Functions continuous on twice differentiable curves, discontinuous on large sets

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Abstract

We provide a simple construction of a function $F: \mathbb{R}^2 \to \mathbb{R}$ discontinuous on a perfect set $P$, while having continuous restrictions $F \upharpoonright C$ for all twice differentiable curves $C$. In particular, $F$ is separately continuous and linearly continuous.

While it has been known that the projection $\pi[P]$ of any such set $P$ onto a straight line must be meager, our construction allows $\pi[P]$ to have arbitrarily large measure. In particular, $P$ can have arbitrarily large 1-Hausdorff measure, which is the best possible result in this direction, since any such $P$ has Hausdorff dimension at most 1.

1 Introduction

In this paper, a \textit{curve} is understood as the range of a continuous injection $h = \langle h_1, h_2 \rangle$ of an interval $J$ into the plane $\mathbb{R}^2$. A curve $C$ is said to be \textit{smooth} (or $C^1$), if the coordinate functions $h_1$ and $h_2$ are continuously differentiable (i.e., are $C^1$) and $\langle h_1'(t), h_2'(t) \rangle \neq \langle 0, 0 \rangle$ for every $t \in J$; we say that $C$ is \textit{twice differentiable} (or $D^2$), when it is smooth (so, its derivative nowhere vanishes) and the coordinate functions are twice differentiable. It has been proved by Rosenthal [17] that

\textsuperscript{*}E-mail: KCies@math.wvu.edu; web page: \url{http://www.math.wvu.edu/~kcies}. This paper will be a part of Ph.D. thesis of the second author.
(*) For any function $G: \mathbb{R}^2 \to \mathbb{R}$, if its restriction $G \upharpoonright C$ is continuous for every smooth curve $C$, then $G$ is continuous. However, there exists a discontinuous function $F: \mathbb{R}^2 \to \mathbb{R}$ with $F \upharpoonright C$ continuous for all twice differentiable curves $C$.\(^1\)

The function $F$ constructed by Rosenthal was discontinuous at a single point. The function constructed in our Theorem 4 seems to be the first example of a function with continuous restrictions to all twice differentiable curves, which has uncountable set of points of discontinuity.

For a family $\mathcal{C}$ of curves $C$ in the plane $\mathbb{R}^2$, we say that $F: \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{C}$-continuous, provided its restriction $F \upharpoonright C$ is continuous for every $C \in \mathcal{C}$. The $\mathcal{C}$-continuous functions for different classes $\mathcal{C}$ of curves have been studied from the dawn of mathematical analysis. For the class $\mathcal{L}_0$ of straight lines parallel to either of the axis, the $\mathcal{L}_0$-continuity coincides with separate continuity (referring to maps $F$ with section functions $F(\cdot, y)$ and $F(x, \cdot)$ continuous for every $x, y \in \mathbb{R}$). Separately continuous functions have been investigated by many prominent mathematicians: Volterra (see Baire [2, p. 95]), Baire (1899, see [2]), Lebesgue (1905, see [13, pp. 201-202]), and Hahn (1919, see [9]). For the class $\mathcal{L}$ of all straight lines, $\mathcal{L}$-continuity is known under the name linear continuity. It has been known by J. Thomae (1870, see [20, p. 15] or [11]) that linearly continuous function need not be continuous. A simple example of such a function, which can be traced to a 1884 treatise on calculus by Genocchi and Peano [10], is defined as $F(x, y) = \frac{xy^2}{x^2+y^2}$ for $(x, y) \neq (0, 0)$, and $F(0, 0) = 0$. Scheeffer (1890, see [18]) and Lebesgue (1905, see [13, pp. 199-200]) have also noticed that the continuity along all analytic curves does not implies continuity. The question for what classes $\mathcal{C}$ of curves does $\mathcal{C}$-continuity imply continuity, apparently addressed in all works cited above, has been elegantly answered in 1955 by Rosenthal, as we stated in (*).

A next natural question, in this line of research, is about the structure of the sets $D(F)$ of points of discontinuity of $\mathcal{C}$-continuous functions $F$ for different classes $\mathcal{C}$ of curves. Of course, every set $D(F)$ must be $F_\sigma$. This follows from a well known result (see [14, thm. 7.1]) that, for arbitrary $F: \mathbb{R}^2 \to \mathbb{R}$,

\(^1\)Clearly, for any such $F$, the composition $F \circ h$ is continuous, whenever $h = \langle h_1, h_2 \rangle$ is a coordinate system for a $D^2$ curve. In fact, a little care in constructing such an $F$ (e.g. by using $C^\infty$ functions $h_n$ in Proposition 1) insures that $F \circ h$ is also $D^2$. However, it is important here, that the derivative $h'$ never vanishes, as it has been proved by Boman [3] (see also [11]), that if $F \circ \langle h_1, h_2 \rangle$ is $C^1$ for any $C^\infty$ functions $h_1, h_2$, then $F$ is continuous.
$D(F)$ is a union of the closed sets $D_n(F) = \{z \in \mathbb{R}^2 : \omega_F(z) \geq 2^{-n}\}$, where \(\omega_F(z) = \lim_{\delta \to 0^+} \sup \{|F(z) - F(w)| : ||z - w|| < \delta\}\) is the oscillation of $F$ at $z$.

The structure of sets $D(F)$ for separately continuous functions (i.e., for $C = L_0$) was examined by Young and Young (1910, see [21]) and was fully described in 1943 by Kershner [12] (compare also [4]), who showed that a set $D \subset \mathbb{R}^2$ is equal to $D(F)$ for a separately continuous $F : \mathbb{R}^2 \to \mathbb{R}$ if and only if $D$ is $F_\sigma$ and the projection of $D$ onto each axis is meager. More precisely, the characterization follows from the fact that a bounded set $D \subset \mathbb{R}^2$ is equal to the set $D_n(F) = \{z \in \mathbb{R}^2 : \omega_F(z) \geq 2^{-n}\}$ for a separately continuous $F : \mathbb{R}^2 \to \mathbb{R}$ if and only if $D$ is closed and its projection onto each axis is nowhere dense. Notice, that this characterization implies, in particular, that a set of points of discontinuity a separately continuous $F : \mathbb{R}^2 \to \mathbb{R}$ can have full planar measure.

The structure of sets $D(F)$ for linearly continuous functions $F : \mathbb{R}^2 \to \mathbb{R}$ is considerably more restrictive, as can be seen by the following result of Slobodnik [19]. More on separate continuity can be found in [7, 15, 16].

**Proposition 1** If $D$ is the set of points of discontinuity of a linearly continuous function $F : \mathbb{R}^2 \to \mathbb{R}$, then

1. $D$ is a union of sets $D_n$, $n = 1, 2, 3, \ldots$, where each $D_n$ is a rotation of a graph $h_n \mid P_n$ of a Lipschitz function $h_n : \mathbb{R} \to \mathbb{R}$ restricted to a compact nowhere dense set $P_n$.

Since the graph of a Lipschitz function has Hausdorff dimension 1 (see e.g. [8, sec. 3.2]), this means that so does any set of points of discontinuity of a linearly continuous function. We have recently shown [5] that the condition (1) is actually quite close to the full characterization of sets $D(F)$ for linearly continuous functions $F$, by proving that: if $D$ is as in (1), where each function $h_n$ is either convex or $C^2$, then $D$ is equal to the set of points of discontinuity of some linearly continuous function. This new result implies, in particular, that any meager $F_\sigma$ subset of a line is the set of points of discontinuity of some linearly continuous function; so such a set may have positive 1-Hausdorff measure.

The main goal of this paper is to show that a function $F : \mathbb{R}^2 \to \mathbb{R}$ with continuous restrictions to all twice differentiable curves can also have a set of points of discontinuity with large 1-Hausdorff measure.
Notice, that any smooth curve \( C \), with associated injection \( h = \langle h_1, h_2 \rangle \), is locally (at a neighborhood of an arbitrary point \( \langle h_1(t), h_2(t) \rangle \)) a function of either variable \( x \) (when \( h_1'(t) \neq 0 \)) or of variable \( y \) (when \( h_2'(t) \neq 0 \)). Thus, \( \mathcal{C}(C^1) \)-continuity with respect to the class \( \mathcal{C}(C^1) \) of all smooth curves is the same as the \( C^1 \cup (C^1)^{-1} \)-continuity, where \( C^1 \) is the class of all continuously differentiable functions \( g: \mathbb{R} \to \mathbb{R} \), and \( (C^1)^{-1} = \{ g^{-1} : g \in C^1 \} \), with \( g^{-1} \) understood as an inverse relation, that is, as \( g^{-1} = \{ \langle g(y), y \rangle : y \in \mathbb{R} \} \).

Similarly, \( \mathcal{C}(D^2) \)-continuity, where \( \mathcal{C}(D^2) \) is the class of all (smooth) twice differentiable curves, coincides with \( D^2 \cup (D^2)^{-1} \)-continuity.

2 The main result

Our example will be constructed using the following simple, but general result on \( \mathcal{C} \)-continuous functions. Recall that the support of a function \( F: \mathbb{R}^2 \to \mathbb{R} \), denoted as \( \text{supp}(F) \), is defined as the closure of the set \( \{ x \in \mathbb{R}^2 : f(x) \neq 0 \} \).

Symbol \( \omega \) will be used here to denote the first infinite ordinal number, which is identified with the set of all natural numbers, \( \omega = \{ 0, 1, 2, \ldots \} \).

Lemma 2 Let \( \mathcal{C} \) be a family of curves in \( \mathbb{R}^2 \) and let \( \{ D_j \subset \mathbb{R}^2 : j < \omega \} \) be a pointwise finite family of open sets such that

\( (F) \) the set \( \{ j < \omega : D_j \cap C \neq \emptyset \} \) is finite for every \( C \in \mathcal{C} \).

Then for every sequence \( \langle F_j : j < \omega \rangle \) of continuous functions from \( \mathbb{R}^2 \) into \( \mathbb{R} \) such that \( \text{supp}(F_i) \subset D_i \) for all \( i < \omega \), the function \( F \overset{\text{def}}{=} \sum_{j<\omega} F_j \) is \( \mathcal{C} \)-continuous. Moreover, if

- the diameters of the sets \( D_j \) go to 0, as \( j \to \infty \),
- \( \hat{P} \) is the set of all \( z \in \mathbb{R}^2 \) for which every open \( U \ni z \) intersects infinitely many sets \( D_j \), and
- each function \( F_j \) is onto \([0,1]\),

then \( \hat{P} = D(F) = \{ z \in \mathbb{R}^2 : \omega_F(z) = 1 \} \).

Proof. The first part is obvious. The second follows easily from the fact, that, for any \( z \in \hat{P} \), every open \( U \ni z \) contains infinitely many sets \( D_j \).

Lemma 2 will be used with \( \hat{P} = h \upharpoonright P \), the graph of \( h \) restricted to \( P \), where \( h \) and \( P \) are from the proposition below.
Proposition 3 For every \( M \in [0, 1) \) there exists a \( C^1 \) function \( h: \mathbb{R} \to \mathbb{R} \) and a nowhere dense perfect \( P \subset (0, 1) \) of measure \( M \) such that for every \( \hat{x} \in P \):
\[
h'(\hat{x}) = 0 \quad \text{and} \quad \lim_{x \to \hat{x}} \frac{|h(x) - h(\hat{x})|}{(x - \hat{x})^2} = \infty.
\] (1)

We will postpone the proof of Proposition 3 till the next section. However, we like to notice here, that the limit \( \lim_{x \to \hat{x}} \frac{|h(x) - h(\hat{x})|}{(x - \hat{x})^2} \) is a variant of the limit \( \lim_{x \to \hat{x}} 2 \frac{h(x) - h(\hat{x})}{(x - \hat{x})^2} \), which constitutes a generalized second derivative (related to Peano derivative) of \( h \) at \( \hat{x} \). Indeed, if \( h''(\hat{x}) \) exists, finite or infinite, then, by l’Hôpital’s Rule, \( \lim_{x \to \hat{x}} 2 \frac{h(x) - h(\hat{x})}{(x - \hat{x})^2} = \lim_{x \to \hat{x}} \frac{h'(x) - 0}{2(x - \hat{x})} = \lim_{x \to \hat{x}} \frac{h'(x) - h'(\hat{x})}{x - \hat{x}} = h''(\hat{x}) \). We need Proposition 3 in its current form, since there is no \( C^1 \) function \( h \) having an infinite second derivative on set of positive measure. \footnote{This follows, for example, from [1, thm. 19] (used with \( f = h' \)) which says that: for any real-valued continuous function \( f \) defined on a set \( P \subset \mathbb{R} \) of positive measure there exists a \( C^1 \) function \( g: \mathbb{R} \to \mathbb{R} \) which agrees with \( f \) on an uncountable set.} But see also remarks at the end of this section.

Theorem 4 Let \( h \) and \( P \) be as in Proposition 3. Then \( \hat{P} = h \upharpoonright P \) is the set of points of discontinuity of a \( D^2 \)-continuous function \( F: \mathbb{R}^2 \to \mathbb{R} \). Moreover, \( F \) has oscillation equal 1 at every point from \( \hat{P} \).

Proof. Let \( \{J_j: j < \omega\} \) be an enumeration, without repetitions, of bounded connected components of \( \mathbb{R} \setminus P \). For every \( j < \omega \) let the \( I_j \) be the open middle third subinterval of \( J_j \) and let \( F_j \) be a continuous function from \( \mathbb{R}^2 \) onto \([0, 1]\) with \( \text{supp}(F_j) \) contained in \( D_j = \{ (x, y) \in \mathbb{R}^2: x \in I_j \land |y - h(x)| < |I_j|^3 \} \), where \( |I_j| \) is the length of \( I_j \). We will show that the function \( F = \sum_{j < \omega} F_j \) is as required.

It is enough to show that sets \( D_j \) satisfy property \((F)\) for \( \mathcal{E} = D^2 \cup (D^2)^{-1} \), since all other assumptions of Lemma 2 are clearly satisfied. To see this, fix a \( D^2 \) function \( g: \mathbb{R} \to \mathbb{R} \). We need to prove that both \( g \) and \( g^{-1} \) intersect only finitely many sets \( D_j \).

To see that \( g \) intersects only finitely many sets \( D_j \), by way of contradiction, assume that there is an infinite set \( \{ j_n: n < \omega \} \) such that \( g \cap D_{j_n} \neq \emptyset \). For \( n < \omega \) choose \( (x_n, y_n) \in g \cap D_{j_n} \). Then \( g(x_n) = y_n \) for all \( n < \omega \). Choosing a subsequence, if necessary, we can assume that \( \lim_{n \to \infty} x_n = \hat{x} \in P \). Then, by the definition of sets \( D_j \), we have
\[
\lim_{n \to \infty} (y_n - h(x_n)) = \lim_{n \to \infty} \frac{y_n - h(x_n)}{x_n - \hat{x}} = \lim_{n \to \infty} \frac{y_n - h(x_n)}{(x_n - \hat{x})^2} = 0,
\] (2)
as \(\lim_{n \to \infty} \left| \frac{y_n - h(x_n)}{(x_n - \hat{x})^2} \right| \leq \lim_{n \to \infty} \left| \frac{y_n - h(x_n)}{I_{j_n}} \right| \leq \lim_{n \to \infty} |I_{j_n}| = 0\). In particular,
\[
g(\hat{x}) = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} y_n = \lim_{n \to \infty} (y_n - h(x_n)) + \lim_{n \to \infty} h(x_n) = h(\hat{x})
\]
and
\[
g'(\hat{x}) = \lim_{n \to \infty} \frac{y_n - h(\hat{x})}{x_n - \hat{x}} = \lim_{n \to \infty} \frac{y_n - h(x_n)}{x_n - \hat{x}} + \lim_{n \to \infty} \frac{h(x_n) - h(\hat{x})}{x_n - \hat{x}} = h'(\hat{x}) = 0.
\]
Hence, by l'Hôpital's Rule, \(\lim_{x \to \hat{x}} \frac{g(x) - g(\hat{x})}{(x - \hat{x})^2} = \lim_{x \to \hat{x}} \frac{g'(x) - g'(\hat{x})}{2(x - \hat{x})} = \frac{1}{2}g''(\hat{x})\) and, using (2) once more,
\[
\lim_{n \to \infty} \frac{h(x_n) - h(\hat{x})}{(x_n - \hat{x})^2} = \lim_{n \to \infty} \frac{h(x_n) - y_n}{(x_n - \hat{x})^2} + \lim_{n \to \infty} \frac{g(x_n) - g(\hat{x})}{(x_n - \hat{x})^2} = \frac{1}{2}g''(\hat{x}),
\]
where the first equation is justified by \(y_n = g(x_n)\) and \(h(\hat{x}) = g(\hat{x})\). But this contradicts the assumption on \(h\) that \(\lim_{x \to \hat{x}} \frac{|h(x) - h(\hat{x})|}{(x - \hat{x})^2} = \infty\).

To see that \(g^{-1}\) intersects only finitely many sets \(D_j\), by way of contradiction, assume that there is an infinite set \(\{j_n : n < \omega\}\) such that \(g^{-1} \cap D_{j_n} \neq \emptyset\). For \(n < \omega\) choose \((x_n, y_n) \in g^{-1} \cap D_{j_n}\). Then \(g(y_n) = x_n\) for all \(n < \omega\). Choosing a subsequence, if necessary, we can assume that \(\lim_{n \to \infty} x_n = \hat{x} \in P\). Then, \(\hat{y} \overset{\text{def}}{=} \lim_{n \to \infty} y_n = \lim_{n \to \infty} (y_n - h(x_n)) + \lim_{n \to \infty} h(x_n) = h(\hat{x})\) and also \(g(\hat{y}) = \lim_{n \to \infty} g(y_n) = \lim_{n \to \infty} x_n = \hat{x}\). Since, by the assumptions from Proposition 3, \(h'(\hat{x}) = 0\) we obtain
\[
1 = \lim_{n \to \infty} \frac{g(y_n) - g(\hat{y})}{y_n - \hat{y}} \cdot \frac{y_n - \hat{y}}{g(y_n) - g(\hat{y})} = \lim_{n \to \infty} \frac{g(y_n) - g(\hat{y})}{y_n - \hat{y}} \cdot \lim_{n \to \infty} \frac{y_n - h(\hat{x})}{x_n - \hat{x}} = g'(\hat{y}) \cdot h'(\hat{x}) = g'(\hat{y}) \cdot 0 = 0,
\]
a contradiction.

It is also worth to notice here, that if \(h : \mathbb{R} \to \mathbb{R}\) is a \(C^1\) homeomorphism and \(P\) is a perfect set such that \(h''(\hat{x}) = \lim_{x \to \hat{x}} \frac{h'(x) - h'(\hat{x})}{x - \hat{x}} = \infty\) for every \(\hat{x} \in P\), then a small modification of the above proof gives a \(D^2\) continuous function \(F : \mathbb{R}^2 \to \mathbb{R}\) with \(D(F) = h \upharpoonright P\). This remark is of interest here, since such an \(h\) is easily constructed with standard calculus tools, see e.g. [6, Example 4.5.1]. However, as mentioned above, for such an \(h\), neither can \(P\)
have positive measure, nor can we have \( h'(x) = 0 \) for more than finitely many points \( x \) from \( P \). So, in the modified argument for \( g \), the fraction \( \frac{h(x_n) - h(\hat{x})}{(x_n - \hat{x})^2} \) would need to be replaced with \( \frac{h(x_n) - [h'(\hat{x})(x_n - \hat{x}) + h(\hat{x})]}{(x_n - \hat{x})^2} \). Moreover, the same argument that we used to show that \( g \not\in D^2 \) would need to be repeated for \( g^{-1} \), however, this would require more restrictions in the definition of the sets \( D_j \) to allow for the reversed role of the variables \( x \) and \( y \).

### 3 Proof of Proposition 3

Function \( h \) described below is a minor modification of a map \( f \) from [1, thm. 18].

Let \( \varepsilon \in (0, 1) \) be such that \( M < 1 - \varepsilon \) and let \( K \) be a symmetrically defined Cantor-like subset of \([0, 1]\) of measure \( 1 - \varepsilon \). More precisely, the set \( K \) is defined as \( K = \bigcap_{n<\omega} \bigcup_{s \in 2^n} I_s = [0, 1] \setminus \bigcup_{s \in 2^{n} < \omega} J_s \), where: \( 2^n \) denotes the set of all sequences from \( n = \{0, 1, \ldots, n - 1\} \) into \( \{0, 1\} \); \( 2^{<\omega} = \bigcup_{n<\omega} 2^n \) is the set of all finite 0-1 sequences; \( I_0 = [0, 1] \), and, for any \( s \in 2^n \), \( J_s \) is an open interval of length \( \varepsilon/3^{n+1} \) sharing the center with \( I_s \), while \( I_{s0} \) and \( I_{s1} \) are the left and right component intervals of \( I_s \setminus J_s \), respectively. Note that \( |J_s| = \frac{\varepsilon}{3^{n+1}} < \frac{1}{3^{n+1}} < |I_s| \leq \frac{1}{2^n} \) for every \( s \in 2^n \), so the choice of \( J_s \) is always possible. Clearly the set \( K \) has the desired measure of \( 1 - \sum_{s \in 2^{<\omega}} |J_s| = 1 - \sum_{n<\omega} 2^n \frac{\varepsilon}{3^{n+1}} = 1 - \varepsilon \).

For every \( s \in 2^n \) let \( f_s \) be a function from \( \mathbb{R} \) onto \([0, 1/(n+1)]\) defined as \( f_s(x) = \frac{2}{(n+1)|J_s|} \text{dist}(x, \mathbb{R} \setminus J_s) \), where \( \text{dist}(x, T) = \inf\{|x-t|: t \in T\} \) denotes the distance from \( x \) to \( T \). Then, the function \( h_0 = \sum_{s \in 2^{<\omega}} f_s : \mathbb{R} \to [0, 1] \) is continuous and our \( C^1 \) function \( h : \mathbb{R} \to \mathbb{R} \) is defined as \( h(x) = \int_0^x h_0(t) \, dt \). Note that \( h \) is strictly increasing on \([0, 1]\).

Let \( P \) be an arbitrary perfect subset of \( K \) of measure \( M \), which is disjoint with the set of all endpoints of the intervals \( J_s \), \( s \in 2^{<\omega} \). We will show that \( h \) and \( P \) are as required.

Clearly, for every \( \hat{x} \in P \subseteq K \) we have \( h'(|\hat{x}) = \frac{h(x)}{(x_n - \hat{x})^2} \). To see the other condition, first notice that for \( n > 1/\ln(4/3) \)

\[
\text{if } \hat{x}, x_0 \in K \cap I_s \text{ for } s \in 2^n \text{ and } \hat{x} \neq x_0, \text{ then } \left| \frac{h(x_0) - h(\hat{x})}{(x_0 - \hat{x})^2} \right| \geq \frac{\varepsilon}{6(n+1)}. \tag{3}
\]

To argue for (3), choose the largest \( m < \omega \) such that \( \hat{x}, x_0 \in I_t \) for some
increasing on $J$. To argue for (4), let $x / \hat{x}$ and $x_0$ be the end point between $x_0$ and $x$. Then $h_0$ is linear on the interval between $x_0$ and $x_1$ with the slope $\pm \frac{2}{(n+1)|J_s|}$. Hence, indeed,

$$\frac{|h(x) - h(x_0)|}{(x - x_0)^2} > \frac{|h(x_1) - h(x_0)|}{4(x_1 - x_0)^2} = \frac{\frac{1}{2}(x_1 - x_0)^2}{4(x_1 - x_0)^2} = \frac{3^{n+1}}{4(n+1)\varepsilon}.$$ 

Finally, fix an $\hat{x} \in P$. We need to show that $\lim_{x \to \hat{x}} \frac{|h(x) - h(\hat{x})|}{(x - \hat{x})^2} = \infty$. For this, we fix an arbitrarily large $N$ and show that $\frac{|h(x) - h(\hat{x})|}{(x - \hat{x})^2} \geq N$ for the points $x$ close enough to $\hat{x}$.

Let $n_0$ be such that $\min \left\{ \frac{\varepsilon (4/3)^n}{6(n+1)}, \frac{3^{n+1}}{4(n+1)\varepsilon} \right\} \geq 4N$ for all $n \geq n_0$ and let $s \in 2^{n_0}$ be such that $\hat{x} \in I_s$. Notice that $\hat{x}$ belongs to the interior $U$ of $I_s$, as $\hat{x} \in P$. Hence, it is enough to show that $\frac{|h(x) - h(\hat{x})|}{(x - \hat{x})^2} \geq N$ for every $x \neq \hat{x}$ from $U$. So, fix such an $x$.

If $x \in K$, then $\frac{|h(x) - h(\hat{x})|}{(x - \hat{x})^2} \geq N$ follows immediately from (3). So, assume that $x \notin K$. Then $x \in J_t$ for some $t \supset s$. Let $x_0$ be the end point of $J_t$ between $x$ and $\hat{x}$. Notice, that $x_0 \neq \hat{x}$, since $\hat{x} \in P$. Then, since $h$ is increasing on $[0, 1]$, properties (3) and (4) imply

$$\frac{|h(x) - h(\hat{x})|}{(x - \hat{x})^2} = \frac{|h(x) - h(x_0)|}{(x - x_0)^2} \frac{(x - x_0)^2}{(x - \hat{x})^2} + \frac{|h(x_0) - h(\hat{x})|}{(x_0 - \hat{x})^2} \frac{(x_0 - \hat{x})^2}{(x - \hat{x})^2} \geq 4N \frac{(x - x_0)^2}{(x - \hat{x})^2} + 4N \frac{(x_0 - \hat{x})^2}{(x - \hat{x})^2} \geq N,$$

finishing the proof.
References


