example 3.6.1 in such a way that $h(a_0 - a) < b_0$. Define $g_0\colon [0, c] \to \mathbb{R}$ by putting $g_0(x) = 0$ for $x \in [0, a]$, $g_0(x) = x$ for $x \in [b_0, c]$, $g_0(x) = h(x - a)$ for $x \in [a, a_0]$ and extend g_0 on $[a_0, b_0]$ in a continuous manner as a linear function.

Evidently g_0 is $\mathcal{I}$-density continuous. It is also $\mathcal{C}^\infty$ at all points with the exception of a_0 and b_0. To obtain the desired function g we will modify g_0 on $[a, b]$ by "rounding its corners" at the points a_0 and b_0. But notice that g_0 is increasing and linear on some left and right-hand neighborhoods of a_0 and b_0. Using an appropriate homothetic transformation of a function $L(x) + rh(x)$, where L is a linear function, $r \in R$ and h is from Example 3.6.1, g_0 can readily be modified to be a $\mathcal{C}^\infty$ function g with the property that $g'(x) > 0$ on the set where the modification takes place. Thus, g is $\mathcal{I}$-density continuous. □

EXAMPLE 3.6.3. *There exists a $\mathcal{C}^\infty$ function f which is deep-$\mathcal{I}$-density continuous and density continuous but is not $\mathcal{I}$-density continuous; i.e.,*

$$\mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}} \cap \mathcal{C}^\infty \not\subset \mathcal{C}_{\mathcal{I}\mathcal{I}}.$$

PROOF. Let $g\colon [0, 1] \to [0, 1]$ be as in Example 3.6.2, with $0 = k < l < m < 1/2 < a < b < c = 1$. Choose constants $c_k \le \frac{1}{k}$ such that $c_k g^{(i)}(x) \le \frac{1}{k}$ for every $x \in [0, 1]$ and $i \le k$.

Choose a right interval set $E = \bigcup_{k\in\mathbb{N}} [p_k, q_k] \subset [0, 1]$ for which 0 is a right $\mathcal{I}$-density point Let h and $\{d_n\}$ be chosen as in Example 3.6.1 for the sequence $\{c_k\}$ and the set E. Let us also extend h onto $\mathbb{R}$ by putting $h(x) = 0$ for $x < 0$ and as a linear function on $[q_1, \infty)$ by the same formula as on $[p_1, q_1]$. Then, by Theorem 1.4.3, it is easy to see that

$$h \in \mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \cap \mathcal{C}^\infty.$$

Notice also that for every $t > 0$ the function $g_{k,t}\colon [0, t] \to [0, t]$ defined by $g_{k,t}(x) = t\, d_k g(x/t)$ has the properties that

$$g_{k,t}^{(i)}(x) \le \frac{1}{k} \text{ for every } x \in [0, t] \text{ and } i \le k$$

and

$$g_{k,t}(x) = d_k x \text{ for every } x \in t\left([0, l] \cup [b, 1]\right).$$

Moreover, let $\{s_n\}_{n\in\mathbb{N}}$ be a decreasing enumeration of the set S from Example 2.2.5. Without loss of generality we may assume that $S \subset \operatorname{int}(E)$, as 0 is a right deep-$\mathcal{I}$-density point of E. Notice, that each component of E contains only finitely many points from S. Denote the component of E which contains s_n by $[p_{k(n)}, q_{k(n)}]$.

Choose nonempty pairwise disjoint intervals

$$(a_n, b_n) \subset [a_n, b_n] \subset (p_{k(n)}, q_{k(n)}),$$

centered at s_n such that 0 is a dispersion point of $P = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ and

$$\lim_{n\to\infty} \frac{h(b_n) - h(a_n)}{h(a_n)} = 0. \tag{48}$$

Define $h_n(x) = g_{k(n),b_n-a_n}(x - a_n)$ for $x \in (a_n, b_n)$, with the interval in (a_n, b_n) on which h_n is constant denoted by I_n. Notice that $s_n \in I_n$. Define f by putting $f|_{P^c} = h|_{P^c}$ and $f(x) = h_n(x) + h(a_n)$ for $x \in (a_n, b_n)$. (See Figure 3.3.)

Obviously f is $\mathcal{C}^\infty$, $\mathcal{I}$-density continuous and density continuous at any point $\neq 0$. It is also clear, by the choice of the constants c_k, that f is infinitely differentiable at 0.

To see that f is density continuous at 0 it is enough to notice that

$$f|_{(-\infty,0]\cup P^c} = h|_{(-\infty,0]\cup P^c},$$

while $h \in \mathcal{C}_{\mathcal{N}\mathcal{N}}$ and $(-\infty, 0] \cup P^c \in \mathcal{T}_{\mathcal{N}}$.

It is also easy to see that $f \notin \mathcal{C}_{\mathcal{I}\mathcal{I}}$, as 0 is an $\mathcal{I}$-dispersion point of the countable set $f(S)$, while, by Example 2.2.5, 0 is not an $\mathcal{I}$-dispersion point of

$$f^{-1}(f(S)) \supset \bigcup_{n\in\mathbb{N}} I_n \supset S.$$

It remains to prove that f is right deep-$\mathcal{I}$-density continuous at 0, as the left hand case is obvious. So, let 0 be an $\mathcal{I}$-dispersion point of a right interval set $B = \bigcup_{n\in\mathbb{N}}(v_n, w_n)$.

We claim that the lengths of some of the intervals (v_n, w_n) can be increased slightly to satisfy the condition

$$(h(a_k), h(b_k)) \cap (v_n, w_n) = \emptyset \quad \text{or} \quad (h(a_k), h(b_k)) \subset (v_n, w_n), \tag{49}$$

while still preserving the property that 0 is an $\mathcal{I}$-dispersion point of B.

To see this, define

$$B' = \bigcup \{(h(a_k), h(b_k)) \colon (h(a_k), h(b_k)) \cap B \neq \emptyset\} \cup B.$$

Evidently, B' satisfies (49). It must only be shown that 0 is an $\mathcal{I}$-dispersion point of B'.

So, let $(a, b) \subset (0, 1)$ and $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers. Using Theorem 2.2.2(viii), it is possible to find an increasing subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ and a nonempty interval $(c, d) \subset (a, b)$ such that

$$(c, d) \cap n_{k_p} B = \emptyset \quad \text{for all } p \in \mathbb{N}.$$

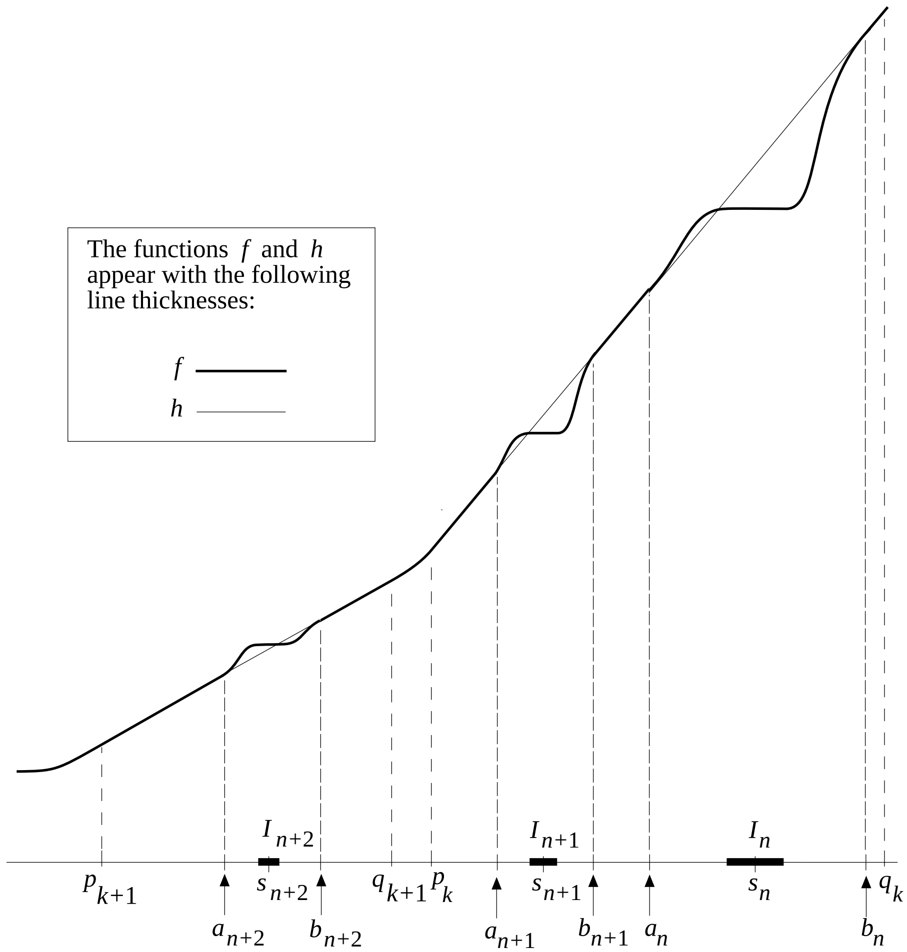

FIGURE 3.3. This is a section of the function f from Example 3.6.3.

According to (48), there is a $p_0 \in \mathbb{N}$ such that

$$\frac{h(b_{n_{k_p}}) - h(a_{n_{k_p}})}{h(a_{n_{k_p}})} < \frac{d-c}{3} \quad \text{for all } p \geq p_0.$$

Hence, if $(h(a_{n_{k_p}}), h(b_{n_{k_p}})) \subset B'$ and $(c,d) \cap n_{k_p}(h(a_{n_{k_p}}), h(b_{n_{k_p}})) \neq \emptyset$ then $n_{k_p} h(a_{n_{k_p}}) < 1$ and

$$n_{k_p}(h(b_{n_{k_p}}) - h(a_{n_{k_p}})) < \frac{h(b_{n_{k_p}}) - h(a_{n_{k_p}})}{h(a_{n_{k_p}})} < \frac{d-c}{3} \quad \text{for all} \quad p \geq p_0.$$

Moreover, $(h(a_{n_{k_p}}), h(b_{n_{k_p}})) \not\subset (c,d)$. Hence,

$$(c + \frac{d-c}{3}, d - \frac{d-c}{3}) \cap n_{k_p} B' = \emptyset \quad \text{for all} \quad p \geq p_0.$$

So, by Theorem 2.2.2(viii), 0 is an $\mathcal{I}$-dispersion point of B'.

But (49) implies that $h^{-1}(B) = f^{-1}(B)$. Therefore, because h is deep-$\mathcal{I}$-density continuous, 0 is a deep-$\mathcal{I}$-dispersion point of $f^{-1}(B)$. $\square$

EXAMPLE 3.6.4. *There exists an $\mathcal{I}$-density continuous function which is not continuous; i.e.,*

$$\mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{C}.$$

PROOF. Let $E = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$ be a right interval set such that 0 is an $\mathcal{I}$-dispersion point of E (see Corollary 2.1.5) and define $f\colon \mathbb{R} \to \mathbb{R}$ by

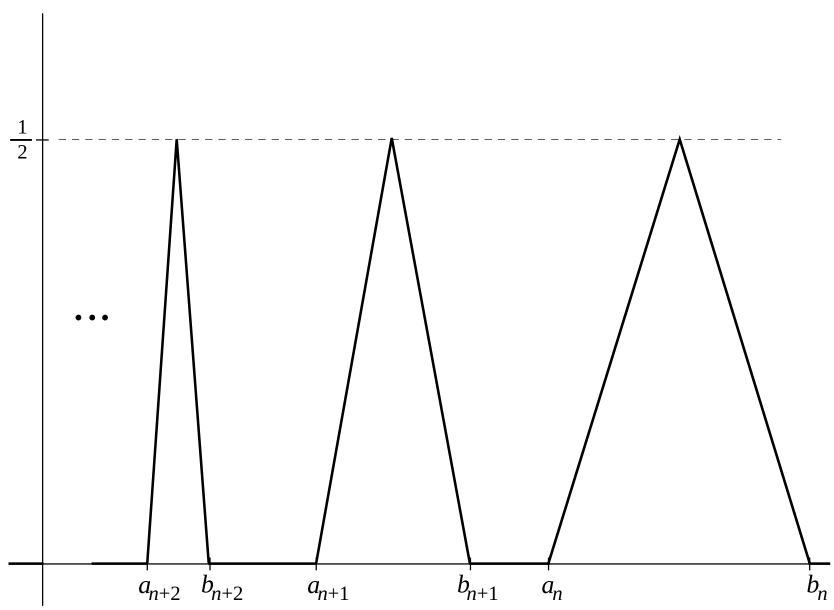

FIGURE 3.4. The function f from Example 3.6.4.

$$f(x) = \begin{cases} 0 & \text{if } x \notin E \\ \frac{\text{dist}(\{x\},(a_n,b_n)^c)}{b_n - a_n} & \text{for } x \in [a_n, b_n]. \end{cases} \tag{50}$$

See Figure 3.4.

By Corollary 3.1.8, f is $\mathcal{I}$-density continuous at every point $x \neq 0$. Also, f is $\mathcal{I}$-density continuous at 0 as it is constant on the $\mathcal{I}$-density open neighbourhood, E^c, of 0. Finally, f is not continuous as the value of f equals $\frac{1}{2}$ at the center of every (a_n, b_n), while $f(0) = 0$. □

EXAMPLE 3.6.5. *There exists an $\mathcal{I}$-approximately continuous and approximately continuous function which is not in Baire*1. In particular,*

$$\mathcal{C}_{\mathcal{IO}} \cap \mathcal{C}_{\mathcal{NO}} \not\subset \mathcal{DB}_1^*.$$

PROOF. Let $\{q_n \colon n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. Let f be as in (50) and put

$$h_n(x) = \frac{2}{3^n} f(x - q_n).$$

Then it is easy to see that $h_n(\mathbb{R}) = [0, 1/3^n]$ and that h_n is piecewise linear at points $\neq q_n$ while $\omega(h_n, q_n) = 1/3^n$. Define $g_n = \Sigma_{i \leq n} h_i$ and let g be a limit of $\{g_n\}$. Notice that

$$|g - g_n| < \frac{1}{2}\frac{1}{3^n}$$

so, $\{g_n\}$ converges to g uniformly. The same argument as in Example 3.6.4 shows that $g_n \in \mathcal{C}_{\mathcal{IO}}\, \mathcal{C}_{\mathcal{NO}}$. Thus, by Theorems 1.3.1(ii) and 2.5.5(ii), $g \in \mathcal{C}_{\mathcal{IO}}\, \mathcal{C}_{\mathcal{NO}}$. Notice also that, for every $n \in \mathbb{N}$,

$$\omega(h, q_n) \geq \omega(h_n, q_n) - \sup_{x \in \mathbb{R}} (g(x) - g_n(x)) \geq \frac{1}{3^n} - \frac{1}{3^{n+1}} > 0$$

so, g is discontinuous at every rational number. But, Baire*1 functions are continuous on a dense open set. Thus, $g \notin \mathcal{DB}_1^*$. □

All of the following charts use the convention that, for the classes of functions $\mathcal{X}$ and $\mathcal{Y}$, the symbol $\mathcal{X}\,\mathcal{Y}$ stands for the class $\mathcal{X} \cap \mathcal{Y}$. The results of this section can be summarized in the following theorem.

THEOREM 3.6.6. *The chart below shows the relationships between the following classes.*

$$
\begin{array}{ccccccc}
 & & & & \mathcal{C}_{\mathcal{I}\mathcal{O}} & \subset & \mathcal{DB}_1 \\
 & & & & \cup & & \cup \\
\mathcal{C}_{\mathcal{I}\mathcal{I}} & \subset & \mathcal{C}_{\mathcal{D}\mathcal{D}} & \subset & \mathcal{C}_{\mathcal{I}\mathcal{O}}\,\mathcal{DB}_1^* & \subset & \mathcal{DB}_1^* \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{I}\mathcal{I}}\,\mathcal{C} & \subset & \mathcal{C}_{\mathcal{D}\mathcal{D}}\,\mathcal{C} & \subset & \mathcal{C} & & \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{I}\mathcal{I}}\,\mathcal{C}^\infty & \subset & \mathcal{C}_{\mathcal{D}\mathcal{D}}\,\mathcal{C}^\infty & \subset & \mathcal{C}^\infty & & \\
\cup & & & & & & \\
\mathcal{A} & & & & & &
\end{array}
$$

All the containments are proper.

PROOF. The vertical containments follow immediately from the obvious containments $\mathcal{A} \subset \mathcal{C}^\infty \subset \mathcal{C} \subset \mathcal{DB}_1^* \subset \mathcal{DB}_1$, $\mathcal{C} \subset \mathcal{C}_{\mathcal{I}\mathcal{O}}$ and from $\mathcal{A} \subset \mathcal{C}_{\mathcal{I}\mathcal{I}}$, which is a restatement of Theorem 3.4.2. The horizontal inclusions follow from $\mathcal{C}_{\mathcal{I}\mathcal{I}} \subset \mathcal{C}_{\mathcal{D}\mathcal{D}} \subset \mathcal{C}_{\mathcal{I}\mathcal{O}}$ (Theorem 3.1.5), $\mathcal{C}_{\mathcal{D}\mathcal{D}} \subset \mathcal{DB}_1^*$ (Theorem 3.5.2) and $\mathcal{C}_{\mathcal{I}\mathcal{O}} \subset \mathcal{DB}_1$ (Corollary 2.5.3).

The fact that all the horizontal inclusions are proper follows from the following observations.

- $\mathcal{C}_{\mathcal{D}\mathcal{D}}\,\mathcal{C}^\infty \not\subset \mathcal{C}_{\mathcal{I}\mathcal{I}}$ – see Example 3.6.3;
- $\mathcal{C}^\infty \not\subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$ – see Corollary 3.2.7;
- $\mathcal{DB}_1^* \not\subset \mathcal{C}_{\mathcal{I}\mathcal{O}}$ – construct a function f as in Example 3.6.4, but with 0 not an $\mathcal{I}$-dispersion point of E. (We could also use Example 3.7.1.)

For the vertical inclusions it is enough to notice that:

- $\mathcal{C}_{\mathcal{I}\mathcal{I}}\,\mathcal{C}^\infty \not\subset \mathcal{A}$ – is easily seen using any $\mathcal{C}^\infty$ non analytic function with nonzero derivative everywhere (Corollary 3.2.12); for example, if f is from Corollary 3.2.6, then $h(x) = f(x) + x$ has the desired property;
- $\mathcal{C}_{\mathcal{I}\mathcal{I}}\,\mathcal{C} \not\subset \mathcal{C}^\infty$ – is easily shown with $f(x) = |x|$, which is a continuous, nondifferentiable and piecewise linear function (Corollary 3.1.8);
- $\mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{C}$ – see Example 3.6.4;
- $\mathcal{C}_{\mathcal{I}\mathcal{O}} \not\subset \mathcal{DB}_1^*$ – see Example 3.6.5.

This finishes the proof of Theorem 3.6.6. □

We finish this section with the following theorem in which $\mathcal{C}([0,1])$ is equipped with the topology of uniform convergence.

THEOREM 3.6.7. *Let $\mathcal{C}_{\mathcal{I}}$ denote the class of continuous functions $f\colon [0,1] \to \mathbb{R}$ which have at least one point of $\mathcal{I}$-density continuity. Then $\mathcal{C}_{\mathcal{I}}$ is a first category subset of $\mathcal{C}([0,1])$.*

PROOF. We will show that there exists a dense $\mathbf{G}_\delta$ subset E of $\mathcal{C} = \mathcal{C}([0,1])$ such that every $f \in E$ is nowhere $\mathcal{I}$-density continuous.

For every $n \in \mathbb{N}$ denote by D_n the set of all $f \in \mathcal{C}$ such that for every $i = 1, 2, \dots, 2^n$, f is linear and nonconstant on every interval $[(i-1)2^{-n}, i2^{-n}]$. Note that $D_{n+1} \supset D_n$ for every $n \in \mathbb{N}$ and

$$D = \bigcup_{n \in \mathbb{N}} D_n$$

is a dense subset of $\mathcal{C}$.

For $f \in \mathcal{C}$ define

$$\|f\|_n = \max_{i=1,2,\dots,2^n} \left| f(i2^{-n}) - f((i-1)2^{-n}) \right|.$$

We claim that for each open set U in $\mathcal{C}$, there exists an $n \in \mathbb{N}$ and a function $f \in D_n$ such that the open ball in $\mathcal{C}$ centered at f of radius $\|f\|_n$ is entirely contained in U. To see this, first find an $m \in \mathbb{N}$ and an $f \in D_m$ such that $f \in U$. Since U is open, there is a $\delta > 0$ such that the open ball of radius δ centered at f is contained in U. Using the uniform continuity of f, we can find an $n \geq m$ such that if $|x - y| \leq 2^{-n}$, then $|f(x) - f(y)| \leq \delta$. From this it is clear that $f \in D_n$ and $\|f\|_n \leq \delta$. The claim becomes evident.

We now start the construction of the $\mathbf{G}_\delta$ set E as an intersection of dense open sets W_k.

Let $k \geq 1$ be an integer and let U be a nonempty open subset of $\mathcal{C}$. Choose $n = n_k \geq k$ and $f \in D_n$ as above. For $j = 0, 1, 2, \dots, 2^{n+1}$, define

$$g\left(\frac{j}{2^{n+1}}\right) = f\left(\frac{j}{2^{n+1}}\right).$$

If

$$i2^{-n} \leq j2^{-n-1} < (j+1)2^{-n-1} \leq (i+1)2^{-n},$$

where $i \in \{0, 1, 2, \dots, 2^n - 1\}$, put

$$L_i = (i2^{-n}, (i+1)2^{-n}),$$

$$M_j = (j2^{-n-1}, (j+1)2^{-n-1})$$

and let $K_j = [a_j, b_j]$ be an interval centered in the center of M_j and such that

$$\frac{\mathrm{m}(K_j)}{\mathrm{m}(M_j)} = 1 - \frac{1}{2^n} = \frac{2\mathrm{m}(K_j)}{\mathrm{m}(L_i)}.$$

Let us choose $I_j^0 = [c_j, d_j]$ centered in the center of the interval $f(M_j)$ and such that

$$\frac{\mathrm{m}(I_j^0)}{\mathrm{m}(f(M_j))} = \frac{1}{2^n}.$$

Define the function g to be monotone on L_i and linear on each of the intervals

$$[j2^{-n-1}, a_j], [a_j, b_j], \text{ and } [b_j, (j+1)2^{-n-1}]$$

in such a way that

$$g([a_j, b_j]) = [c_j, d_j] = I_j^0.$$

(See Figure 3.5.)

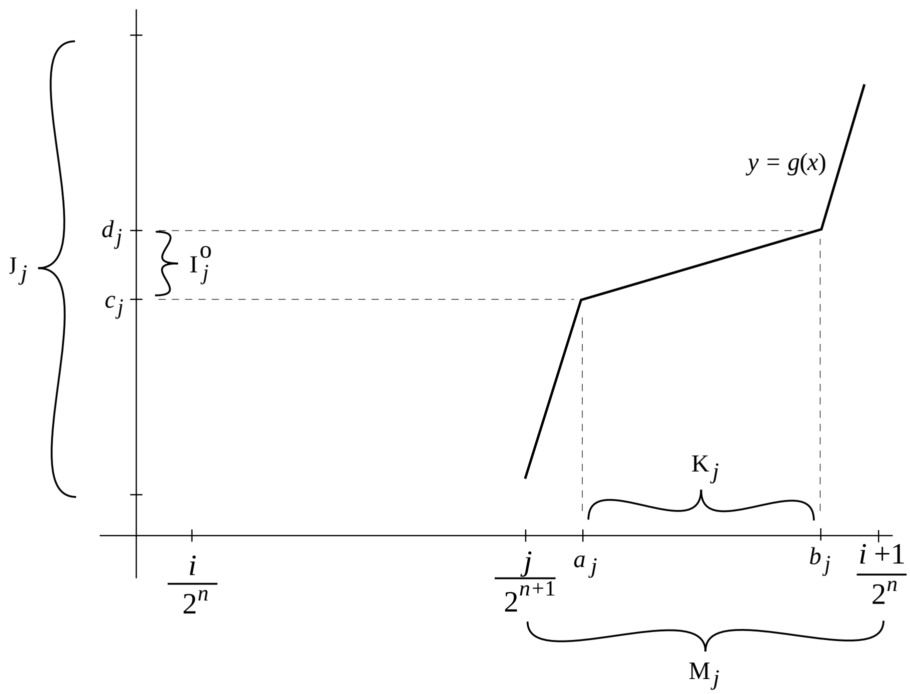

FIGURE 3.5. This is a section of the function g from Theorem 3.6.7.

Thus, if

$$J_j = f(M_j) = g(M_j),$$

then

$$\frac{\mathrm{m}(g(K_j))}{\mathrm{m}(g(M_j))} = \frac{\mathrm{m}(I_j^0)}{\mathrm{m}(J_j)} = \frac{1}{2^n}$$

and

$$\frac{\mathrm{m}(g^{-1}(I_j^0))}{\mathrm{m}(g^{-1}(J_j))} = \frac{\mathrm{m}(K_j)}{\mathrm{m}(M_j)} = 1 - \frac{1}{2^n}.$$

Note that g is contained in the open ball centered at f of radius $\|f\|_n$. Thus, $g \in U$.

Let W_U^k be the open ball centered at g of radius

$$\varepsilon_k = 2^{-n-1} \min_{i=1,2,\dots,2^n} \left| f\left(\frac{i}{2^n}\right) - f\left(\frac{i-1}{2^n}\right) \right| > 0, \tag{51}$$

where $n = n_k$ is as above. Obviously

$$W_k = \bigcup \{W_U^k \colon U \text{ is open and nonempty in } \mathcal{C}\}$$

is open and dense in $\mathcal{C}$, so that

$$E = \bigcap_{k\in\mathbb{N}} W_k$$

is a residual set in $\mathcal{C}$. We will show that if $h \in E$ then h is nowhere $\mathcal{I}$-density continuous.

Let $x \in [0,1]$ be arbitrary. We will choose intervals I_m, $m \in \mathbb{N}$, such that $h(x)$ is an $\mathcal{I}$-dispersion point of $\bigcup_{m\in\mathbb{N}} I_m$, but x is not an $\mathcal{I}$-dispersion point of $h^{-1}\left(\bigcup_{m\in\mathbb{N}} I_m\right)$. This will prove that h is not $\mathcal{I}$-density continuous at x.

Let $m \in \mathbb{N}$. We have $h \in W_m$, so there exists a set U, open in $\mathcal{C}$, such that $h \in W_U^m$. Let g be the center of W_U^m, $n \geq m$ be the number given in the construction of W_U^m and let $i \in \{0,1,2,\ldots,2^n-1\}$ be such that $x \in [i2^{-n},(i+1)2^{-n}]$.

Put

$$L_m = [i2^{-n},(i+1)2^{-n}].$$

Let

$$M^1 = \left((2i)2^{-n-1},(2i+1)2^{-n-1}\right),$$
$$M^2 = \left((2i+1)2^{-n-1},2(i+1)2^{-n-1}\right),$$

and let $M_m \in \{M^1, M^2\}$ be such that $h(x) \notin g(M_m)$. This can be done, since g is monotone on L_m. Put $J_m = g(M_m)$ and let $I_m^0 = [c_j,d_j]$, $K_m = [a_j,b_j]$ be as in the construction of g.

Thus we have

$$\frac{\mathrm{m}(I_m^0)}{\mathrm{m}(J_m)} = \frac{1}{2^n} \leq \frac{1}{2^m} \text{ and } \frac{\mathrm{m}(K_m)}{\mathrm{m}(M_m)} = 1-\frac{1}{2^n} \geq 1-\frac{1}{2^m}.$$

Define

$$I_m = [c_j-\varepsilon_m, d_j+\varepsilon_m],$$

where ε_m is the radius of W_U^m defined by (51). As $h(x) \notin J_m$ and $\mathrm{m}(M_m) \to 0$, we can choose a subsequence $\{I_{m_i}\}_{i\in\mathbb{N}}$ of $\{I_m\}_{m\in\mathbb{N}}$ such that $\bigcup_{i\in\mathbb{N}} I_{m_i}$ is a left or right interval set at $h(x)$. Without loss of generality we may assume that it is a right interval set at $h(x)$. For each $i \in \mathbb{N}$, I_{m_i} and J_{m_i} have a common center and, by (51),

$$\lim_{i\to\infty} \frac{\mathrm{m}(I_{m_i})}{\mathrm{m}(J_{m_i})} = 0.$$

Thus, according to Lemma 2.1.7, there is a subsequence $\{I_{m_{i_j}}\}_{j\in\mathbb{N}}$ of $\{I_{m_i}\}_{i\in\mathbb{N}}$ such that $h(x)$ is an $\mathcal{I}$-dispersion point of $\bigcup_{j\in\mathbb{N}} I_{m_{i_j}}$.

On the other hand, since $|h(x)-g(x)| < \varepsilon_m$ for all $x \in [0,1]$,

$$K_n = g^{-1}([c_j,d_j]) \subset h^{-1}(I_n).$$

Thus, using Lemma 2.2.6, the fact that $x \in L_m$ for every $m \in \mathbb{N}$ and

$$\lim_{j\to\infty} \frac{\mathrm{m}(K_{n_{i_j}})}{\mathrm{m}(L_{n_{i_j}})} = \lim_{j\to\infty} \frac{\mathrm{m}(K_{n_{i_j}})}{2\mathrm{m}(M_{n_{i_j}})} = \frac{1}{2} > 0$$

we conclude that x is not an $\mathcal{I}$-dispersion point of $\bigcup_{j\in\mathbb{N}} K_{n_{i_j}}$. Thus x is not an $\mathcal{I}$-dispersion point of $h^{-1}\left(\bigcup_{j\in\mathbb{N}} I_{n_{i_j}}\right)$. This finishes the proof of Theorem 3.6.7. □

3.7. $\mathcal{I}$-density Versus Density Continuous Functions

The purpose of this section is to discuss the inclusions of Theorem 3.6.6 in connection with the class of density continuous functions and to summarize the comparison of different properties of the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ and $\mathcal{C}_{\mathcal{N}\mathcal{N}}$.

We start with the following three examples, the first two of which are done similarly to Example 3.6.4.

Example 3.7.1. *There exists a density continuous function which is not $\mathcal{I}$-approximately continuous; i.e.,*

$$\mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}_{\mathcal{I}\mathcal{O}}.$$

Proof. Let $S = \{s_n\}_{n\in\mathbb{N}}$ be as in Example 2.2.5 and let $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ be a right interval set such that s_n is the center of each interval $[a_n, b_n]$ and 0 is a dispersion point of E. (For example, taking $b_n - a_n = s_n/(n!)$.) Define f as in (50) by

$$f(x) = \begin{cases} 0 & \text{if } x \notin E \\ \dfrac{\operatorname{dist}(\{x\}, (a_n, b_n)^c)}{b_n - a_n} & \text{for } x \in [a_n, b_n]. \end{cases}$$

(See Figure 3.4.)

Then, by a method similar to that in Example 3.6.4, we can show that f is density continuous. But, f is not $\mathcal{I}$-approximately continuous at 0, as $f^{-1}(0) = 0 \in f^{-1}((-\infty, 1/4))$ and 0 is not an $\mathcal{I}$-dispersion point of the open set

$$B = f^{-1}((1/4, \infty)) \subset f^{-1}((-\infty, 1/4))^c,$$

which contains S. □

Example 3.7.2. *There exists a function which is $\mathcal{I}$-density continuous, but not approximately continuous; i.e.,*

$$\mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{C}_{\mathcal{N}\mathcal{O}}.$$

Proof. Let $C \subset [\frac{1}{2}, 1]$ be a closed nowhere dense set with positive Lebesgue measure and let $\{d_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} d_{n+1}/d_n = 0$. Then, by Lemma 2.8.1, 0 is a deep-$\mathcal{I}$-dispersion point of $Q = \bigcup_{n\in\mathbb{N}} d_n C$. Thus, there exists an open set $U \supset Q$ such that 0 is an $\mathcal{I}$-dispersion point of U. Since every set $d_n C$ is compact, we can decrease U, if necessary, in such a way that, for some sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers, $U = \bigcup_{n\in\mathbb{N}}(a_n + \varepsilon_n, b_n - \varepsilon_n)$ and 0 is a deep-$\mathcal{I}$-dispersion point of the right interval set $D = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$. Then, $D^c \in \mathcal{T}_{\mathcal{I}}$. Moreover, as in Theorem 2.8.2 ($\mathcal{T}_D \not\subset \mathcal{T}_{\mathcal{N}}$), 0 is not a dispersion point of Q. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin D \\ \frac{\operatorname{dist}(\{x\},(a_n,b_n)^c)}{\varepsilon_n} & \text{for } x \in [a_n, b_n]. \end{cases}$$

Then, as in Example 3.6.4, f is $\mathcal{I}$-density continuous. But, f is not approximately continuous at 0, as $f^{-1}(0) = 0 \in f^{-1}((-\infty, 1))$ and 0 is not a dispersion point of the set $f^{-1}([1,\infty)) \supset U \supset Q$. □

Example 3.7.3. *There exists a function which is $\mathcal{C}^\infty$ and $\mathcal{I}$-density continuous, but is not density continuous; i.e.,*

$$\mathcal{C}^\infty \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{C}_{\mathcal{N}\mathcal{N}}.$$

Proof. We will proceed in a method similar to Example 3.6.3.

Let $g\colon [0,1] \to [0,1]$ be as in Example 3.6.2 such that g is constant on $[1/3, 2/3]$. Choose constants $c_k \le \frac{1}{k}$ such that $c_k g^{(i)}(x) \le \frac{1}{k}$ for every $x \in [0,1]$ and $i \le k$.

Choose a right interval set $E = \bigcup_{k\in\mathbb{N}}[p_k, q_k] \subset [0,1]$ for which 0 is a right density and $\mathcal{I}$-density point. Let h and $\{d_n\}$ be chosen as in Example 3.6.1 for the sequence $\{c_k\}$ and the set E. Extend h onto $\mathbb{R}$ by putting $h(x) = 0$ for $x < 0$ and as a linear function on $[q_1, \infty)$ by the same formula as on $[p_1, q_1]$. It is easy to see that

$$h \in \mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \cap \mathcal{C}^\infty.$$

Notice also that for every $t > 0$ the function $g_{k,t}\colon [0,t] \to [0,t]$ defined by $g_{k,t}(x) = t\, d_k g(x/t)$ has the properties that

$$g^{(i)}_{k,t}(x) \le \frac{1}{k} \text{ for every } x \in [0,t] \text{ and } i \le k \tag{52}$$

and

$$g_{k,t}(x) = d_k x \text{ for every } x \in t\left([0,l] \cup [b,1]\right). \tag{53}$$

Moreover, let $D = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ be a right interval set as in Example 3.7.2, i.e., such that $D^c \in \mathcal{T}_D \setminus \mathcal{T}_{\mathcal{N}}$. Without loss of generality we may assume that $D \subset \operatorname{int}(E)$, as 0 is a right density and deep-$\mathcal{I}$-density point of E. Notice, that each component of E contains only finitely many intervals $[a_n, b_n]$. Denote the component of E which contains $[a_n, b_n]$ by $[p_{k(n)}, q_{k(n)}]$.

Let $h_n = g_{k(n),b_n-a_n}$, with the interval in $(0, b_n - a_n)$ on which h_n is constant denoted by I_n. Define f by putting $f|_{\left(\bigcup_{n\in\mathbb{N}}(a_n,b_n)\right)^c} = h|_{\left(\bigcup_{n\in\mathbb{N}}(a_n,b_n)\right)^c}$ and $f(x) = h_n(x - a_n) + h(a_n)$ for $x \in [a_n, b_n]$.

Obviously f is $\mathcal{C}^\infty$, $\mathcal{I}$-density continuous and density continuous at any point $\ne 0$. It is also clear, by the choice of the constants c_k, that f is infinitely differentiable at 0. Also, f is $\mathcal{I}$-density continuous at 0 as $f|_{D^c} = h|_{D^c}$, while $h \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $D^c \in \mathcal{T}_{\mathcal{I}}$.

Finally, f is not density continuous at 0, as 0 is a dispersion point of the countable set $f(\bigcup_{n\in\mathbb{N}} I_n)$, while 0 is not a dispersion point of $f^{-1}(f(\bigcup_{n\in\mathbb{N}} I_n)) = \bigcup_{n\in\mathbb{N}} I_n$. This last statement is correct, as $\bigcup_{n\in\mathbb{N}} I_n$ is obtained by taking the middle third from every component of $D = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$. $\square$

THEOREM 3.7.4. *The chart from Theorem* 3.6.6

$$\begin{array}{ccccccc}
 & & & & \mathcal{C}_{\mathcal{IO}} & \subset & \mathcal{DB}_1 \\
 & & & & \cup & & \cup \\
\mathcal{C}_{\mathcal{II}} & \subset & \mathcal{C}_{\mathcal{DD}} & \subset & \mathcal{C}_{\mathcal{IO}}\ \mathcal{DB}_1^* & \subset & \mathcal{DB}_1^* \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\ \mathcal{C} & \subset & \mathcal{C}_{\mathcal{DD}}\ \mathcal{C} & \subset & \mathcal{C} & & \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\ \mathcal{C}^\infty & \subset & \mathcal{C}_{\mathcal{DD}}\ \mathcal{C}^\infty & \subset & \mathcal{C}^\infty & & \\
\cup & & & & & & \\
\mathcal{A} & & & & & &
\end{array}$$

relativized to (i.e., intersecting each of the element of the chart by) $\mathcal{C}_{\mathcal{NO}}$ *and* $\mathcal{C}_{\mathcal{NN}}$ *gives respectively*

$$\begin{array}{ccccccc}
 & & & & \mathcal{C}_{\mathcal{IO}}\ \mathcal{C}_{\mathcal{NO}} & \subset & \mathcal{C}_{\mathcal{NO}} \\
 & & & & \cup & & \cup \\
\mathcal{C}_{\mathcal{II}}\ \mathcal{C}_{\mathcal{NO}} & \subset & \mathcal{C}_{\mathcal{DD}}\ \mathcal{C}_{\mathcal{NO}} & \subset & \mathcal{C}_{\mathcal{IO}}\ \mathcal{DB}_1^*\ \mathcal{C}_{\mathcal{NO}} & \subset & \mathcal{DB}_1^*\ \mathcal{C}_{\mathcal{NO}} \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\ \mathcal{C} & \subset & \mathcal{C}_{\mathcal{DD}}\ \mathcal{C} & \subset & \mathcal{C} & & \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\ \mathcal{C}^\infty & \subset & \mathcal{C}_{\mathcal{DD}}\ \mathcal{C}^\infty & \subset & \mathcal{C}^\infty & & \\
\cup & & & & & & \\
\mathcal{A} & & & & & &
\end{array}$$

and

$$\begin{array}{ccccccc}
 & & & & \mathcal{C}_{\mathcal{IO}}\ \mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{NN}} \\
 & & & & \| & & \| \\
\mathcal{C}_{\mathcal{II}}\ \mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{DD}}\ \mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{IO}}\ \mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{NN}} \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\ \mathcal{C}\ \mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{DD}}\ \mathcal{C}\ \mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}\ \mathcal{C}_{\mathcal{NN}} & & \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\ \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{DD}}\ \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{NN}} & & \\
\cup & & & & & & \\
\mathcal{A} & & & & & &
\end{array}$$

Moreover, the inclusions in the charts and the nontrivial inclusions between corresponding parts of the charts, i.e.,

$$\mathcal{X}\ \mathcal{C}_{\mathcal{NO}} \subset \mathcal{X} \quad \textit{and} \quad \mathcal{Y}\ \mathcal{C}_{\mathcal{NN}} \subset \mathcal{Y}\ \mathcal{C}_{\mathcal{NO}}$$

for

$$\mathcal{X} \in \{\mathcal{C}_{\mathcal{I}\mathcal{I}}, \mathcal{C}_{\mathcal{D}\mathcal{D}}, \mathcal{C}_{\mathcal{I}\mathcal{O}}\ \mathcal{DB}_1^*, \mathcal{DB}_1^*, \mathcal{C}_{\mathcal{I}\mathcal{O}}, \mathcal{DB}_1\}$$

and

$$\mathcal{Y} \in \{\mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}^\infty, \mathcal{C}_{\mathcal{D}\mathcal{D}}\ \mathcal{C}^\infty, \mathcal{C}^\infty, \mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}, \mathcal{C}_{\mathcal{D}\mathcal{D}}\ \mathcal{C}, \mathcal{C}, \mathcal{C}_{\mathcal{I}\mathcal{I}}, \mathcal{C}_{\mathcal{D}\mathcal{D}}, \mathcal{C}_{\mathcal{I}\mathcal{O}}\ \mathcal{DB}_1^*, \mathcal{DB}_1^*\},$$

are proper.

Proof. All the inclusions follow immediately from Theorem 3.6.6 and the inclusion $\mathcal{C}_{\mathcal{N}\mathcal{N}} \subset \mathcal{C}_{\mathcal{N}\mathcal{O}}$. The fact that the second and third charts are appropriate relativizations follows from the inclusion $\mathcal{C} \subset \mathcal{C}_{\mathcal{N}\mathcal{O}}$ and Theorems 1.4.2 and 1.4.3.

To prove the inclusions in the main charts are proper it is enough to reduce our task to the following inclusions. To argue for the inclusions between first and second rows it is enough to show that $\mathcal{C}_{\mathcal{I}\mathcal{O}}\ \mathcal{C}_{\mathcal{N}\mathcal{O}} \not\subset \mathcal{DB}_1^*$. For the other inclusions it is enough to consider the inclusions in the third chart. Vertical containments are proper on the basis of $\mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{A}$, $\mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}^\infty$ and $\mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}$. To prove that the horizontal inclusions are proper it is enough to show that $\mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}_{\mathcal{I}\mathcal{O}}$, $\mathcal{C}^\infty\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}\ \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}_{\mathcal{I}\mathcal{I}}$. To argue the additional part it is enough to show that $\mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{C}_{\mathcal{N}\mathcal{O}}$ and $\mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}^\infty \not\subset \mathcal{C}_{\mathcal{N}\mathcal{N}}$.

$\mathcal{C}_{\mathcal{I}\mathcal{O}}\ \mathcal{C}_{\mathcal{N}\mathcal{O}} \not\subset \mathcal{DB}_1^*$ is a restatement of Example 3.6.5.

$\mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{A}$ by the same function as used in the proof of $\mathcal{C}^\infty\ \mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{A}$ in Theorem 3.6.6.

$\mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}^\infty$ is easily shown by $f(x) = |x|$.

$\mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}$ by the function f from Example 3.6.4, if we additionally make sure that 0 is a dispersion point of E. (See Corollary 2.1.5.)

$\mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}_{\mathcal{I}\mathcal{O}}$ is a restatement of Example 3.7.1.

$\mathcal{C}^\infty\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$ follows from Corollary 3.2.7 and Theorem 1.4.3.

$\mathcal{C}_{\mathcal{D}\mathcal{D}}\ \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}_{\mathcal{I}\mathcal{I}}$ is a restatement of Example 3.6.3.

$\mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{C}_{\mathcal{N}\mathcal{O}}$ is a restatement of Example 3.7.2.

$\mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}^\infty \not\subset \mathcal{C}_{\mathcal{N}\mathcal{N}}$ is a restatement of Example 3.7.3.

This finishes the proof of Theorem 3.7.4. □

We finish this section with the following theorem, which compares several properties of the $\mathcal{I}$-density and deep-$\mathcal{I}$-density continuous functions with the class of density continuous functions.

Theorem 3.7.5. *The following statements show relationships between various function classes discussed above.*

- **($\mathcal{N}$1):** $\mathcal{A} \subset \mathcal{C}_{\mathcal{N}\mathcal{N}} \subset \mathcal{DB}_1^*$; $\mathcal{C}^\infty \not\subset \mathcal{C}_{\mathcal{N}\mathcal{N}}$;
- **($\mathcal{I}$1):** $\mathcal{A} \subset \mathcal{C}_{\mathcal{I}\mathcal{I}} \subset \mathcal{C}_{\mathcal{D}\mathcal{D}} \subset \mathcal{DB}_1^*$; $\mathcal{C}^\infty \not\subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$; $\mathcal{H} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} = \mathcal{H} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$;
- **($\mathcal{N}$2):** *if h and h^{-1} fulfill a local Lipschitz condition, then $h \in \mathcal{C}_{\mathcal{N}\mathcal{N}}$;*
- **($\mathcal{I}$2):** *if h and h^{-1} fulfill a local Lipschitz condition, then $h \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$;*
- **($\mathcal{N}$3):** *locally convex functions are density continuous;*
- **($\mathcal{I}$3):** *there is a $\mathcal{C}^\infty$ convex function which is not deep-$\mathcal{I}$-density continuous;*

($\mathcal{N}$4): *the class $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ is not closed under uniform convergence;*

($\mathcal{I}$4): *the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ are not closed under uniform convergence;*

($\mathcal{N}$5): *the class $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ is closed under the supremum and infimum operations; thus, it forms a lattice;*

($\mathcal{I}$5): *the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ are closed under the supremum and infimum operations; thus, they form lattices;*

($\mathcal{N}$6): *the class $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ is not closed under addition; and also,*

($\mathcal{I}$6): *the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ are not closed under addition; however,*

($\mathcal{N}$7): *if f and g are increasing homeomorphisms such that $f, g \in \mathcal{C}_{\mathcal{N}\mathcal{N}}$ ($f^{-1}, g^{-1} \in \mathcal{C}_{\mathcal{N}\mathcal{N}}$), then $f + g \in \mathcal{C}_{\mathcal{N}\mathcal{N}}$ ($(f+g)^{-1} \in \mathcal{C}_{\mathcal{N}\mathcal{N}}$);*

($\mathcal{I}$7): *if f and g are increasing homeomorphisms such that $f, g \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$ ($f^{-1}, g^{-1} \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$), then $f + g \in \mathcal{C}_{\mathcal{D}\mathcal{D}}$ ($(f+g)^{-1} \in \mathcal{C}_{\mathcal{D}\mathcal{D}}$);*

($\mathcal{N}$8): *the space $\mathcal{C}_{\mathcal{N}\mathcal{N}}$, equipped with the topology of uniform convergence, is of the first category in itself;*

($\mathcal{I}$8): *the spaces $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$, equipped with the topology of uniform convergence, are of the first category in themselves;*

($\mathcal{N}$9): $S \in \{f^{-1}(\{x\}^c) \colon x \in \mathbb{R} \quad \text{and} \quad f \in \mathcal{C}_{\mathcal{N}\mathcal{N}}\}$ *if, and only if,*

$$S \in \boldsymbol{F}_\sigma \cap \boldsymbol{G}_\delta \cap \mathcal{T}_{\mathcal{N}};$$

($\mathcal{N}$10): $Z = \{x \in \mathbb{R} \colon f \text{ is discontinuous at } x\}$ *for some* $f \in \mathcal{C}_{\mathcal{N}\mathcal{N}}$ *if, and only if,* Z *is nowhere dense* $\boldsymbol{F}_\sigma$ *set.*

Proof. The proofs can be found as follows: for ($\mathcal{N}$1) and ($\mathcal{I}$1) in Theorem 3.7.4 and, for $\mathcal{C}^\infty \not\subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$, in Theorem 3.2.1; for ($\mathcal{N}$2) in Theorem 1.4.4; for ($\mathcal{I}$2) in Corollary 3.4.4; for ($\mathcal{N}$3) in Theorem 1.4.3; for ($\mathcal{I}$3) in Corollary 3.2.7.

($\mathcal{N}$4) follows from Example 1.4.1 and Theorem 1.4.3. Similarly, ($\mathcal{I}$4) follows from Corollary 3.2.7 and Theorem 3.4.2. ($\mathcal{N}$5) can be found in [**18**], while ($\mathcal{I}$5) is a restatement of Proposition 3.2.15. ($\mathcal{I}$6) is proved in Corollary 3.2.13. ($\mathcal{N}$6) can be proved in a much the same way. Its proof can be found in [**13**].

The proofs of the condition ($\mathcal{N}$7) can be found in [**9**] and in [**50**], respectively. ($\mathcal{I}$7) is a restatement of Theorems 3.3.6 and 3.3.10. The proof of ($\mathcal{N}$8) is exactly the same as that of ($\mathcal{I}$8) which, in turn, is a restatement of Theorem 3.5.4.

The proofs of ($\mathcal{N}$9) and ($\mathcal{N}$10) can be found in [**18**] and in [**21**]. □

Problem 3.7.6. *Find characterizations for the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ similar to ($\mathcal{N}$9) and ($\mathcal{N}$10).*

3.8. Other Continuities

For $\mathcal{J}, \mathcal{K} \in \{\mathcal{N}, \mathcal{I}, \mathcal{D}, \mathcal{O}\}$ let $C_{\mathcal{J}K}$ stand for the family of all continuous functions

$$f \colon (\mathbb{R}, \mathcal{T}_{\mathcal{J}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{K}}).$$

It is easy to see that there are sixteen different classes:

$$\begin{array}{cccc}
\mathcal{C}_{\mathcal{O}\mathcal{O}} & \mathcal{C}_{\mathcal{D}\mathcal{D}} & \mathcal{C}_{\mathcal{I}\mathcal{O}} & \mathcal{C}_{\mathcal{N}\mathcal{O}} \\
C_{\mathcal{O}D} & C_{\mathcal{D}D} & C_{\mathcal{I}D} & C_{\mathcal{N}D} \\
C_{\mathcal{O}I} & C_{\mathcal{D}I} & C_{\mathcal{I}I} & C_{\mathcal{N}I} \\
C_{\mathcal{O}N} & C_{\mathcal{D}N} & C_{\mathcal{I}N} & C_{\mathcal{N}N}
\end{array}$$

that represent different continuities formed with respect to the ordinary, density, $\mathcal{I}$-density and deep-$\mathcal{I}$-density topologies on the domain and range of functions. In the charts of Theorem 3.7.4 only six of those classes are considered: $\mathcal{C}_{\mathcal{O}\mathcal{O}} = \mathcal{C}$, $\mathcal{C}_{\mathcal{N}\mathcal{N}}$, $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\mathcal{C}_{\mathcal{N}\mathcal{O}}$ and $\mathcal{C}_{\mathcal{I}\mathcal{O}}$. In this short section it is shown that all the other classes are either trivial or coincide with one of the classes listed above.

Theorem 3.8.1. *$\mathcal{C}_{\mathcal{D}\mathcal{O}} = \mathcal{C}_{\mathcal{I}\mathcal{O}}$ and $C_{\mathcal{I}D} = \mathcal{C}_{\mathcal{D}\mathcal{D}}$.*

Proof. The first equation is a restatement of Corollary 2.7.7.

To argue the second equation let $f \in C_{\mathcal{I}D}$. We must prove that f is continuous as a function from $(\mathbb{R}, \mathcal{T}_D)$ to $(\mathbb{R}, \mathcal{T}_D)$. But, evidently, $g \circ f \in \mathcal{C}_{\mathcal{I}\mathcal{O}} = \mathcal{C}_{\mathcal{D}\mathcal{O}}$ for every $g \in \mathcal{C}_{\mathcal{D}\mathcal{O}}$. Hence, by Lemma 3.1.3, $f \in \mathcal{C}_{\mathcal{D}\mathcal{D}}$. So, $C_{\mathcal{I}D} \subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$. The opposite inclusion is obvious. □

Theorem 3.8.2. *If Const stands for the class of all constant functions then,*

$$C_{\mathcal{O}D} = C_{\mathcal{O}I} = C_{\mathcal{O}N} = C_{\mathcal{D}I} = C_{\mathcal{D}N} = C_{\mathcal{I}N} = C_{\mathcal{N}D} = C_{\mathcal{N}I} = Const.$$

Proof. It is obvious that the constant functions are members of all these classes, as they are universally continuous. Thus, the only thing left to show is that the functions from the above classes $C_{\mathcal{J}K}$ are constant.

By way of contradiction, assume that there is a non-constant function f in some class $C_{\mathcal{J}K}$ from the theorem.

Case 1^o. $C_{\mathcal{J}K} \in \{C_{\mathcal{O}D}, C_{\mathcal{O}I}, C_{\mathcal{O}N}\}$. By assumption there are $a < b$ such that $f([a,b])$ has more than one point. But, by Theorems 1.2.3(vi), 2.6.2(vi) and 2.7.8(vi), $f([a,b])$ is finite, being the continuous image of a compact set. This means that $f([a,b])$ is disconnected, which is impossible because the interval $[a,b]$ is connected in $\mathcal{T}_{\mathcal{O}}$.

Case 2^o. $C_{\mathcal{J}K} \in \{C_{\mathcal{D}I}, C_{\mathcal{D}N}\}$. Let D be a countable dense subset of $(\mathbb{R}, \mathcal{T}_{\mathcal{O}})$ such that $A = f(D)$ has more that one point. By Theorems 1.2.3(ii) and 2.6.2(ii), A is closed and discrete in $\mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}_{\mathcal{N}}$. We show that $B = f^{-1}(A)$ is not closed in $\mathcal{T}_D$. First notice that A, as a discrete set having more that one element, is disconnected and thus, B is disconnected as the continuous preimage of disconnected set. So, by Theorem 2.7.8(v), $B \neq \mathbb{R}$. But, $B \supset D$ is dense in $\mathcal{T}_D$, by Theorem 2.7.8(iii). Hence, B is not closed in $\mathcal{T}_D$.

Case 3^o. $C_{\mathcal{J}K} = C_{\mathcal{I}N}$. As in the previous case, let D be a countable dense subset of $(\mathbb{R}, \mathcal{T}_{\mathcal{O}})$ such that $A = f(D)$ has more than one point and let $a \notin B = f^{-1}(A)$. Without loss of generality we may assume that $a = 0 = f(a)$. Let $S = \{s_n\}_{n \in \mathbb{N}}$ be as in Example 2.2.5 and, for every $n \in \mathbb{N}$, let $d_n \in B \cap [s_{n+1}, s_n]$. Now, choose $a_n < f(d_n) < b_n$ such that 0 is a dispersion point of $E = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$.

Let $U \in \mathcal{T}_{\mathcal{N}}$ be such that $0 \in U \subset E^c$. Then, $f^{-1}([a_n, b_n])$ is of second category at every interval $[s_{n+1}, s_n]$. Hence, the same argument as in Example 2.2.5 shows that 0 is not an $\mathcal{I}$-dispersion point of $f^{-1}(E) \subset f^{-1}(U)$; i.e., $f^{-1}(U) \notin \mathcal{T}_{\mathcal{I}}$.

Case 4^o. $C_{\mathcal{J}K} \in \{C_{\mathcal{N}D}, C_{\mathcal{N}I}\}$. As $C_{\mathcal{N}I} \subset C_{\mathcal{N}D}$ we may assume that $C_{\mathcal{J}K} = C_{\mathcal{N}D}$. The proof of this case is quite complicated and will take the rest of this Section. For this we need the following lemmas.

LEMMA 3.8.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f(0) = 0$ and let $c \in (0,1)$. If $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ is a right interval set such that, for every $n \in \mathbb{N}$, $a_n \geq c\, b_n$ and $\{d_n\}_{n\in\mathbb{N}}$ is a sequence from $(0,1)$ such that*

$$\text{(54)} \qquad \mathrm{m}(f^{-1}([a_n, b_n]) \cap (0, d_n)) \geq d_n/2,$$

then $f \notin C_{\mathcal{N}D}$.

PROOF. The sets $f^{-1}([a_n, b_n]) \cap (0,1)$ are bounded and pairwise disjoint. This implies $\lim_{n\to\infty} \mathrm{m}(\bigcup_{k\geq n} f^{-1}([a_k, b_k]) \cap (0,1)) = 0$. From this and (54) it follows that $\lim_{n\to\infty} d_n = 0$.

Taking a subsequence, if necessary, we can assume that

$$\lim_{n\to\infty} b_{n+1}/b_n = 0.$$

We may also assume that $a_n = c\, b_n$, because decreasing a_n does not change (54).

There are two cases to consider:

- **(1):** there is a point $x \in (c, 1)$ and an $\varepsilon > 0$ such that such that for every nontrivial interval $I \subset (c,1)$ containing x and for every $k \in \mathbb{N}$ there is $n_k \geq k$ with $\mathrm{m}(f^{-1}(b_{n_k} I) \cap (0, d_{n_k})) \geq \varepsilon d_{n_k}$; and
- **(2):** for every $x \in (c,1)$ and every $\varepsilon > 0$ there exists an interval $I \subset (c,1)$ containing x and a $k \in \mathbb{N}$ such that $\mathrm{m}(f^{-1}(b_n I) \cap (0, d_n)) < \varepsilon d_n$ for every $n \geq k$.

Case (1). Let x and ε be as in the assumption. Put $n_0 = 0$ and, by induction on $k \in \mathbb{N}$, define a closed interval $I_k \subset (x - 1/k, x + 1/k) \cap (c, 1)$ and an $n_k > n_{k-1}$ such that

$$\text{(55)} \qquad \mathrm{m}(f^{-1}(b_{n_k} I_k) \cap (0, d_{n_k})) \geq \varepsilon d_{n_k}.$$

Then, by Lemma 2.1.4, 0 is a deep-$\mathcal{I}$-dispersion point of the interval set $D = \bigcup_{k\in\mathbb{N}} b_{n_k} I_k$, while 0 is not a density point of $f^{-1}(D^c)$, because

$$\liminf_{k\to\infty} \frac{\mathrm{m}(f^{-1}(D) \cap (0, d_{n_k}))}{d_{n_k}} \geq \varepsilon > 0.$$

Hence, $D^c \in \mathcal{T}_D$ and $f^{-1}(D^c) \notin \mathcal{T}_{\mathcal{N}}$; i.e., $f \notin C_{\mathcal{N}D}$.

Case (2). Let $\{q_i \colon i \in \mathbb{N}\}$ be a dense countable subset of $(c,1)$ such that

$$\mathrm{m}(f^{-1}(\{b_n q_i\})) = 0 \quad \text{for every} \quad i, n \in \mathbb{N}$$

and let $\delta \in (0, 1/2)$. For every $i \in \mathbb{N}$ choose an open interval I_i containing q_i and $k_i \in \mathbb{N}$ such that

$$\mathrm{m}(f^{-1}(b_n I_i) \cap (0, d_n)) < \frac{\delta}{2^i} d_n \quad \text{for every} \quad n \geq k_i.$$

Now, since for every $n < k_i$

$$\lim_{m\to\infty} \mathrm{m}\left(f^{-1}\left(b_n\left(q_i - \frac{1}{m}, q_i + \frac{1}{m}\right)\right) \cap (0, d_n)\right) \leq \mathrm{m}(f^{-1}(\{b_n q_i\})) = 0$$

we can decrease I_i, if necessary, to obtain

$$\mathrm{m}(f^{-1}(b_n I_i) \cap (0, d_n)) < \frac{\delta}{2^i} d_n \quad \text{for every} \quad n \in \mathbb{N}.$$

Let $C = [c, 1] \setminus \bigcup_{i\in\mathbb{N}} I_i$. Then, C is closed and nowhere dense. By Lemma 2.8.1, 0 is a deep-$\mathcal{I}$-dispersion point of $E = \bigcup_{n\in\mathbb{N}} b_n C$. So, $E^c \in \mathcal{T}_D$. On the other hand, 0 is not a dispersion point of $f^{-1}(E)$, since for every $n \in \mathbb{N}$

$$\begin{aligned} \frac{\mathrm{m}(f^{-1}(E) \cap (0, d_n))}{d_n} &\geq \frac{\mathrm{m}(f^{-1}(b_n C) \cap (0, d_n))}{d_n} \\ &= \frac{\mathrm{m}(f^{-1}\left([a_n, b_n] \setminus \bigcup_{i\in\mathbb{N}} b_n I_i\right) \cap (0, d_n))}{d_n} \\ &\geq \frac{1}{2} - \sum_{i=1}^{\infty} \frac{\delta}{2^i} = \frac{1}{2} - \delta > 0. \end{aligned}$$

Thus, $f^{-1}(E^c) \notin \mathcal{T}_{\mathcal{N}}$ and $f \notin C_{\mathcal{N}D}$. □

LEMMA 3.8.4. *$\mathcal{C} \cap C_{\mathcal{N}D} = Const$.*

PROOF. Evidently, $Const \subset \mathcal{C} \cap C_{\mathcal{N}D}$. To prove the opposite inclusion, let $f \in \mathcal{C} \setminus Const$. We will show that $f \notin C_{\mathcal{N}D}$ by using Lemma 3.8.3. Let $a < b$ be such that $f(a) \neq f(b)$. We may assume that $f(a) < f(b)$ and, by the continuity of f, that $f((a, b)) = (f(a), f(b))$. We may also assume, by modifying f in a linear way, if necessary, that $f(a) = a = -1$ and $f(b) = b = 1$. Then, we obtain $f(-1) = -1$, $f(1) = 1$ and $f((-1, 1)) = (-1, 1)$.

We construct, by induction on $n \in \mathbb{N}$, the sequences: $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ of real numbers and sequences $\{I_n\}$ and $\{J_n\}$ of intervals. We start by putting $a_0 = c_0 = -1$, $b_0 = d_0 = 1$ and $I_0 = [-1, 1]$. Then we proceed inductively to obtain the following conditions:

(a): $I_n = [a_n, b_n]$;
(b): $f(c_n) = a_n$ and $f(d_n) = b_n$;
(c): $f((c_n, d_n)) = (a_n, b_n)$;
(d): $I_n \in \{[a_{n-1}, (a_{n-1} + b_{n-1})/2], [(a_{n-1} + b_{n-1})/2, b_{n-1}]\}$;
(e): $J_n = \mathrm{cl}\,(I_{n-1} \setminus I_n)$;
(f): $\mathrm{m}(f^{-1}(J_n) \cap [c_{n-1}, d_{n-1}]) \geq (d_{n-1} - c_{n-1})/2$.

The inductive step is self-explanatory. First, select I_n as in (d) to satisfy (f). If $I_n = [a_{n-1}, (a_{n-1} + b_{n-1})/2]$ then we put $c_n = c_{n-1}$ and $d_n = \min\{x \in [c_{n-1}, d_{n-1}] \colon f(x) = (a_{n-1} + b_{n-1})/2\}$. In the other case, proceed similarly.

Let $x \in \bigcap_{n\in\mathbb{N}} [c_n, d_n]$. Then, $f(x) \in \bigcap_{n\in\mathbb{N}} I_n$. We may assume, translating f, if necessary, that $x = 0 = f(x)$.

Evidently, $\mathrm{m}(I_n) = 2\mathrm{m}(I_{n+1})$. A simple argument shows that for every $n \in \mathbb{N}$ either $\mathrm{dist}(J_n, I_j) \geq \mathrm{m}(J_n)/4$ or $\mathrm{dist}(J_{n+1}, I_j) \geq \mathrm{m}(J_{n+1})/4$ for all $j \geq n + 2$. This allows us to choose a subsequence $\{n_k\}$ such that

$$\mathrm{dist}(J_{n_k}, 0) \geq \mathrm{m}(J_{n_k})/4 \tag{56}$$

for all $k \in \mathbb{N}$. It is easy to assume that the subsequence $\{n_k\}$ is chosen in such a way that the intervals $\{J_{n_k}\}$ are monotone and on one side of 0. For simplicity, assume that $E = \bigcup_{k\in\mathbb{N}} J_{n_k}$ is a right interval set. Then, condition (56) implies that $\min J_{n_k} \geq (1/5) \max J_{n_k}$. Thus, the first of the assumptions from Lemma 3.8.3 is satisfied for the set E with $c = 1/5$. To finish the proof we will show that the second part is satisfied as well.

First notice that for infinitely many k we have either

$$\frac{\mathrm{m}(f^{-1}(J_{n_k}) \cap [0, d_{n_k-1}])}{d_{n_k-1}} \geq \frac{\mathrm{m}(f^{-1}(J_{n_k}) \cap [c_{n_k-1}, 0])}{-c_{n_k-1}} \tag{57}$$

or the converse inequality (where, 0/0 is considered to be 0.) Without loss of generality we may assume that (57) holds for every k. But this, together with (f), implies that

$$\mathrm{m}(f^{-1}(J_{n_k}) \cap [0, d_{n_k-1}]) \geq d_{n_k-1}/2.$$

Thus, the assumptions from Lemma 3.8.3 are satisfied and Lemma 3.8.4 has been established. □

The next lemma combines the proofs of the Theorems 1.4.2 and 3.5.2.

Lemma 3.8.5. *$C_{\mathcal{N}D} \subset$ Baire*1.*

Proof. Assume to the contrary that for some perfect set P the set

$$Z = \{x \in P \colon f|_P \text{ is not continuous at } x\}$$

is dense in P.

We will construct sequences: $\{x_n\}_{n\in\mathbb{N}}$ of points of P, $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ of open intervals, $\{J_n\}_{n\in\mathbb{N}}$ of compact intervals, and $\{I_n\}_{n\in\mathbb{N}}$ of open intervals having the same midpoint as the corresponding J_n, and contained in that corresponding J_n. The construction is inductive, and aimed at having all the objects obtained satisfy the conditions (a) through (f) listed below.

Start by choosing $x_0 \in Z$, $(a_0, b_0) = (x_0 - 1, x_0 + 1)$ and $I_0 = J_0 = \emptyset$. Assume that for all $n \in \mathbb{N}$ and all $i \in \mathbb{N}$, $i \leq n$, it holds that:

(a): $f(x_i) \in I_i \subset J_i$;

(b): $J_{i-1} \cap J_i = \emptyset$ and, for $i > 2$,

$$\mathrm{m}(J_i) \leq \frac{1}{3} \min\{\mathrm{dist}(J_k, J_{k+1})\colon k \in \mathbb{N}, k < i-1\};$$

(c): $\mathrm{m}(J_i) < \omega(f|_P, x_i)$ and $0 < \mathrm{m}(I_i) < 2^{-i}\mathrm{m}(J_i)$;
(d): $x_i \in (a_i, b_i) \cap Z \subset [a_i, b_i] \subset (a_{i-1}, b_{i-1})$;
(e): $(b_i - a_i) < 2^{-i}$; and,
(f): $\mathrm{m}(f^{-1}(I_i) \cap (a_i, b_i)) > (1 - 2^{-i})(b_i - a_i)$.

To continue with the inductive step, note that by (c) and (d), we are able to choose

$$y \in P \cap f^{-1}(J_n^c) \cap (a_n, b_n).$$

If $y \in Z$, then let $x_{n+1} = y$. Otherwise, $f|_P$ is continuous at y. In this case, the fact that Z is dense in P guarantees the existence of

$$x_{n+1} \in P \cap f^{-1}(J_n^c) \cap (a_n, b_n) \cap Z.$$

Because J_n is closed and $x_{n+1} \in Z$, there is a closed interval J_{n+1} centered at $f(x_{n+1})$ such that

$$J_{n+1} \cap J_n = \emptyset, \quad 0 < \mathrm{m}(J_{n+1}) < \omega(f|_P, x_{n+1})$$

and, for $i > 2$,

$$\mathrm{m}(J_i) \leq \frac{1}{3} \min\{\mathrm{dist}(J_k, J_{k+1})\colon k \in \mathbb{N}, k < i-1\}.$$

If I_{n+1} is the closed interval centered at $f(x_{n+1})$ with length equal $\mathrm{m}(J_{n+1})/2^{n+1}$, it follows that (a), (b) and (c) are true with $i = n+1$. Next, use the approximate continuity of f at x_{n+1} to find an interval $(a_{n+1}, b_{n+1}) \subset (a_n, b_n)$ containing x_{n+1} such that (d), (e) and (f) are satisfied. The induction is complete.

Let

$$\{x\} = \bigcap_{n \in \mathbb{N}} [a_n, b_n] = \bigcap_{n \in \mathbb{N}} \left([a_n, b_n] \cap Z\right).$$

We show that there is an increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ of natural numbers such that

(1): $f(x)$ is a deep-$\mathcal{I}$-dispersion point of $\bigcup_{i \in \mathbb{N}} I_{n_i}$, and
(2): x is not a dispersion point of $f^{-1}\left(\bigcup_{i \in \mathbb{N}} I_{n_i}\right)$.

This implies that $f \notin C_{\mathcal{N}D}$.

First notice that x is not a dispersion point of $f^{-1}\left(\bigcup_{i \in \mathbb{N}} I_{n_i}\right)$ for every sequence $\{n_i\}_{i \in \mathbb{N}}$ as, by condition (f),

$$\lim_{i \to \infty} \frac{\mathrm{m}(f^{-1}(I_{n_i}) \cap (a_{n_i}, b_{n_i}))}{\mathrm{m}((a_{n_i}, b_{n_i}))} = 1.$$

Condition **(1)** follows immediately from Lemma 3.5.3 for $z = f(x)$ since (A), (B) and (C) from the lemma follows from (b) and (c).

This finishes the proof of Lemma 3.8.5. $\square$

Now, we are ready to finish the proof of Theorem 3.8.1.

Evidently, $Const \subset C_{\mathcal{N}D}$. To prove the opposite inclusion, let $f \in C_{\mathcal{N}D}$. By Lemma 3.8.5 the set

$$U = \operatorname{int}(\{x \in \mathbb{R}\colon f \text{ is continuous at } x\})$$

is dense. Let $\{(a_n, b_n)\colon n \in \mathbb{N}\}$ be an enumeration of all components of U. Notice that, by Lemma 3.8.4, f is constant on any interval (a_n, b_n). Moreover, f is approximately continuous, so f has the Darboux property. Hence, f is also constant on any interval $[a_n, b_n]$. This immediately implies that U^c does not have any isolated points; i.e., the set $P = U^c$ is perfect. We prove that $P = \emptyset$.

By way of contradiction, assume that $P \neq \emptyset$ and use Lemma 3.8.5 for f and P. Then, there is a nonempty portion $Q = P \cap (c, d)$ on which f is continuous. The set P is nowhere dense, so there exists an n such that $(c, d) \cap (a_n, b_n) \neq \emptyset$. Then, $(c, d) \cup (a_n, b_n)$ is an interval properly containing (a_n, b_n). We will obtain a contradiction with the assumption that (a_n, b_n) is a component of U by showing that f is continuous on $J = (c, d) \cup (a_n, b_n)$. So, let $x \in J$. If $x \in U$, then evidently f is continuous at x. If $x \in P$, then choose a sequence $\{x_i\}_{i\in\mathbb{N}}$ converging to x and define

$$y_i = \begin{cases} x_i & \text{if } \ x_i \in P \\ a_k & \text{for } x_i \in (a_k, b_k). \end{cases}$$

Then, $y_i \in P$, $f(x_i) = f(y_i)$ for $i \in \mathbb{N}$ and $\lim_{i\to\infty} y_i = x$. Moreover,

$$\lim_{i\to\infty} f(x_i) = \lim_{i\to\infty} f(y_i) = f(x)$$

as $f|_P$ is continuous at x. Hence, f is continuous at x.

This finishes the proof of 3.8.1. □

3.9. Historical and Bibliographic Notes

Homeomorphisms preserving $\mathcal{I}$-density points were studied by Aversa and Wilczyński [**1**]. In particular, they proved Lemma 3.2.4, Example 3.4.6 and the equivalence of (i) and (ii) in Lemma 3.2.2. Because the rest of Lemma 3.2.2 is a pure translation of their result, we can also attribute to them Proposition 3.1.7, Corollary 3.4.4 and the part of Theorem 3.2.1 that concerns homeomorphisms. The general form of Theorem 3.2.1 has been proved by Ciesielski and Larson in [**14**].

All the results from Section 3.3 were proved by Ciesielski [**9**]. Ciesielski [**10**] also proved the part of Theorem 3.8.2 which states that $C_{\mathcal{N}D} = Const$. Essentially all the other results from this Chapter were proved by Ciesielski and Larson in papers [**16, 17, 12, 15**].

Lemma 3.1.3 is a compilation of Theorems 14.12 and 8.10 from [**69**].

CHAPTER 4

Semigroups

In this chapter we examine the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ of $\mathcal{I}$-density continuous functions and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ of deep-$\mathcal{I}$-density continuous functions as semigroups with composition as the operation. We also analyze some of their subsemigroups. In particular, we are interested in showing that the semigroups considered have the inner automorphism property.

4.1. Preliminaries

For a topological space X, let $\mathrm{S}(X)$ be the set of all continuous selfmaps of X; i.e., continuous functions $f : X \to X$. If there is no topology defined on X, then the discrete topology is used in the definition of $\mathrm{S}(X)$ so that $\mathrm{S}(X) = X^X$.

It is apparent that $\mathrm{S}(X)$ is a semigroup[1] with composition as the operation. In particular, the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}} = \mathrm{S}((\mathbb{R}, \mathcal{T}_{\mathcal{I}}))$, $\mathcal{C}_{\mathcal{D}\mathcal{D}} = \mathrm{S}((\mathbb{R}, \mathcal{T}_D))$ and $\mathcal{C}_{\mathcal{N}\mathcal{N}} = \mathrm{S}((\mathbb{R}, \mathcal{T}_{\mathcal{N}}))$ form semigroups.

For a semigroup G we denote by $\mathrm{Aut}(G)$ the group of all automorphisms of G.

We say that a subsemigroup G of $\mathrm{S}(X)$ has the *inner automorphism property*, if for every automorphism $\Psi \in \mathrm{Aut}(G)$ there is an $h \in G$ such that $\Psi(g) = h \circ g \circ h^{-1}$ for every $g \in G$.

For a topological space X the property that $\mathrm{S}(X)$ has the inner automorphism property is strongly related to the following definitions and facts.

Two topological spaces X and Y are said to belong to the same S-admissible class if whenever there exists an isomorphism $\Phi : \mathrm{S}(X) \to \mathrm{S}(Y)$, then there must exist a homeomorphism $h : X \to Y$ which induces Φ in the sense that $\Phi(f) = h \circ f \circ h^{-1}$.

Of course, it is always true that the topology on a space X determines the structure of $\mathrm{S}(X)$, but the converse need not be true. For example, let $X = \{0, 1\}$ and suppose $\mathcal{T}_1$ is the discrete topology on X and $\mathcal{T}_2 = \{X, \emptyset\}$. Then $(X, \mathcal{T}_1)$ is not homeomorphic to $(X, \mathcal{T}_2)$, but $\mathrm{S}((X, \mathcal{T}_1)) = \mathrm{S}((X, \mathcal{T}_2)) = X^X$. Within an

[1] By a semigroup we understand a set with a binary associative operation and an identity element.

S-admissible class, the algebraic structure of S(X) determines the topology on X. For more information on this subject see Magill [**44**].

It is clear that a necessary condition for a space X to belong to any S-admissible class is that S(X) must have the inner automorphism property.

The basis for studying the inner automorphism property of subsemigroups G of S(X) is given by the following theorem of Schreier [**59**]. For completeness sake, we include an easy proof.

PROPOSITION 4.1.1. *Let X be a set and let G be a subsemigroup of S(X) such that every constant mapping is in G. Then, for every $\Psi \in Aut(G)$ there exists a unique bijection h of X such that $\Psi(g) = h \circ g \circ h^{-1}$ for every $g \in G$.*

PROOF. For $a \in X$ let $f_a \in$ S(X) be the constant function defined by $f_a(x) = a$. Choose an arbitrary $x_0 \in X$ and define h by $h(a) = \Psi(f_a)(x_0)$ for $a \in X$.

We must prove that $\Psi(g) = h \circ g \circ h^{-1}$ for every $g \in G$. So let us fix $g \in G$. Then $\Psi(g) = h \circ g \circ h^{-1}$ is equivalent to the fact that $(\Psi(g) \circ h)(a) = (h \circ g)(a)$; i.e., $\Psi(g)(h(a)) = h(g(a))$ for every $a \in X$. But,

$$\begin{aligned} \Psi(g)(h(a)) &= \Psi(g)(\Psi(f_a)(x_0)) \\ &= (\Psi(g) \circ \Psi(f_a))(x_0) \\ &= \Psi(g \circ f_a)(x_0) \\ &= \Psi(f_{g(a)})(x_0) \\ &= h(g(a)). \end{aligned}$$

Thus, $\Psi(g) = h \circ g \circ h^{-1}$ for every $g \in G$.

To prove the of uniqueness of h, assume that for some $h_1 \in$ S(X) we also have $\Psi(g) = h_1 \circ g \circ h_1^{-1}$ for every $g \in G$. Then, for every $a \in X$,

$$h(a) = \Psi(f_a)(x_0) = (h_1 \circ f_a \circ h_1^{-1})(x_0) = h_1(f_a(h_1^{-1}(x_0))) = h_1(a),$$

because $f_a(h_1^{-1}(x_0)) = a$. This finishes the proof of Proposition 4.1.1. □

Every semigroup considered in this chapter contains all constant functions. In particular, if Ψ is an automorphism of a semigroup $G \subset$ S(X) containing all constant functions then the unique bijection h of X given by Proposition 4.1.1 for which $\Psi(g) = h \circ g \circ h^{-1}$ for all $g \in G$ is called the *generating bijection* of Ψ.

A topological space X is *generated* if it is T_1 and the collection of complements of level sets for its continuous selfmaps

$$\left\{ \left(f^{-1}(\{x\})\right)^c : x \in X \text{ and } f \in \mathrm{S}(X) \right\}$$

forms a subbase for X. (Compare also [**62**].)

It is known that the class of all generated spaces is S–admissible [**44**, Theorem 2.3, p. 198]. In particular, there is the following.

PROPOSITION 4.1.2. *If a topological space X is generated, then S(X) has the inner automorphism property.*

4.2. Density Continuous Functions

Let Δ be the class of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Extending this notation, let $\Delta^{(\mathcal{N})}$ stand for the class of all approximately differentiable functions and let $\Delta^{(\mathcal{N})}_{\mathcal{N}}$ be the class of all almost everywhere approximately differentiable functions.

The classes defined above are connected to the class $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ of density continuous functions by the following theorem.

THEOREM 4.2.1. *If $\mathcal{F}_m$ stands for the class of measurable functions, then*

$$\begin{array}{ccccccc} \Delta & \subset & \Delta^{(\mathcal{N})} & \subset & \Delta^{(\mathcal{N})}_{\mathcal{N}} & \subset & \mathcal{F}_m \\ \cup & & \cup & & \cup & & \cup \\ \Delta \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} & \subset & \Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} & \subset & \Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} & \subset & \mathcal{C}_{\mathcal{N}\mathcal{N}} \end{array}$$

and all the inclusions are proper.

PROOF. The inclusions are obvious. The vertical inclusions are proper because there is even a $\mathcal{C}^\infty$ function which is not density continuous. (See Example 1.4.1.)

$\mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \Delta^{(\mathcal{N})}_{\mathcal{N}}$ follows immediately from Example 1.4.5.

$\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \Delta^{(\mathcal{N})}$ is proved by the function $h(x) = |x|$.

$\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \Delta$ is easily justified by the function f defined by $f(x) = 0$ for $x \in E^c$ and

$$f(x) = \frac{(x-a_n)^2(x-b_n)^2}{(b_n-a_n)^4}$$

for $x \in [a_n, b_n]$ and $n \in \mathbb{N}$, where $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ is a right interval set for which 0 is a dispersion point. □

The motivation for the study of the inner automorphism property of these classes was taken from the following theorem of Magill [**43**].

THEOREM 4.2.2. *The semigroup Δ of all differentiable functions has the inner automorphism property.*

Following this path, Ostaszewski [**54**] proved the following.

THEOREM 4.2.3. *The semigroups $\mathcal{C}_{\mathcal{N}\mathcal{N}}$, $\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$, $\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$ and $\Delta \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$ have the inner automorphism property.*

Notice that in the above theorem the fact that $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ has the inner automorphism property cannot be deduced from Proposition 4.1.2, as the density topology is not generated (Theorem 1.2.3(vii)). In fact, the real numbers equipped with the density topology is the only example known to us of a completely regular topological space which is not generated such that the semigroup of its continuous selfmaps has the inner automorphism property.

Theorem 4.2.3 is proved by showing the following facts: h is a generating bijection of $\Psi \in \mathrm{Aut}(\mathcal{C}_{\mathcal{N}\mathcal{N}}) \cup \mathrm{Aut}(\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}})$ if, and only if, $h, h^{-1} \in \mathcal{H} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \subset$

$\Delta^{(\mathcal{N})}$, and h is a generating bijection of $\Psi \in \mathrm{Aut}(\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}) \cup \mathrm{Aut}(\Delta \cap \mathcal{C}_{\mathcal{N}\mathcal{N}})$ if, and only if, $h, h^{-1} \in \mathcal{H} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \Delta$. In particular, the observations listed above allow us to identify the automorphisms of the function classes listed with certain groups of homeomorphisms. This implies the following theorem.

Corollary 4.2.4. *The following relations hold:*

$$\begin{aligned} Aut(\Delta \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}) &= Aut(\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}) \\ &\subset Aut(\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}) = Aut(\mathcal{C}_{\mathcal{N}\mathcal{N}}), \end{aligned}$$

and the inclusion is proper.

Theorems 4.2.2 and 4.2.3 imply that the classes Δ, $\Delta \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$, $\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$, $\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$ and $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ have the inner automorphism property. For the remaining three classes of Theorem 4.2.1 the question about their inner automorphism property makes no sense because these classes do not form semigroups. For the class $\Delta^{(\mathcal{N})}$ this is shown by Example 4.3.15. For the other two classes it follows from the example below.

Example 4.2.5. *There exist almost everywhere differentiable functions f and g such that $g \circ f$ is not measurable. In particular, the classes $\Delta^{(\mathcal{N})}_{\mathcal{N}}$ and $\mathcal{F}_m$ are not closed under composition.*

Proof. Let $P \subset \mathbb{R}$ be any perfect nowhere dense set with positive measure, $S \subset P$ be nonmeasurable and let C be a Cantor set of measure 0. Let f be a homeomorphism such that $f^{-1}(C) = P$ and let $g = \chi_S \circ f^{-1}$. Then, f is differentiable almost everywhere, as a homeomorphism and g is differentiable almost everywhere, as $g = \chi_{f(S)}$ and $f(S) \subset C$ has measure 0. On the other hand, $g \circ f = \chi_S \circ f^{-1} \circ f = \chi_S$ is not measurable, because S is nonmeasurable. □

4.3. The $\mathcal{I}$-approximate Derivative

In this section we introduce the notions of the $\mathcal{I}$-approximate derivative and the deep-$\mathcal{I}$-approximate derivative as category analogues of the approximate derivative and examine their properties.

We begin with the following definitions. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *$\mathcal{I}$-approximately differentiable at the point x*, if there exists a number $D^{(\mathcal{I})} f(x)$, called the *$\mathcal{I}$-approximate derivative* of f at x, such that for every $\varepsilon > 0$, x is an $\mathcal{I}$-density point of some Baire subset of

$$\left\{ t \in \mathbb{R} \colon \frac{f(t) - f(x)}{t - x} \in \left(D^{(\mathcal{I})} f(x) - \varepsilon, D^{(\mathcal{I})} f(x) + \varepsilon \right) \right\}. \tag{58}$$

(Compare also [**41**] and [**68**, Definition 8].) Similarly, we say that $f\colon \mathbb{R} \to \mathbb{R}$ is *deep-$\mathcal{I}$-approximately differentiable at a point x* whenever there exists a number

$D^{(\mathcal{D})}f(x)$, called the *deep-$\mathcal{I}$-approximate derivative* of f at x, such that for every $\varepsilon > 0$, x is a deep-$\mathcal{I}$-density point of

$$\left\{t \in \mathbb{R}\colon \frac{f(t)-f(x)}{t-x} \in \left(D^{(\mathcal{D})}f(x)-\varepsilon, D^{(\mathcal{D})}f(x)+\varepsilon\right)\right\}. \tag{59}$$

If a function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{I}$-approximately differentiable at every point, then it is said to be *$\mathcal{I}$-approximately differentiable*. The definition of a (*deep-$\mathcal{I}$-approximately differentiable*) function is similar. The class of all $\mathcal{I}$-approximately differentiable (deep-$\mathcal{I}$-approximately differentiable) functions is denoted by $\Delta^{(\mathcal{I})}$ ($\Delta^{(\mathcal{D})}$).

The function $f\colon \mathbb{R} \to \mathbb{R}$ is *$\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere* (or *deep-$\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere*) if there is a set $A \in \mathcal{I}$ such that f is $\mathcal{I}$-approximately differentiable (deep-$\mathcal{I}$-approximately differentiable) at every point of A^c. The class of all functions $\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere (deep-$\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere) is denoted by $\Delta^{(\mathcal{I})}_{\mathcal{I}}$ ($\Delta^{(\mathcal{D})}_{\mathcal{I}}$).

To compare these classes we need the following facts.

PROPOSITION 4.3.1. *If a function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{I}$-approximately differentiable (deep-$\mathcal{I}$-approximately differentiable) at the point x then f is $\mathcal{I}$-approximately continuous (deep-$\mathcal{I}$-approximately continuous) at x.*

PROOF. We prove this only for f being $\mathcal{I}$-approximately differentiable at x. The proof of the other case is essentially the same.

Let $d = D^{(\mathcal{I})}f(x)$ and choose $\varepsilon > 0$. We must prove that x is an $\mathcal{I}$-density point of the set $\{t \in \mathbb{R}\colon |f(t)-f(x)| \in (-\varepsilon,\varepsilon)\}$. But, if we choose $\delta > 0$ such that $\delta\,(d-\varepsilon, d+\varepsilon) \subset (-\varepsilon,\varepsilon)$, then

$$\begin{gathered}\{t \in \mathbb{R}\colon |f(t)-f(x)| \in (-\varepsilon,\varepsilon)\} \supset \\ \{t \in \mathbb{R}\colon |t-x| < \delta \;\&\; |f(t)-f(x)| \in \delta\,(d-\varepsilon, d+\varepsilon)\} \supset \\ \{t \in \mathbb{R}\colon |t-x| < \delta \;\&\; |f(t)-f(x)| \in |t-x|\,(d-\varepsilon, d+\varepsilon)\} = \\ \left\{t \in \mathbb{R}\colon \frac{|f(t)-f(x)|}{|t-x|} - d \in (-\varepsilon,\varepsilon)\right\} \cap (x-\delta, x+\delta).\end{gathered}$$

By the assumption, x is an $\mathcal{I}$-density point of some Baire subset of the last set. This finishes the proof. □

PROPOSITION 4.3.2. *If f is $\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere, then f is a Baire function.*

PROOF. This is an immediate consequence of Proposition 4.3.1 and Theorem 2.5.6. □

PROPOSITION 4.3.3. *Let f be an $\mathcal{I}$-approximately continuous function. Then f is deep-$\mathcal{I}$-approximately differentiable at x if, and only if, f is $\mathcal{I}$-approximately differentiable at x. In this case, $D^{(\mathcal{I})}f(x) = D^{(\mathcal{D})}f(x)$.*

PROOF. The only nontrivial part is to prove that the existence of $D^{(\mathcal{I})}f(x)$ implies the existence of $D^{(\mathcal{D})}f(x)$. So assume that $D^{(\mathcal{I})}f(x)$ exists. We will prove that $D^{(\mathcal{D})}f(x)$ exists. The proof is essentially the same as the proof of Theorem 2.7.6.

Define $g\colon \mathbb{R}\setminus\{x\}\to\mathbb{R}$ by

$$g(t)=\frac{f(t)-f(x)}{t-x}.$$

It is easy to see that g is $\mathcal{I}$-approximately continuous on $\mathbb{R}\setminus\{x\}$. Let us choose $a<D^{(\mathcal{I})}f(x)<b$. We must prove that x is a deep-$\mathcal{I}$-density point of

$$g^{-1}((a,b))=\left\{t\in\mathbb{R}\colon \frac{f(t)-f(x)}{t-x}\in(a,b)\right\}.$$

Let $c<d$ be such that $D^{(\mathcal{I})}f(x)\in(c,d)\subset[c,d]\subset(a,b)$ and define $E=g^{-1}([c,d])$. Then, $E^c\setminus\{x\}=g^{-1}([c,d]^c)\in\mathcal{T}_{\mathcal{I}}$. Moreover, x is an $\mathcal{I}$-density point of $g^{-1}((c,d))\subset E$ and so x is a deep-$\mathcal{I}$-density point of $\tilde{E}$ and of $\tilde{E}\setminus\{x\}$ as well. But, $\tilde{E}\cap(E^c\setminus\{x\})\in\mathcal{I}\cap\mathcal{T}_{\mathcal{I}}$ so $E^c\cap\tilde{E}\setminus\{x\}=\emptyset$; i.e., $\tilde{E}\setminus\{x\}\subset E$. Hence, x is a deep-$\mathcal{I}$-density point of $E\subset g^{-1}((a,b))$. This finishes the proof. □

As an immediate corollary of Propositions 4.3.1 and 4.3.3 we obtain

COROLLARY 4.3.4. $\Delta\subseteq\Delta^{(\mathcal{D})}=\Delta^{(\mathcal{I})}\subseteq\Delta^{(\mathcal{D})}_{\mathcal{I}}\subseteq\Delta^{(\mathcal{I})}_{\mathcal{I}}$.

Because of Corollary 4.3.4 we need not work with the class $\Delta^{(\mathcal{D})}$ and instead concentrate on the classes

$$\Delta\subseteq\Delta^{(\mathcal{I})}\subseteq\Delta^{(\mathcal{D})}_{\mathcal{I}}\subseteq\Delta^{(\mathcal{I})}_{\mathcal{I}}.$$

We will prove that the inclusions given above are proper. For this purpose it will be convenient to construct the following example.

EXAMPLE 4.3.5. *For every $a<b\le c<d$ there exists a $\mathcal{C}^\infty$ $\mathcal{I}$-density continuous and density continuous function $f\colon\mathbb{R}\to[0,1]$ such that $f(x)=0$ for $x\in(a,d)^c$ and $f(x)=1$ for $x\in[b,c]$. (See Figure 4.6.)*

PROOF. It is sufficient to construct the portion of the function f which is defined on the interval $((b+c)/2,+\infty)$, as the portion of it defined on $(-\infty,(b+c)/2)$ can be obtained via symmetry. Also, without loss of generality, we can assume that $c=0$ and $d=1$. In view of that, it is enough to show that there is a $\mathcal{C}^\infty$ deep-$\mathcal{I}$-density continuous and density continuous function $g\colon\mathbb{R}\to[0,1]$ such that

$$g(x)=1 \text{ for } x\le 0 \text{ and } g(x)=0 \text{ for } x\ge 1. \tag{60}$$

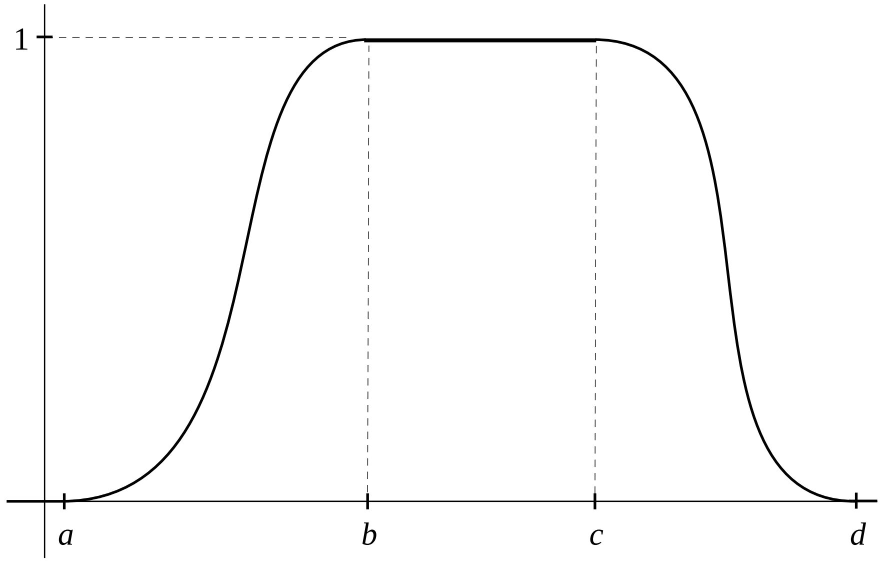

FIGURE 4.6. The function f from Example 4.3.5

To see this let $h\colon [0,\frac{1}{2}] \to \mathbb{R}$ be as in Example 3.6.1 such that $h(\frac{1}{2}) = \frac{1}{2}$ and let us extend h by putting $h(x) = 0$ for $x \le 0$. Define

$$g_1(x) = h\left(\frac{1}{2} - x\right) - \frac{1}{2}$$

for $x \ge 0$ and put $g_1(x) = -g_1(-x)$ for $x \le 0$. Then, it is easy to see that $g(x) = g_1(x - \frac{1}{2}) + \frac{1}{2}$ satisfies condition (60). □

The previous example easily implies the existence of the following.

EXAMPLE 4.3.6. *There exists an approximately and $\mathcal{I}$-approximately differentiable, density continuous and $\mathcal{I}$-density continuous function which is not continuous. In particular,*

$$\Delta^{(\mathcal{N})} \cap \Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{C}.$$

PROOF. Let $E = \bigcup_{n\in\mathbb{N}}[a_n, d_n]$ be an interval set for which 0 is both a dispersion and $\mathcal{I}$-dispersion point. For each n, let f_n be as in Example 4.3.5 with $[a, d] = [a_n, d_n]$. (The choice of b and c can be arbitrary.) Define $f(x) = 0$ on E^c and $f = f_n$ on $[a_n, d_n]$. The rest is obvious. □

Now we are ready to prove

THEOREM 4.3.7. *The following containments are true:*

$$\Delta \subset \Delta^{(\mathcal{I})} \subset \Delta^{(\mathcal{D})}_{\mathcal{I}} \subset \Delta^{(\mathcal{I})}_{\mathcal{I}}.$$

Moreover, they are all proper.

PROOF. $\Delta^{(\mathcal{I})} \not\subset \Delta$ follows from Example 4.3.6.
$\Delta_{\mathcal{I}}^{(\mathcal{D})} \not\subset \Delta^{(\mathcal{I})}$ is justified by the function $f(x) = |x|$.
$\Delta_{\mathcal{I}}^{(\mathcal{I})} \not\subset \Delta_{\mathcal{I}}^{(\mathcal{D})}$ is shown by the characteristic function $\chi_{\mathbb{Q}}$. □

In what follows we will need also the following example.

EXAMPLE 4.3.8. *There exists a continuous, density continuous and $\mathcal{I}$-density continuous function f which is nowhere approximately and $\mathcal{I}$-approximately differentiable. In particular,*

$$\mathcal{C}_{\mathcal{O}\mathcal{O}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \Delta_{\mathcal{N}}^{(\mathcal{N})} \cup \Delta_{\mathcal{I}}^{(\mathcal{I})}.$$

PROOF. We begin by defining a version of the classical Peano area-filling curve $P : [0,1] \to [0,1] \times [0,1]$. To do this, a sequence of continuous paths $P_n : [0,1] \to [0,1] \times [0,1]$ for $n = 0, 1, \ldots$, are defined which converge uniformly to P. This definition is facilitated by the following basic construction, which will be referred to as BCP.

Given a square $[a,b] \times [c,d]$ with one of its diagonals a parametrized constant speed path $\lambda : [\alpha, \beta] \to [a,b] \times [c,d]$, we construct a new path $\lambda_1 : [\alpha, \beta] \to [a,b] \times [c,d]$ as shown in Figure 4.7, where the speed of the new path is constant and three times the speed of λ.

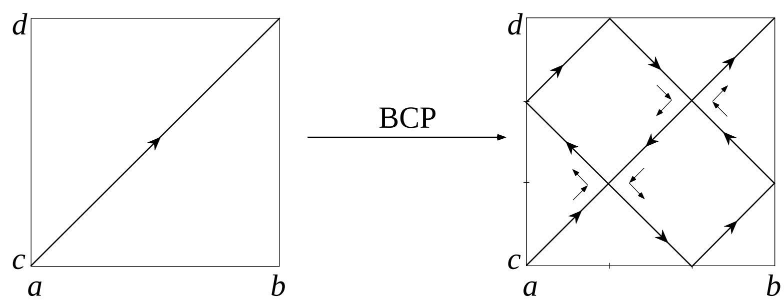

FIGURE 4.7. Basic construction BCP.

Using symmetries, BCP can be applied to either of the two diagonals of any square with either path orientation. Also, if $\|G\|_\infty = \sup_x |G(x)|$, then it is clear that

$$\|\lambda - \lambda'\|_\infty \leq \sqrt{(b-a)^2 + (d-c)^2} \tag{61}$$

for every $\lambda' : [\alpha, \beta] \to [a,b] \times [c,d]$.

To construct the Peano curve, let $P_0(t) = (t,t)$ and define P_1 by applying BCP to P_0. The image of P_1 consists of a diagonal from each of the nine squares

$$\left[\frac{i}{3}, \frac{i+1}{3}\right] \times \left[\frac{j}{3}, \frac{j+1}{3}\right], \quad i,j = 0,1,2.$$

(See Figure 4.7 with $a = c = 0$ and $b = d = 1$.) Construct P_2 by applying BCP to each of the diagonals of these squares as shown in Figure 4.8.

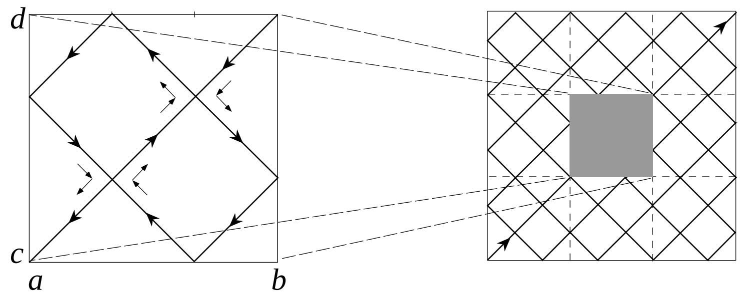

FIGURE 4.8. Construction of P_2.

This process can be continued inductively in the obvious way to form the sequence P_n, $n \in \mathbb{N}$. From (61) it follows that

$$\|P_n - P_m\|_\infty \leq \sqrt{2}\, 3^{-\min(n,m)}.$$

This shows that P_n converges uniformly to P. It is also easy to see that the image of P is a dense, compact subset of $[0,1] \times [0,1]$, so P is an area-filling curve.

If $P = (p_1, p_2)$, where $p_i : [0,1] \to [0,1]$, $i = 1, 2$ are the coordinate functions for P, then we claim $f = p_1$ is a function satisfying the conditions of Example 4.3.8.

To see this, it might be helpful to see how f can be defined directly as a uniformly convergent sequence of continuous functions $f_n : [0,1] \to [0,1]$, where each f_n is the first coordinate of P_n. The first coordinate of BCP can be represented by the construction shown in Figure 4.9. A similar construction can be done with either diagonal via an obvious reflection. This construction is denoted BCX.

Notice that Figure 4.9(B) also represents $f_1 : [0,1] \to [0,1]$, if we take $a = \alpha = 0$ and $b = \beta = 1$. To form f_2 it is enough to apply BCX to each linear segment of f_1. Then, apply BCX to each linear segment of f_2 to arrive at f_3, etc.

Evidently, f is continuous, as the first coordinate of the continuous function P. Also, by Example 1.4.5, it is density continuous and nowhere approximately differentiable.

In the rest of the proof, we will need the following easy observations.

The function P is self-similar in the sense that for every $n \in \mathbb{N}$ and every $i = 0, 1, \ldots, 9^n - 1$, there exist $l(i), r(i) \in \{0, 1, \ldots, 3^n - 1\}$ such that the image

$$P\left(\left[\frac{i}{9^n}, \frac{i+1}{9^n}\right]\right) = \left[\frac{l(i)}{3^n}, \frac{l(i)+1}{3^n}\right] \times \left[\frac{r(i)}{3^n}, \frac{r(i)+1}{3^n}\right], \tag{62}$$

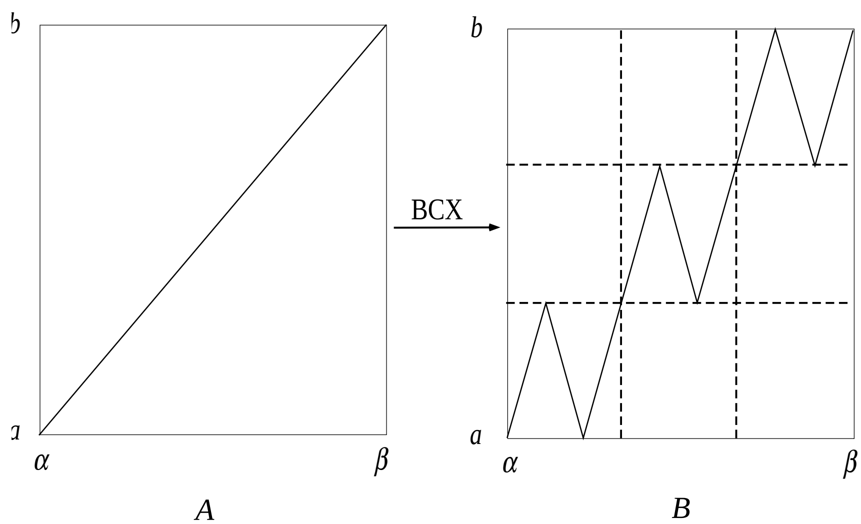

FIGURE 4.9. Basic construction BCX.

and the path followed is a scaled and reflected copy of the entire path of P in $[0,1]\times[0,1]$. Since f is a first coordinate of P, condition (62) implies also that for each integer $i \in \{0,1,\dots,9^n-1\}$, there is an integer $l(i) \in \{0,1,\dots,3^n-1\}$ such that

$$f\left(\left[\frac{i}{9^n},\frac{i+1}{9^n}\right]\right)=\left[\frac{l(i)}{3^n},\frac{l(i)+1}{3^n}\right]. \tag{63}$$

Also notice the following easy geometrical fact.

For every $t \in \mathbb{N}$, $t>1$, and nonempty interval $(a,b)\subset[0,1]$ there are $i,n\in\mathbb{N}$ such that

$$K=\left[\frac{i}{t^n},\frac{i+1}{t^n}\right]\subset(a,b) \quad \text{and} \quad \frac{\mathrm{m}(K)}{b-a}\geq\frac{1}{2t} \tag{64}$$

To see this, let n be the smallest natural number such that $1/t^n<(b-a)/2$. Thus, $2/t^{n-1}\geq(b-a)$ and there exists i such that $i/t^n\in(a,(b+a)/2)$. Hence, $K=[i/t^n,(i+1)/t^n]\subset(a,b)$ and $\mathrm{m}(K)/(b-a)\geq(1/t^n)/(2/t^{n-1})=1/2t$. This finishes the proof of (64).

Notice that (63) implies $f^{-1}(E)$ is nowhere dense for every nowhere dense set E. So,

$$f^{-1}(E)\in\mathcal{I} \ \text{ for every } \ E\in\mathcal{I}.$$

Thus, by Theorem 3.2.1, to show that f is $\mathcal{I}$-density continuous it is enough to prove that f is deep-$\mathcal{I}$-density continuous.

Let $x \in [0,1]$ and let $A \subset \mathbb{R} \setminus \{f(x)\}$ be a set such that $f(x)$ is a deep-$\mathcal{I}$-density point of A. By Corollary 2.7.11, A^c is superporous at $f(x)$. It must be shown that x is a deep-$\mathcal{I}$-dispersion point of $(f^{-1}(A))^c$; i.e., that $(f^{-1}(A))^c$ is superporous at x. This will be done with the aid of Lemma 2.4.2.

Let $s = 1/9^k \in (0,1)$. We must find $D_s > 0$ and $R_s \in (0,1)$ such that whenever $0 < D < D_s$ and an interval $I \subset (x-D, x+D) \setminus \{x\}$ with $\mathrm{m}(I)/D > s$, then there is an interval $J \subset I \cap f^{-1}(A)$ with

$$\frac{\mathrm{m}(J)}{\mathrm{m}(I)} > R_s. \tag{65}$$

Let $s' = s/9^3$. Using Lemma 2.4.2 with A^c and $f(x)$, there exists $D_{s'} > 0$ and $R_{s'} = 1/3^l \in (0,1)$ such that

- whenever $0 < D' \le D_{s'}$ and an interval

$$I' \subset (f(x) - D', f(x) + D') \setminus \{f(x)\}$$

with $\mathrm{m}(I')/D' \ge s'$, then there is an interval $J' \subset I' \cap A$ with

$$\mathrm{m}(J')/\mathrm{m}(I') > R_{s'}. \tag{66}$$

Let $D_s > 0$ be such that

$$|f(x) - f(y)| < D_{s'} \quad \text{for} \quad |x - y| < D_s \tag{67}$$

and let $R_s = 1/9^{l+5}$. Let $0 < D < D_s$ and choose an interval $I \subset (x - D, x + D) \setminus \{x\}$ with $\mathrm{m}(I)/D > s$. We will find an interval $J \subset I \cap f^{-1}(A)$ with $\mathrm{m}(J)/\mathrm{m}(I) > R_s$.

Assume that $I \subset (x, x + D)$. The other case is similar.

Using (64), we can find $I_0 = [j/9^{n-1}, (j+1)/9^{n-1}] \subset I$ such that

$$\frac{\mathrm{m}(I_0)}{\mathrm{m}(I)} \ge \frac{1}{18}. \tag{68}$$

Moreover, using (62), it is easy to find $I_1 = [i/9^n, (i+1)/9^n] \subset I_0$ such that $f(x) \notin f(I_1)$. Thus,

$$\frac{\mathrm{m}(I_1)}{D} = \frac{1}{9}\frac{\mathrm{m}(I_0)}{D} \ge \frac{1}{9}\frac{1}{18}\frac{\mathrm{m}(I)}{D} > \frac{s}{9^3} = s'.$$

In particular, there exist $p = (s')^{-1}$ contiguous intervals $I^1, I^2, \dots, I^p$ of length $1/9^n$, one of which is I_1 and such that $x \in I^1 \cup I^2 \cup \ldots \cup I^p$.

Define

$$D' = \max\{|f(x) - f(i/9^n)|, |f(x) - f((i+1)/9^n)|\} > 0$$

and $I' = f(I_1)$. By (67) we see that $D' < D_s$ and, by (62), $f(i/9^n)$ and $f((i+1)/9^n)$ are the end points of I' so that $I' \subset [f(x) - D', f(x) + D'] \setminus \{f(x)\}$.

Moreover, since $x, i/9^n, (i+1)/9^n \in I^1 \cup I^2 \cup \ldots \cup I^p$ then, by (63), we have

$$D' \leq \mathrm{m}\left(f\left(\bigcup_{j=1}^{p} I^j\right)\right) \leq \sum_{j=1}^{p} \mathrm{m}(f(I^j)) = p\mathrm{m}(I').$$

Hence,

$$\frac{\mathrm{m}(I')}{D'} \geq \frac{\mathrm{m}(I')}{p\mathrm{m}(I')} = p^{-1} = s'.$$

Thus, by (66), there is an interval $J' \subset I' \cap A$ such that $\mathrm{m}(J')/\mathrm{m}(I') > R_{s'}$.

Using (64), we can find an interval

$$J_1' = [j_0/3^m, (j_0+1)/3^m] \subset J'$$

such that $\mathrm{m}(J_1')/\mathrm{m}(J') \geq 1/6 > 1/9$. Hence,

$$\frac{\mathrm{m}(J_1')}{\mathrm{m}(f(I_1))} = \frac{\mathrm{m}(J_1')}{\mathrm{m}(I')} = \frac{\mathrm{m}(J_1')}{\mathrm{m}(J')}\frac{\mathrm{m}(J')}{\mathrm{m}(I')} > \frac{1}{9}R_{s'} = \frac{1}{3^{l+2}}$$

and $J_1' = [j_0/3^m, (j_0+1)/3^m] \subset f(I_1) = f([i/9^n, (i+1)/9^n])$. But now condition (62) implies easily that there exists an interval $J = [j/9^m, (j+1)/9^m] \subset I_1 = [i/9^n, (i+1)/9^n]$ such that $f(J) = J_1'$ and

$$\frac{\mathrm{m}(J)}{\mathrm{m}(I_1)} > \left(\frac{1}{3^{l+2}}\right)^2 = \frac{1}{9^{l+2}}.$$

Hence, by (68),

$$\frac{\mathrm{m}(J)}{\mathrm{m}(I)} \geq \frac{\mathrm{m}(J)}{18\mathrm{m}(I_0)} = \frac{1}{9}\frac{\mathrm{m}(J)}{18\mathrm{m}(I_1)} > \frac{1}{9^3}\frac{1}{9^{l+2}} = R_s.$$

Condition (65) is proved. This finishes the proof that f is $\mathcal{I}$-density continuous.

To see that f is not $\mathcal{I}$-approximately differentiable at a point $x \in [0,1]$ let us do the following construction for each $n \in \mathbb{N}$. Choose $i \in \mathbb{N}$ such that $x \in [i/9^n, (i+1)/9^n]$. Then, by (63), $f([i/9^n, (i+1)/9^n]) = [j/3^n, (j+1)/3^n]$ for some $j \in \mathbb{N}$. It is also not difficult to see that condition (62) implies that

$$\left\{f\left(\left[\frac{9i}{9^{n+1}}, \frac{9i+1}{9^{n+1}}\right]\right), f\left(\left[\frac{9i+8}{9^{n+1}}, \frac{9i+9}{9^{n+1}}\right]\right)\right\}$$
$$= \left\{\left[\frac{3j}{3^{n+1}}, \frac{3j+1}{3^{n+1}}\right], \left[\frac{3j+2}{3^{n+1}}, \frac{3j+3}{3^{n+1}}\right]\right\}.$$

This implies, in particular, that for every $y \in [9i/9^{n+1}, (9i+1)/9^{n+1}]$ and $y' \in [(9i+8)/9^{n+1}, (9i+9)/9^{n+1}]$ we have

$$\frac{|f(y) - f(y')|}{|y - y'|} \geq \frac{1/3^{n+1}}{1/9^n} = 3^{n-1}.$$

Hence, an easy geometrical argument implies that for one of the intervals

$$[9i/9^{n+1}, (9i+1)/9^{n+1}] \text{ or } [(9i+8)/9^{n+1}, (9i+9)/9^{n+1}],$$

which we denote by $[a_n, b_n]$, we have $x \notin [a_n, b_n]$ and

$$\frac{|f(y) - f(x)|}{|y - x|} \geq 3^{n-1} \quad \text{for every} \quad y \in [a_n, b_n].$$

But, by Lemma 2.2.6, x is not an $\mathcal{I}$-dispersion point of $\bigcup_{n\in\mathbb{N}}[a_n, b_n]$. Thus, for every $\mathcal{I}$-density open set U containing x, for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there is an $y \in (x - \varepsilon, x + \varepsilon) \cap U \cap \bigcup_{m>n}[a_m, b_m]$ for which

$$\frac{|f(y) - f(x)|}{|y - x|} \geq 3^n.$$

This implies that f is not $\mathcal{I}$-approximately differentiable. Also notice that the construction of the intervals $[a_n, b_n]$ given above also implies that f is not approximately differentiable. This finishes the proof of Example 4.3.8. $\square$

Notice that Proposition 4.3.3 immediately gives the following corollary.

Corollary 4.3.9. *$\Delta_{\mathcal{I}}^{(\mathcal{D})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}} = \Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ and $\Delta_{\mathcal{I}}^{(\mathcal{D})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} = \Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$.*

Because of the corollary given above we no longer need consider the classes $\Delta_{\mathcal{I}}^{(\mathcal{D})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta_{\mathcal{I}}^{(\mathcal{D})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$.

Now, we are ready to discuss the inclusions between the other classes.

Theorem 4.3.10. *If $\mathcal{F}_c$ stands for the Baire functions, then*

$$\begin{array}{ccccccc}
\Delta & \subset & \Delta^{(\mathcal{I})} & \subset & \Delta_{\mathcal{I}}^{(\mathcal{I})} & \subset & \mathcal{F}_c \\
\cup & & \cup & & \cup & & \cup \\
\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}} & \subset & \Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}} & \subset & \Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}} & \subset & \mathcal{C}_{\mathcal{D}\mathcal{D}} \\
\cup & & \cup & & \cup & & \cup \\
\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} & \subset & \Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} & \subset & \Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} & \subset & \mathcal{C}_{\mathcal{I}\mathcal{I}}
\end{array}$$

and all the inclusions are proper.

Proof. The inclusions are obvious.

The vertical inclusions are proper because $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}} \not\subset \mathcal{C}_{\mathcal{I}\mathcal{I}}$ (Example 3.6.3) and $\Delta \not\subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$ (Corollary 3.2.6).

The fact that the horizontal inclusions are proper follows from $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \Delta$ (Example 4.3.6), from $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \Delta^{(\mathcal{I})}$ which is proved by $f(x) = |x|$ and from $\mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \Delta_{\mathcal{I}}^{(\mathcal{I})}$ (Example 4.3.8).

This finishes the proof. $\square$

To prove the next theorem we need the following two lemmas. The first one is a version of the chain rule. The analogous theorem for approximate derivatives can be found in [**53**].

Lemma 4.3.11. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be functions such that for some x_0, $D^{(\mathcal{I})}f(x_0)$ and $D^{(\mathcal{I})}g(f(x_0))$ both exist, then $D^{(\mathcal{I})}(g \circ f)(x_0)$ also exists and*

$$D^{(\mathcal{I})}(g \circ f)(x_0) = D^{(\mathcal{I})}g(f(x_0))\, D^{(\mathcal{I})}f(x_0). \tag{69}$$

PROOF. First assume that $D^{(\mathcal{I})}g(f(x_0)) > 0$ and $D^{(\mathcal{I})}f(x_0) > 0$. Put $z_0 = f(x_0)$ and let u_1, u_2, v_1, v_2 be arbitrary positive numbers such that $0 < u_1 < D^{(\mathcal{I})}f(x_0) < u_2$ and $0 < v_1 < D^{(\mathcal{I})}g(z_0) < v_2$. Then, there exist sets $U, V \in \mathcal{T}_D$, $x_0 \in U$, $z_0 \in V$, such that

$$u_1 \le \frac{f(x) - f(x_0)}{x - x_0} \le u_2 \tag{70}$$

for all $x \in U$ and

$$v_1 \le \frac{g(z) - g(z_0)}{z - z_0} \le v_2 \tag{71}$$

for all $z \in V$. Since, by Proposition 4.3.1, f is $\mathcal{I}$-density continuous at x_0, there exists $W \in \mathcal{T}_\mathcal{I}$ such that $x_0 \in W \subset f^{-1}(V)$. If $f(x_0)$ is not an $\mathcal{I}$-dispersion point of $f^{-1}(f(x_0))$, then $D^{(\mathcal{I})}f(x_0) = 0$ contradicting our assumption. So, we can assume that $f(x) \ne f(x_0)$ for every $x \in W$, $x \ne x_0$. In particular, by (71), for $x \in W \setminus \{x_0\}$ we have $z = f(x) \in V \setminus \{f(x_0)\}$ and

$$v_1 \le \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \le v_2. \tag{72}$$

Multiplying (70) by (72) gives

$$u_1 v_1 < \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \, \frac{f(x) - f(x_0)}{x - x_0} < u_2 v_2;$$

i.e.,

$$u_1 v_1 < \frac{g(f(x)) - g(f(x_0))}{x - x_0} < u_2 v_2,$$

for every $x \in U \cap W \in \mathcal{T}_\mathcal{I}$. But then,

$$u_1 v_1 < D^{(\mathcal{I})}g(f(x_0))\, D^{(\mathcal{I})}f(x_0) < u_2 v_2$$

where the numbers $u_1 v_1$ and $u_2 v_2$ can be chosen as close to the value of

$$D^{(\mathcal{I})}g(f(x_0))\, D^{(\mathcal{I})}f(x_0)$$

as desired. This implies (69).

The remaining cases, when $D^{(\mathcal{I})}g(f(x_0)) \le 0$ or $D^{(\mathcal{I})}f(x_0) \le 0$, are very similar, modulo some little technical problems with the signs of the inequalities. This finishes the proof. □

LEMMA 4.3.12. *If $f \in \mathcal{C}_{\mathcal{D}\mathcal{D}} \cap \Delta_\mathcal{I}^{(\mathcal{I})}$ and $g \in \Delta_\mathcal{I}^{(\mathcal{I})}$, then $g \circ f \in \Delta_\mathcal{I}^{(\mathcal{I})}$.*

PROOF. Let $A, B \in \mathcal{I}$ be $\mathbf{F}_\sigma$ sets such that f is $\mathcal{I}$-approximately differentiable on A^c and g is $\mathcal{I}$-approximately differentiable on B^c.

Let $E_x = f^{-1}(x)$ for every $x \in \mathbb{R}$. Note that

$$\widetilde{E}_x \subset E_x \quad \text{for every} \quad x \in \mathbb{R}. \tag{73}$$

This is true because any point $y \in \widetilde{E}_x$ is an $\mathcal{I}$-density point of the set E_x on which f is equal x. So $f(y) = x$. Hence, $g \circ f$ is $\mathcal{I}$-approximately differentiable on every set $\widetilde{E}_x \in \mathcal{T}_\mathcal{I}$ because $g \circ f$ is constant on this set. This implies that $g \circ f$ is $\mathcal{I}$-approximately differentiable on the set $E = \bigcup_{x \in \mathbb{R}} \widetilde{E}_x$. Moreover, by Lemma 4.3.11 the function $g \circ f$ is also $\mathcal{I}$-approximately differentiable on the set $D = A^c \cap f^{-1}(B^c)$.

To finish the proof it is enough to show that $(D \cup E)^c \in \mathcal{I}$. But

$$(D \cup E)^c = D^c \cap E^c = \left[A \cup f^{-1}(B)\right] \cap E^c \subset A \cup \left(f^{-1}(B) \setminus E\right),$$

so it is enough to show that $P = f^{-1}(B) \setminus E \in \mathcal{I}$.

By way of contradiction let us assume that $P \notin \mathcal{I}$. Then, there exists a nontrivial closed interval $I \subset \tilde{P}$. By Theorem 3.5.2, choosing a subinterval of I, if necessary, we can assume that f is continuous on I. But $(f|_I)^{-1}(B) = f^{-1}(B) \cap I$ is residual in I, while $B \in \mathcal{I}$. The only way it can happen for a continuous function $f|_I$ is when $(f|_I)^{-1}(F) = f^{-1}(F) \cap I \notin \mathcal{I}$ for some closed nowhere dense set $F \subset B$. But this easily implies that there exists $x \in F$ such that $(f|_I)^{-1}(x) = f^{-1}(x) \cap I$ has a nonempty interior. This yields a the contradiction to the fact that

$$f^{-1}(x) \cap I \subset E_x \cap \tilde{P} \in \mathcal{I},$$

which follows from $P \cap \tilde{E}_x \subset P \cap E = \emptyset$. This finishes the proof. □

As a corollary we obtain

Theorem 4.3.13. *The classes Δ, $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta^{(\mathcal{I})}_\mathcal{I} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta^{(\mathcal{I})}_\mathcal{I} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ are closed under composition. In particular, they form semigroups.*

Proof. The classes Δ, $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ are obviously closed under composition.

The classes $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ form semigroups, as Δ is closed under composition.

For the classes $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ the conclusion follows immediately from Lemma 4.3.11, while for the classes $\Delta^{(\mathcal{I})}_\mathcal{I} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta^{(\mathcal{I})}_\mathcal{I} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ it follows from Lemma 4.3.12. □

Problem 4.3.14. *Are the classes $\Delta^{(\mathcal{D})}_\mathcal{I}$ and $\Delta^{(\mathcal{I})}_\mathcal{I}$ closed under composition? If so, do they have the inner automorphism property?*

We finish this section by showing that the remaining classes $\Delta^{(\mathcal{I})}$ and $\mathcal{F}_c$ of Theorem 4.3.10 are not closed under composition.

Example 4.3.15. *There exists a $\mathcal{C}^\infty$ function f and a function g which is approximately differentiable and $\mathcal{I}$-approximately differentiable such that $g \circ f$ is neither approximately nor $\mathcal{I}$-approximately continuous. In particular, the classes $\Delta^{(\mathcal{N})}$ and $\Delta^{(\mathcal{I})}$ are not closed under composition.*

PROOF. Let $D = \bigcup_{n\in\mathbb{N}}[p_n, q_n]$ be a right interval set for which 0 is a dispersion and $\mathcal{I}$-dispersion point. Let us define $h\colon \mathbb{R} \to \mathbb{R}$ by putting $h(x) = 0$ for $x \in D^c$ and

$$h(x) = u_n e^{(-(x-p_n)^{-2}-(x-q_n)^{-2})}$$

for $x \in [p_n, q_n]$, where u_n are chosen in such a way that $h^{(i)}(x) \le 1/n$ for $x \in [p_n, q_n]$ and $i \le n$. Define f by $f(x) = \int_0^x h(y)\,dy$. It is easy to see that f is $\mathcal{C}^\infty$ and constant on each interval contained in D^c. Decreasing the constants u_n, if necessary, we may also assume that $\lim_{n\to\infty} f(p_{n+1})/f(p_n) = 0$.

Choose intervals (a_n, b_n) centered at $f(p_n)$ such that 0 is a dispersion and $\mathcal{I}$-dispersion point of a right interval set $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$. Define $g(x) = 0$ for $x \in E^c$ and

$$g(x) = v_n e^{(-(x-a_n)^{-2}-(x-b_n)^{-2})}$$

for $x \in (a_n, b_n)$, where the v_n are chosen in such a way that $g(f(p_n)) = 1$. Then g has derivatives of all orders at every point $\neq 0$ and g constantly equals 0 on $E^c \in \mathcal{T}_{\mathcal{I}} \cap \mathcal{T}_{\mathcal{N}}$. Hence, $g \in \Delta^{(\mathcal{N})} \cap \Delta^{(\mathcal{I})}$. On the other hand, $g \circ f = 1$ on the set $D^c \cap (0, q_1)$ while, $g \circ f(0) = 0$. Thus, $g \circ f \notin \Delta^{(\mathcal{N})} \cup \Delta^{(\mathcal{I})}$. □

EXAMPLE 4.3.16. *The class $\mathcal{F}_c$ is not closed under composition.*

PROOF. Let h be an embedding of the irrationals $\mathbb{R} \setminus \mathbb{Q}$ into the Cantor set $C \subset [0, 1]$. Put $f(q) = 2$ for $q \in \mathbb{Q}$ and $f(x) = h(x)$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Then, $f \in \mathcal{F}_c$. Moreover, let $S \subset \mathbb{R} \setminus \mathbb{Q}$ be a set without the Baire property and let g be the characteristic function of $f(S)$; i.e., $g = \chi_{f(S)}$. Then $g \in \mathcal{F}_c$, since $f(S) \subset C$ and $g \circ f = \chi_S \notin \mathcal{F}_c$. □

4.4. Semigroups of $\mathcal{I}$-density Continuous Functions

For the remainder of this section let $\mathcal{E}$ stand for any of the semigroups of Theorem 4.3.13 except for Δ; e.g., for one of the classes $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta^{(\mathcal{I})}_{\mathcal{I}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ or $\Delta^{(\mathcal{I})}_{\mathcal{I}} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$. In particular, $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \subset \mathcal{E} \subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$. We will show that these semigroups, with the possible exception of $\Delta^{(\mathcal{I})}_{\mathcal{I}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta^{(\mathcal{I})}_{\mathcal{I}} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, have the inner automorphism property.

We start with the following

LEMMA 4.4.1. *If h is a generating bijection of an automorphism Ψ of $\mathcal{E}$, then h is $\mathcal{I}$-approximately continuous.*

PROOF. Let $r \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. We will show that h is $\mathcal{I}$-approximately continuous at r.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be as in Example 4.3.5 for $a = h(r) - \varepsilon$, $b = c = h(r)$ and $d = h(r) + \varepsilon$. Let $g\colon \mathbb{R} \to \mathbb{R}$ with $g(x) = f(x) + h(r)$. It is easy to see that $g \in \Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \subset \mathcal{E}$.

Ψ is an automorphism of $\mathcal{E}$, thus there is an $\alpha \in \mathcal{E} \subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$ such that $\Psi(\alpha) = g$. We have $h \circ \alpha = g \circ h$. Note that $h(\alpha(r)) = g(h(r)) = f(h(r)) + h(r) = 1 + h(r)$. Therefore, $h(\alpha(r)) \neq h(r)$. But, h is a bijection, so $\alpha(r) \neq r$.

The function α is deep-$\mathcal{I}$-density continuous. Thus, there exists a set $U \in \mathcal{T}_D$ such that $\alpha(x) \neq r$ for all $x \in U$. Then, for all $x \in U$, $f(h(x)) + h(r) = g(h(x)) = h(\alpha(x)) \neq h(r)$, which implies $f(h(x)) \neq 0$; i.e., $h(x) \in (a, c) = (h(r) - \varepsilon, h(r) + \varepsilon)$. So $|h(x) - h(r)| < \varepsilon$ for $x \in U$. This means precisely that h is $\mathcal{I}$-approximately continuous at r. □

Corollary 4.4.2. *If h is a generating bijection of an automorphism Ψ of $\mathcal{E}$, then h is a homeomorphism.*

Proof. h is an $\mathcal{I}$-approximately continuous bijection of $\mathbb{R}$. Since h is also a Darboux Baire one function, it must be a homeomorphism. □

As a next step we need the following

Lemma 4.4.3. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism which is not $\mathcal{I}$-density continuous at 0. Then there exists a function $f \in \mathcal{C}^\infty \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \subset \mathcal{E}$ such that $h^{-1} \circ f \circ h$ is not deep-$\mathcal{I}$-density continuous.*

Proof. Without any loss of generality we may assume that $h(0) = 0$.

As h is a homeomorphism, h is also not deep-$\mathcal{I}$-density continuous at 0. Again, without any loss of generality we may further assume that h is not right deep-$\mathcal{I}$-density continuous at 0. The left-hand side argument is essentially the same.

Thus, there exists a right interval set $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ such that 0 is an $\mathcal{I}$-dispersion point of E, while 0 is not an $\mathcal{I}$-dispersion point of

$$D = \bigcup_{n\in\mathbb{N}} [h^{-1}(a_n), h^{-1}(b_n)].$$

Using the definition of a deep-$\mathcal{I}$-dispersion point, it is possible to choose sequences α_n and β_n such that $h^{-1}(a_n) < \alpha_n < \beta_n < h^{-1}(b_n)$ for $n \in \mathbb{N}$ and 0 is still not an $\mathcal{I}$-dispersion point of $\bigcup_{n\in\mathbb{N}}[\alpha_n, \beta_n]$, while 0 is an $\mathcal{I}$-dispersion point of

$$\bigcup_{n\in\mathbb{N}} [h(\alpha_n), h(\beta_n)] \subset \bigcup_{n\in\mathbb{N}} [a_n, b_n].$$

For each $n \in \mathbb{N}$, let $f_n \in \mathcal{C}_{\mathcal{I}\mathcal{I}} \cap \mathcal{C}^\infty$ be a function such that $f_n(x) = 0$ for $x \notin (a_n, b_n)$ and $f_n(x) = 1$ for $x \in [h(\alpha_n), h(\beta_n)]$. This can be done by Example 4.3.5.

Define $f(x) = 0$ for $x \in E^c$ and $f(x) = c_n f_n(x)$ for $x \in [a_n, b_n]$ and $n \in \mathbb{N}$, where $c_n > 0$ are choosen in such a way that

$$c_n f_n^{(i)}(x) \leq \frac{1}{n} \quad \text{for every } x \in [a_n, b_n] \quad \text{and} \quad i \leq n$$

and

$$\lim_{n\to\infty} \frac{h^{-1}(c_{n+1})}{h^{-1}(c_n)} = 0. \tag{74}$$

Then f is a $\mathcal{C}^\infty$ function. It is also easy to see that f is $\mathcal{I}$-density continuous, because 0 is an $\mathcal{I}$-dispersion point of E.

On the other hand

$$T = h^{-1} \circ f \circ h \left(\bigcup_{n \in \mathbb{N}} (\alpha_n, \beta_n) \right) = \bigcup_{n \in \mathbb{N}} h^{-1}(c_n).$$

Now, by (74) and Lemma 2.1.6, it is apparent that there is a right interval set containing T which has 0 as a deep-$\mathcal{I}$-dispersion point, while 0 is not an $\mathcal{I}$-dispersion point of $\bigcup_{n\in\mathbb{N}}[\alpha_n, \beta_n]$. Hence, $h^{-1} \circ f \circ h$ is not deep-$\mathcal{I}$-density continuous. □

Corollary 4.4.4. *If h is a generating bijection of an automorphism of $\mathcal{E}$, then h and h^{-1} are $\mathcal{I}$-density continuous homeomorphisms of $\mathbb{R}$.*

Proof. If h is a generating bijection for Ψ, then h^{-1} is a generating bijection for Ψ^{-1}. Thus, by Corollary 4.4.2, h and h^{-1} are the homeomorphisms of $\mathbb{R}$. So, by Lemma 4.4.3 they must be $\mathcal{I}$-density continuous.

As an immediate corollary we obtain the following.

Theorem 4.4.5. *The semigroups $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ have the inner automorphism property. Moreover,*

$$Aut(\mathcal{C}_{\mathcal{I}\mathcal{I}}) = Aut(\mathcal{C}_{\mathcal{D}\mathcal{D}}). \quad \square$$

Let us notice also that we can also use Proposition 4.1.2 and Theorem 2.7.8(vii) to conclude immediately that the class $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ has the inner automorphism property. However, this argument does not work for $\mathcal{C}_{\mathcal{I}\mathcal{I}}$.

To discuss the inner automorphism properties of the other semigroups we need the following two lemmas.

Lemma 4.4.6. *The operations ordinary diffferentiation, approximate differentiation and $\mathcal{I}$-approximate differentiation coincide on the class of all homeomorphisms. In other words, if $f\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism and $x \in \mathbb{R}$ then $f'(x) = D^{(\mathcal{I})}f(x) = D^{(\mathcal{N})}f(x)$ whenever any of the three quantities in this equation exists.*

Proof. There is no generality lost with the assumption that f is an increasing homeomorphism, $x = 0$ and $f(0) = 0$. It is obvious that if $f'(0)$ exists, then both the other derivatives exist and are equal to it. It suffices to show that if $D^{(\mathcal{I})}f(0)$ or $D^{(\mathcal{N})}f(0)$ exists, then so does $f'(0)$.

So, suppose that $D^{(\mathcal{I})}f(0)$ exists, but $f'(0)$ fails to exist. At least one of the four Dini derivatives of f at 0 must be unequal to $D^{(\mathcal{I})}f(0)$. So, for example, assume there are constants $M > m > 0$ such that

$$\overline{D}^{+} f(0) > M > m > D^{(\mathcal{I})}f(0) \geq 0.$$

There must exist a sequence of numbers x_n decreasing to 0 such that $f(x_n) > Mx_n$ for all n. Since f is nondecreasing, an elementary calculation shows that

(75) $$f(x) > mx \text{ for all } x \text{ from the interval } [x_n, Mx_n/m].$$

Let $E = \{x : f(x) < mx\}$. By assumption, 0 is an $\mathcal{I}$-density point of E. Consider the sequence of sets

$$E_n = \frac{m}{Mx_n}E.$$

An easy calculation with (75) shows that $E_n \cap [m/M, 1] = \emptyset$ for all n. Therefore, $\liminf_{n\to\infty} E_n$ cannot be residual in $(0, 1)$. Since m/Mx_n increases monotonically to infinity with n, this violates the fact that 0 is an $\mathcal{I}$-density point of E. This contradiction shows that $\overline{D}^+ f(0) = D^{(\mathcal{I})} f(0)$. The arguments with the other three Dini derivatives are essentially the same.

The proof in the case of the approximate derivative is similar. □

It is interesting to note that the proof of Lemma 4.4.6 is true with almost no change in the case of the generalized density topologies of Chapter 1, as long as the underlying ideal contains no intervals.

LEMMA 4.4.7. *Let Ψ be an automorphism of $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ or $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ and let h be its generating bijection. Then h is differentiable; i.e., $h \in \Delta$.*

PROOF. It is already known that h is a homeomorphism. Thus h is differentiable almost everywhere [**7**]. Let x_0 be a point of differentiability of h and let $x \in \mathbb{R}$ be any other point. Define $f(t) = t + (x - x_0)$. Then $f \in \Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, so it belongs to any of the semigroups under consideration. For any positive δ we have

$$\begin{aligned}(76) \qquad \frac{h(x+\delta) - h(x)}{\delta} &= \frac{h \circ f(x_0 + \delta) - h \circ f(x_0)}{\delta} \\ &= \frac{\Psi(f) \circ h(x_0 + \delta) - \Psi(f) \circ h(x_0)}{\delta}.\end{aligned}$$

If Ψ is an automorphism of either $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ or $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, then $\Psi(f)$ is differentiable, and since h is differentiable at x_0, the quotient in (76) converges to $\Psi(f)'(h(x_0))\, h'(x_0)$.

If Ψ is an automorphism of either $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ or $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ then $\Psi(f)$ is $\mathcal{I}$-approximately differentiable. So, by Lemma 4.3.11, $\Psi(f) \circ h$ is $\mathcal{I}$-approximately differentiable at x_0. Hence, condition (76) guarantees that $D^{(\mathcal{I})}h(x)$ exists and is equal $D^{(\mathcal{I})}(\Psi(f) \circ h)(x_0)$. But, h is a homeomorphism. Thus, by Lemma 4.4.6, h must be differentiable at x as well. This ends the proof. □

As an immediate corollary we obtain

THEOREM 4.4.8. *The semigroups $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ have the inner automorphism property. Moreover,*

$$Aut(\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}) = Aut(\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}) = Aut(\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}) = Aut(\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}})$$

$$\subset Aut(\mathcal{C}_{\mathcal{I}\mathcal{I}}) = Aut(\mathcal{C}_{\mathcal{D}\mathcal{D}})$$

and the inclusion is proper.

We are not able to prove or disprove whether the semigroups $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ have the inner automorphism property. We proved that each automorphism of $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ is generated by a homeomorphism h of the real line so it must be differentiable almost everywhere. However, to be an element of $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, h would need to be differentiable outside of a set of first category. In general, a homeomorphism of the real line need not be $\mathcal{I}$-a.e. differentiable. In fact, Belna, Cargo, Evans and Humke [**3**] show that there exists a strictly increasing homeomorphism $h : [0, 1] \to [0, 1]$ and $\tau \in (0, 1)$ such that

$$\underline{D}^- h(x) = \underline{D}^+ h(x) = \tau \text{ and } \overline{D}^- h(x) = \overline{D}^+ h(x) = \infty$$

for a residual set of points $x \in [0, 1]$. On the other hand, there are certain restrictions on the "severity" of the nondifferentiability of h on sets of the second category. Neugebauer [**49**] shows that a continuous function f has $\underline{D}^- f(x) = \underline{D}^+ f(x)$ and $\overline{D}^- f(x) = \overline{D}^+ f(x)$ on a residual set.

We are not able to find a satisfactory answer to our question. Therefore, the following remains.

PROBLEM 4.4.9. *Let h be a homeomorphism of $\mathbb{R}$ such that for every $\mathcal{I}$-density continuous, $\mathcal{I}$-a.e. $\mathcal{I}$-approximately differentiable f, $h \circ f \circ h^{-1}$ is also $\mathcal{I}$-density continuous, $\mathcal{I}$-a.e. and $\mathcal{I}$-approximately differentiable. Is h differentiable on a residual set?*

Clearly, an affirmative answer to this problem is equivalent to the fact that $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ have the inner automorphism property.

4.5. Historical and Bibliographic Notes

The definition of generated space we are using can be found in Magill [**44**, p. 198]. The term "generated space" was originally introduced by Warndof [**62**] in 1969. However, the notion existed implicitly several years earlier. In particular, the theorem that the class of all generated spaces is S-admissible can be found in a 1962 paper of Shneperman [**60**]. The same result, or at least a close approximation to it, was also proved by Gavrilov [**27**] in 1964, by Mal'cev [**45**] in 1966 and by Magill [**42**] in 1967, each apparently unaware of the others' work.

The $\mathcal{I}$-approximate derivative was introduced in the early 1980's by Łazarow and Wilczyński [**41**]. Lemma 4.4.6 is well-known in the case of the approximate derivative [**7**, Theorem 2.4, p. 155] and was first stated without proof for the $\mathcal{I}$-approximate derivative by Łazarow and Wilczyński [**68**, Theorem 38] while citing [**41**]. Related results are contained in [**39**].

Proposition 4.3.3 was stated, without a proof, by the authors in [**22**]. This paper also contains Theorems 4.2.1, 4.3.7, 4.3.10, 4.3.13, 4.4.5 and most of 4.4.8. Lemmas 4.3.11, 4.4.1, 4.4.7, Propositions 4.3.1, 4.3.2 and Examples 4.2.5, 4.3.6 as well as weaker versions of Examples 4.3.5, 4.3.15 and Lemmas 4.4.3, 4.3.12

are also presented in that paper. Example 4.3.8 appears there as a problem. It was also published in [**14**].

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APPENDIX A

Notation

Sets

- $\mathbb{R}$ – the set of real numbers.
- $\mathbb{Q}$ – the set of rationals numbers.
- $\mathbb{N}$ – the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$.

Algebras

- $\mathcal{L}$ – the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$.
- $\mathcal{B}$ – the σ-algebra of subsets of $\mathbb{R}$ having the Baire property.

Operations on Sets

- $A \triangle B$ – the symmetric difference of the sets A and B, i.e., $A \triangle B = (A \cup B) \setminus (A \cap B)$.
- A^c – the complement of the set A.
- X^Y – the set of all functions $f : X \to Y$.
- $\mathcal{P}(A)$ – the power set of A.
- $a + E = E + a = \{a + x \in \mathbb{R} \colon x \in E\}$ for $a \in \mathbb{R}, E \subset \mathbb{R}$.
- $-a + E = E - a = \{x - a \in \mathbb{R} \colon x \in E\}$ for $a \in \mathbb{R}, E \subset \mathbb{R}$.
- $cE = Ec = \{cx \in \mathbb{R} \colon x \in E\}$ for $a \in \mathbb{R}, E \subset \mathbb{R}$.
- $\mathrm{int}\,(A)$ – the interior of the set $A \subset \mathbb{R}$, in the natural topology.
- $\mathrm{cl}\,(A)$ – the closure of a set $A \subset \mathbb{R}$, in the natural topology.
- $\tilde{E}$ – the unique regular open set for $E \in \mathcal{B}$ such that $E \triangle \tilde{E} \in \mathcal{I}$.
- $\mathrm{m}_i(A)$ – the inner Lebesgue measure of the set $A \subset \mathbb{R}$.
- $\mathrm{m}_o(A)$ – the outer Lebesgue measure of the set $A \subset \mathbb{R}$.
- $\mathrm{m}(A)$ – the Lebesgue measure of the measurable set $A \subset \mathbb{R}$.
- $\mathrm{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$ for $A, B \subset \mathbb{R}$.
- $\mathrm{card}\,(A)$ – the cardinality of the set A.
- χ_A – the characteristic function of the set $A \subset \mathbb{R}$.
- $\mathrm{Aut}(H)$ – the group of authomorphisms of the semigroup H.

Ideals

- $\mathcal{O} = \{\emptyset\}$.
- $\mathcal{I}_\omega$ – the σ-ideal of countable subsets of $\mathbb{R}$.
- $\mathcal{N}$ – the σ-ideal of subsets of $\mathbb{R}$ with Lebesgue measure 0.
- $\mathcal{I}_0$ – the ideal of nowhere dense subsets of $\mathbb{R}$.
- $\mathcal{I}$ – the σ-ideal of the first category subsets of $\mathbb{R}$.

Topologies

- $\mathcal{T}_\mathcal{N}$ – the density topology on $\mathbb{R}$.
- $\mathcal{T}_\mathcal{O}$ – the natural topology on $\mathbb{R}$.
- $\mathcal{T}_Q$ – the qualitative topology on $\mathbb{R}$.
- $\mathcal{T}_\mathcal{I} = \mathcal{P}^*$ – the $\mathcal{I}$-density topology on $\mathbb{R}$.
- $\mathcal{T}_D = \mathcal{P}$ – the deep-$\mathcal{I}$-density topology on $\mathbb{R}$.
- $\mathcal{T}''_\mathcal{J}$ – the abstract $\mathcal{J}$-density topology on $\mathbb{R}$.

Function Classes

- $\mathcal{F}_m$ – the class of all measurable functions from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$.
- $\mathcal{F}_c$ – the class of all Baire functions from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$.
- $\mathcal{C}$ – the class of all continuous functions from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$.
- Δ – the class of all differentiable functions from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$.
- $\mathcal{C}^1$ – the class of all differentiable functions from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$ whose derivatives are continuous.
- $\mathcal{C}^\infty$ – the class of all infinitely many times differentiable functions from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$.
- $\mathcal{A}$ – the class of all analytic functions from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$.
- $\mathcal{H}$ – the class of all homeomorphisms from $\mathbb{R}$ (or any subinterval) to $\mathbb{R}$ (or any subinterval).
- $C_{\mathcal{J}K}$ – the class of all continuous functions $f\colon (\mathbb{R}, \mathcal{T}_\mathcal{J}) \to (\mathbb{R}, \mathcal{T}_\mathcal{K})$.
- $\mathcal{C}_{\mathcal{O}\mathcal{O}}$ – the class of all ordinary continuous functions.
- $\mathcal{C}_{\mathcal{N}\mathcal{O}}$ – the class of all approximately continuous functions.
- $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ – the class of all density continuous functions.
- $\mathcal{C}_{\mathcal{I}\mathcal{O}}$ – the class of all $\mathcal{I}$-approximately continuous functions.
- $\mathcal{C}_{\mathcal{D}\mathcal{O}}$ – the class of all deep-$\mathcal{I}$-approximately continuous functions.
- $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ – the class of all $\mathcal{I}$-density continuous functions.
- $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ – the class of all deep-$\mathcal{I}$-density continuous functions.
- $\mathrm{S}(X)$ – the semigroup of all continuous functions $f\colon X \to X$ for a topological space X.
- $\Delta^{(\mathcal{N})}$ – the class of all approximately differentiable functions from $\mathbb{R}$

(or any subinterval) to $\mathbb{R}$.

- $\Delta_{\mathcal{N}}^{(\mathcal{N})}$ – the class of all real functions which are approximately differentiable almost everywhere.
- $\Delta^{(\mathcal{I})}$ – the class of all real functions which are $\mathcal{I}$-approximately differentiable.
- $\Delta_{\mathcal{I}}^{(\mathcal{I})}$ – the class of all real functions which are $\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere.
- $\Delta^{(\mathcal{D})}$ – the class of all real functions which are deep-$\mathcal{I}$-approximately differentiable.
- $\Delta_{\mathcal{I}}^{(\mathcal{D})}$ – the class of all real functions which are deep-$\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere.
- $\mathcal{DB}_1$ – Darboux Baire 1 functions [**7**].
- $\mathcal{DB}_1^*$ – Darboux Baire*1 functions.

Operations on Functions

- $f|_P$ – restriction of the function f to the set P.
- Df – the ordinary derivative of the function f.
- $D^{(\mathcal{N})}f$ – the approximate derivative of the function f.
- $D^{(\mathcal{D})}f$ – the deep-$\mathcal{I}$-approximate derivative of the function f.
- $D^{(\mathcal{I})}f$ – the $\mathcal{I}$-approximate derivative of the function f.
- $\omega(f, x)$ – the oscillation of the function f at the point $x \in \mathbb{R}$ (see [**7**]).
- $\lfloor x \rfloor$ – the greatest integer function.
- $D_-f(x), D_+f(x), D^-f(x), D^+f(x), \bar{f}'(x), \underline{f}'(x), f'(x)$ – the lower left, lower right, upper left, upper right, upper, lower, and ordinary derivatives of the function f at the point x.

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