Modulo 5-orientations and degree sequences

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A R T I C L E   I N F O

Article history:
Received 9 August 2018
Received in revised form 16 January 2019
Accepted 17 January 2019
Available online 16 February 2019

Keywords:
Nowhere-zero flows
Modulo orientations
Strongly group connectivity
Group connectivity
Graphic sequences
Degree sequence realizations

A B S T R A C T

In connection to the 5-flow conjecture, a modulo 5-orientation of a graph G is an orientation of G such that the indegree is congruent to outdegree modulo 5 at each vertex. Jaeger conjectured that every 9-edge-connected multigraph admits a modulo 5-orientation, whose truth would imply Tutte’s 5-flow conjecture. In this paper, we study the problem of modulo 5-orientation for given multigraphic degree sequences. We prove that a multigraphic degree sequence d = (d1, . . . , dn) has a realization G with a modulo 5-orientation if and only if d i = 1, 3 for each i. In addition, we show that every multigraphic sequence d = (d1, . . . , dn) with minE⊆V d |E| ≥ 9 has a 9-edge-connected realization which admits a modulo 5-orientation for every possible boundary function. This supports the above mentioned conjecture of Jaeger.

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1. Introduction

Graphs considered in this paper are finite and loopless. As in [2], a graph is simple if it does not contain parallel edges or loops. For a graph which may contain parallel edges, we call it a multigraph. For a positive integer k, let [k] = {1, 2, . . . , k} and Zk be the set of all integers modulo k, as well as the (additive) cyclic group of order k. Following [2], k ⊓(G) denotes the edge-connectivity of a graph G. Denote a cycle with n vertices by Cn. For vertex subsets U, W ⊆ V(G), let [U, W]k = {uw ∈ E(G) | u ∈ U and w ∈ W}. When U = {u} or W = {w}, we use [u, W]k or [U, w]k for [U, W]k, respectively. The subscript G may be omitted when G is understood from the context. For a graph G and integer k > 0, kG denotes the graph obtained from G by replacing each edge with k parallel edges joining the same pair of vertices.

Let D = D(G) denote an orientation of G. For each v ∈ V(G), let E+(v) (E−(v), resp.) be the set of all arcs directed out (into, resp.) v. As in [2], d+(v) = |E+(v)| and d−(v) = |E−(v)| denote the out-degree and the in-degree of v under the orientation D, respectively. If a graph G has an orientation D such that d+(v) ≡ d−(v) (mod k) for every vertex v ∈ V(G), then we say that G admits a modulo k-orientation. Let Mκ denote the family of all graphs with a modulo k-orientation. Note that, for even k, a graph admits a modulo k-orientation if and only if every vertex has even degree.

Let Γ be an Abelian group, let D be an orientation of G and f : E(G) → Γ. The pair (D, f) is a Γ-flow in G if the net in-flow equals the net out-flow at every vertex. That is, for any vertex v ∈ V(G),

\[ \sum_{e \in E_+(v)} f(e) = \sum_{e \in E_-(v)} f(e). \]

A flow (D, f) is nowhere-zero if f(e) ≠ 0 for every e ∈ E(G). If Γ = Z and −k < f(e) < k then (D, f) is called a k-flow. Tutte’s flow conjectures are perhaps some of the most fascinating conjectures in graph theory. Tutte’s 3-flow conjecture states that

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https://doi.org/10.1016/j.dam.2019.01.021
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every 4-edge-connected graph admits a nowhere-zero 3-flow, which is equivalent to saying that every 4-edge-connected graph admits a modulo 3-orientation (see [2]). The celebrated 5-flow conjecture [15] states that every bridgeless graph admits a nowhere-zero 5-flow. It is well known that the 5-flow conjecture is equivalent to the statement every 3-edge-connected graph \(G\) admits a nowhere-zero \(\mathbb{Z}_5\)-flow. It was observed by Jaeger [6] that if the graph \(3G\) has a modulo 5-orientation, then \(G\) admits a nowhere-zero \(\mathbb{Z}_5\)-flow. Specifically, let \(D\) be a modulo 5-orientation of \(3G\) and \(f = 1\) be a constant mapping from \(E(3G)\) to 1. Then the sum of this flow \((D, f)\) of \(3G\) would give a nowhere-zero \(\mathbb{Z}_5\)-flow of \(G\), and this led Jaeger [6] to propose the following stronger conjecture, whose truth implies Tutte’s 5-flow conjecture.

**Conjecture 1.1** ([6]). Every 9-edge-connected multigraph admits a modulo 5-orientation.

Jaeger [6] also proposed a more general Circular Flow Conjecture that every \(4p\)-edge-connected multigraph admits a modulo \((2p + 1)\)-orientation, however it was disproved for all \(p \geq 3\) in [5]. The concept of strongly \(\mathbb{Z}_5\)-connectedness is introduced in [10] serving as contractible configurations for modulo 5-orientations (see also [9]). For a graph \(G\), let \(Z(G, \mathbb{Z}_5) = \{b : V(G) \to \mathbb{Z}_5 \sum_{v \in V(G)} b(v) \equiv 0 \pmod{5}\}\). A graph \(G\) is strongly \(\mathbb{Z}_5\)-connected if, for every \(b \in Z(G, \mathbb{Z}_5)\), there is an orientation \(D\) such that \(d_D^+(v) - d_D^-(v) \equiv b(v) \pmod{5}\) for every vertex \(v \in V(G)\). Let \(\langle S\mathbb{Z}_5 \rangle\) denote the family of all strongly \(\mathbb{Z}_5\)-connected graphs. Conjecture 1.1 is further strengthened to the following conjecture in [9].

**Conjecture 1.2** ([9]). Every 9-edge-connected multigraph is strongly \(\mathbb{Z}_5\)-connected.

Conjectures 1.1 and 1.2 are confirmed for 12-edge-connected multigraphs by Lovász, Thomassen, Wu and Zhang [12]. We also note that, by a result in [11], the truth of Conjecture 1.2 would imply another conjecture of Jaeger et al. [7] which states that every 3-edge-connected graph is \(\mathbb{Z}_5\)-connected. A graph is called \(\mathbb{Z}_5\)-connected if for any \(b \in Z(G, \mathbb{Z}_5)\), there is an orientation \(D\) and a mapping \(f : E(G) \mapsto \{1, 2, 3, 4\}\) such that for every vertex \(v \in V(G)\),

\[
\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) \equiv b(v) \pmod{5}.
\]

Denote \(\langle Z\mathbb{Z}_5 \rangle\) to be the family of all \(\mathbb{Z}_5\)-connected graphs.

An integral degree sequence \(d = (d_1, d_2, \ldots, d_n)\) is called graphic (multigraphic, resp.) if there is a simple graph (multigraph, resp.) \(G\) so that the degree sequence of \(G\) equals \(d\); such a graph \(G\) is called a realization of \(d\). Graphic and multigraphic sequences with certain flow and group connectivity properties have been extensively studied [3, 11, 13, 14, 16, 17]. Specifically, all graphic sequences with nowhere-zero 3-flow or 4-flow realization are characterized by Luo et al. [13, 14], respectively. The problem of characterizing all degree sequences with \(\mathbb{Z}_5\)-connected properties is proposed and studied by Yang et al. [17], and solved by Dai and Ying [3]. In general, the \(\mathbb{Z}_k\)-connected realization problem is characterized for \(k = 4\) by Wu et al. [16], and it is eventually resolved in [11] for every \(k\).

In this paper, we study the degree sequences with realizations that are strongly \(\mathbb{Z}_5\)-connected or have modulo 5-orientation properties. Our main results are the following characterizations.

**Theorem 1.3.** For any multigraphic sequence \(d = (d_1, d_2, \ldots, d_n)\), \(d\) has a modulo 5-orientation realization if and only if \(d_i \notin \{1, 3\}\) for every \(1 \leq i \leq n\).

**Theorem 1.4.** For any multigraphic sequence \(d = (d_1, d_2, \ldots, d_n)\), \(d\) has a strongly \(\mathbb{Z}_5\)-connected realization if and only if \(\sum_{i=1}^n d_i \geq 8n - 8\) and \(\min_{i \in [n]} d_i \geq 4\).

In addition, we obtain the following theorem, which provides partial evidences for Conjectures 1.1 and 1.2.

**Theorem 1.5.** For any multigraphic sequence \(d = (d_1, d_2, \ldots, d_n)\) with \(\min_{i \in [n]} d_i \geq 9\), \(d\) has a 9-edge-connected strongly \(\mathbb{Z}_5\)-connected realization.

Theorem 1.5 also leads to the following corollary.

**Corollary 1.6.** For any multigraphic sequence \(d = (d_1, d_2, \ldots, d_n)\) with \(\min_{i \in [n]} d_i \geq 8\), \(d\) has a 8-edge-connected modulo 5-orientation realization.

The rest of the paper is organized as follows. In section 2, we present some necessary preliminaries. Our main results are proved in section 3.

### 2. Preliminaries

For an edge set \(X \subseteq E(G)\), the contraction \(G/X\) is the graph obtained from \(G\) by identifying the two ends of each edge in \(X\), and then deleting the resulting loops. If \(H\) is a subgraph of \(G\), then we use \(G/H\) for \(G/E(H)\). As \(K_1\) is strongly \(\mathbb{Z}_5\)-connected, for any graph \(G\), every vertex lies in a maximal strongly \(\mathbb{Z}_5\)-connected subgraph. Let \(H_1, H_2, \ldots, H_t\) denote the collection of all maximal subgraphs in the graph \(G\). Then \(G' = G/(\bigcup_{i=1}^t E(H_i))\) is called the \((SS\mathbb{Z}_5)\)-reduction of \(G\). If \(G\) is strongly \(\mathbb{Z}_5\)-connected, then its \((SS\mathbb{Z}_5)\)-reduction is \(K_1\), a singleton.

The following lemma is a summary of some basic properties stated in [8, 9] and [10].
Lemma 2.1 ([8–10]). Each of the following holds.

(i) If $H \in \langle \mathbb{Z}_5 \rangle$ and $G/H \in \langle \mathbb{Z}_5 \rangle$, then $G \in \langle \mathbb{Z}_5 \rangle$.

(ii) A cycle of length $n$ is in $\langle \mathbb{Z}_5 \rangle$ if and only if $n \leq 4$.

(iii) Let $mK_2$ denote the loopless graph with two vertices and $m$ parallel edges. Then $mK_2$ is strongly $\mathbb{Z}_5$-connected if and only if $m \geq 4$.

(iv) $G \in M_5$ if and only if its $(SZ_5)$-reduction $G' \in M_5$.

(v) $G \in (SZ_5)$ if and only if its $(SZ_5)$-reduction $G' = K_1$.

The following theorem is a special case of the results stated in [11].

Theorem 2.2 ([11]). Let $G$ be a graph. Then each of the following holds.

(i) $G \in \langle \mathbb{Z}_5 \rangle$ if and only if $3G \in (SZ_5)$.

(ii) If $G \in (SZ_5)$, then $G$ contains four edge-disjoint spanning trees, and in particular, $|E(G)| \geq 4|V(G)| - 4$.

For a realization $G$ of a multigraphic degree sequence $d = (d_1, d_2, \ldots, d_n)$, if $G$ is a realization of $d$ with $V(G) = \{v_1, \ldots, v_n\}$ such that $d_i(v_i) = d_i$, then $v_i$ is called the $d_i$-vertex for each $i \in [n]$. As a rearrangement of a degree sequence does not change its realizations, we will just focus on nonincreasing multigraphic sequence in the rest of the paper for convenience.

Theorem 2.3 (Hakimi [4]). Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing integral sequence with $n \geq 2$ and $d_n \geq 0$. Then $d$ is a multigraphic sequence if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$.

Theorem 2.4 (Boesch and Harary [1]). Let $d = (d_1, \ldots, d_n)$ be a nonincreasing integral sequence with $n \geq 2$ and $d_n \geq 0$. Let $j$ be an integer with $2 \leq j \leq n$ such that $d_j \geq 1$. Then the sequence $(d_1, d_2, \ldots, d_n)$ is multigraphic if and only if the sequence $(d_1 - 1, d_2, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n)$ is multigraphic.

Let $G$ be a graph with $w \in E(G)$ and let $w$ be a vertex different from $u$ and $v$, where $w$ may or may not be in $V(G)$. Define $G^{(w,uv)}$ to be the graph containing $w$ obtained from $G - uv$ by adding new edges $wu$ and $wv$. We also say that $G^{(w,uv)}$ is obtained from $G$ by inserting the edge $uv$ to $w$ in this paper. The following observation is straightforward, which indicates the inserting operation would preserve the edge connectivity.

Lemma 2.5. Let $G$ be a connected graph.

(i) Let $w \in V(G) \setminus \{u, v\}$ and $G' = G^{(w,uv)}$. Then $\kappa'(G') = \kappa'(G)$.

(ii) Let $w \notin V(G)$ be a new vertex and $e_1, \ldots, e_t \in E(G)$. Then the graph $G'$ obtained from $G$ by inserting the edges $e_1, \ldots, e_t$ to $w$ satisfies $\kappa'(G') \geq \min(\kappa'(G), 2t)$.

Proof. (i) Let $[X,X']_C$ be an edge cut of $G'$. Observe that either $|[X,X']_C| = |[X,X']_C| = |[X,X']_C|$ or $|[X,X']_C| = |[X,X']_C| + 2$ depending on the position of $u, v, w$ in $X$ or $X'$. So $|[X,X']_C| \geq |[X,X']_C| \geq \kappa'(G)$, and thus $\kappa'(G') \geq \kappa'(G)$.

(ii) The proof of (ii) is similar to (i).

Let $x_1x_2, x_2x_3 \in E(G)$. We use $G_{[x_2,x_3]}$ to denote the graph obtained from $G - \{x_1x_2, x_2x_3\}$ by adding a new edge $x_1x_3$. The operation to get $G_{[x_2,x_3]}$ from $G$ is referred as to lift the edges $x_1x_2, x_2x_3$ in $G$. The next lemma follows from the definition of strongly $\mathbb{Z}_5$-connectedness.

Lemma 2.6. Let $x_1, x_2, x_3$ and $G_{[x_2,x_3]}$ be the same notation as defined above. If $G_{[x_2,x_3]} \in \langle SZ_5 \rangle$, then $G \in \langle SZ_5 \rangle$.

The next lemma shows that the small graphs depicted in Fig. 1 could play a crucial role in the inductive arguments of our proofs.

Lemma 2.7. Each of the graphs $J_1, J_2, J_3, J_4$ in Fig. 1 is strongly $\mathbb{Z}_5$-connected.

Proof. (i) Let $b \in Z/J_1, \mathbb{Z}_5$. If $b(x_1) \neq 0$, we lift two edges $x_3x_1, x_1x_2$ in $J_1$ to obtain the graph $J_{[x_1,x_2,x_3]}$, say $H$. Since $|[x_1, \{x_2, x_3\}]_{Hr} = 3$ and $b(x_1) \neq 0$, we can modify the boundary $b(x_1)$ with the three edges in $[x_1, \{x_2, x_3\}]_{Hr}$.
orient 2, 0, 3, 1 edges toward $x_1$ when $b(x_1) = 4, 3, 2, 1$, respectively. By Lemma 2.1(iii) and $|\{x_2, x_3\}| = 4$, we can also modify the boundaries $b(x_2), b(x_3)$ with four parallel edges $x_2x_3$. By symmetry, we assume that $b(x_1) = b(x_2) = 0$, then $b(x_3) = 0$ since $b \in Z(J_1, Z_3)$. Orient all the edges in $\{x_1, \{x_2, x_3\}\}$ toward $x_1$ and orient all the edges in $\{x_2, \{x_1, x_3\}\}$ from $x_2$ to obtain an orientation of $J_1$, which agrees with the boundary $b(x_1) = b(x_2) = b(x_3) = 0$. Therefore $J_1$ is strongly $Z_3$-connected by definition.

(ii) Let $b \in Z(J_2, Z_3)$. If $b(x_0) = 0$, we lift three pairs of edges $\{x_0x_0, x_0x_3\}$, $\{x_2x_0, x_0x_1\}$ and $\{x_3x_0, x_0x_1\}$ from $J_2$ to obtain the graph $3K_3$. By Lemma 2.1(v) and since $J_1 \in (SZ_3)$ is a spanning subgraph of $3K_3$, we have $3K_3 \in (SZ_3)$, which implies that the boundary $b$ at each vertex can be modified in $J_2$. If $b(x_0) = 2$ or $3$, we lift the edges pair $\{x_0x_0, x_0x_3\}$ twice to obtain the graph $G_1$ and then orient the parallel edges from $x_0$ to $x_1$ or from $x_1$ to $x_0$ in $G_1$, respectively. By Lemma 2.1(iii), we could modify the boundary $b(x_1)$ by two pairs of parallel edges $x_1x_2, x_1x_3$ and then modify the boundaries $b(x_2)$ and $b(x_3)$ by the four parallel edges between $x_2$ and $x_3$. Thus the obtained orientation agrees with the boundary $b$. So we have $b(x_1) = 1, 4$ for each $i$, and by symmetry, we may assume that $b(x_0) = b(x_2) = 1$ and $b(x_1) = b(x_3) = 4$. To agree with the boundary $b$ in this case, we orient two pairs of parallel edges $x_0x_1, x_0x_3$ toward $x_0$, two pairs of parallel edges $x_1x_2, x_3x_2$ toward $x_2$, two parallel edges $x_0x_2$ with opposite directions and two parallel edges $x_1x_3$ with opposite directions. Therefore, all possible boundaries $b$ are examined, and so $J_2$ is strongly $Z_3$-connected by definition.

(iii) Let $b \in Z(J_3, Z_3)$. If $b(x_0) \neq 0$, lift two edges $x_2x_0, x_0x_3$ to obtain $J_3[x_0, x_0x_3]$, say $L$. Since $b(x_0) \neq 0$ and $|\{x_0, \{x_1, x_3\}\}| = 3$, we can modify the boundary $b(x_0)$ with the three edges in $\{x_0, \{x_1, x_3\}\}$. As $|\{x_1, \{x_2, x_3\}\}| = 4$ and by Lemma 2.1(iii), we can modify the boundary $b(x_1)$. Furthermore, as $|\{x_2, x_3\}| = 4$ and by Lemma 2.1(iii), we can modify the boundaries $b(x_2)$ and $b(x_3)$. Thus we assume that $b(x_0) = 0$. We lift the two edges $x_2x_1, x_2x_3$ to obtain $L$. Orient the five edges incident with $x_0$ out from $x_0$ in $L$. If $b(x_1) = 0$, 1, 3 we orient two edges from $x_1$ toward $x_2$, $x_3$, two edges from $x_2, x_3$ toward $x_1$, one edge from $x_2$ and one edge from $x_3$ to $x_1$, respectively. If $b(x_1) = 4, 2$, reverse the above obtained orientation in $L$ corresponding to $b(x_0) = 1, 3$, respectively. Then modify the boundaries $b(x_2)$ and $b(x_3)$, by Lemma 2.1(iii) and $|\{x_2, x_3\}| = 4$. Thus $J_3$ is strongly $Z_3$-connected.

(iv) Since $J_4$ contains $J_1$ as a subgraph, $J_4/J_1 = 4K_2$ and $J_1 \in (SZ_3)$, we conclude that $J_4$ is strongly $Z_3$-connected by Lemma 2.1(iii)(v). $\blacksquare$

3. Proofs of main results

We shall present the proof of Theorem 1.4 first, which will be used in the proof of Theorem 1.3.

3.1. Proof of Theorem 1.4

Define $F_n = \{(d_1, \ldots, d_n) : \sum_{i=1}^{n} d_i = 8n - 8$ and $\min_{i \in [n]} (d_i) \geq 4\}$.

**Lemma 3.1.** Let $d = (d_1, d_2, \ldots, d_n) \in F_n$ be a nonincreasing sequence. Then $d$ is multigraphic. Moreover, each of the following holds.

(i) If $n \geq 4$ and $(d_{n-1}, d_n) \in [(5, 5), (6, 5)]$, then there exist $(d'_1, \ldots, d'_{n-2}) \in F_{n-2}$ and nonnegative integer $c_j$ such that for each $1 \leq j \leq n - 2$, $d^j = d'_j + c_j$ and

$$
\sum_{j=1}^{n-2} c_j = \begin{cases} 6, & \text{if } (d_{n-1}, d_n) = (5, 5); \\ 5, & \text{if } (d_{n-1}, d_n) = (6, 5). \\
\end{cases}
$$

(ii) If $n \geq 5$ and $(d_{n-2}, d_{n-1}, d_n) \in \{(7, 7, 5), (6, 6, 5), (7, 6, 6), (7, 7, 6)\}$, then there exist $(d'_1, \ldots, d'_{n-3}) \in F_{n-3}$ and nonnegative integer $c_j$ such that for each $1 \leq j \leq n - 3$, $d_j = d'_j + c_j$ and

$$
\sum_{j=1}^{n-3} c_j = \begin{cases} 5, & \text{if } (d_{n-2}, d_{n-1}, d_n) = (7, 7, 5); \\ 6, & \text{if } (d_{n-2}, d_{n-1}, d_n) = (6, 6, 5); \\ 5, & \text{if } (d_{n-2}, d_{n-1}, d_n) = (7, 6, 6); \\ 4, & \text{if } (d_{n-2}, d_{n-1}, d_n) = (7, 7, 6). \\
\end{cases}
$$

**Proof.** Since $d_n \geq 4$, we have $\sum_{i=1}^{n} d_i \geq 4n - 4$. Then $d_1 \leq \sum_{i=1}^{n} d_i - (4n - 4) = 4n - 4 \leq \sum_{i=1}^{n} d_i$. By Theorem 2.3, $d$ is multigraphic.

(i) Denote $k = 16 - d_{n-1} - d_n$. If $n \geq 4$, then by $\sum_{i=1}^{n} d_i = 8n - 8$, we have

$$
\sum_{i=1}^{n} d_i = 8n - 8 \geq 4(n - 2) + 16 = 4(n - 2) + (d_n + d_{n-1}) + k.
$$

Thus there exists a minimal integer $i_0 \in [n - 2]$ such that $\sum_{j=1}^{i_0} d_j \geq 4i_0 + k$. Let $c_j = d_j - 4$ for $1 \leq j \leq i_0 - 1$, $c_{i_0} = k - \sum_{j=1}^{i_0-1} d_j$ and $c_j = 0$ if $i_0 + 1 \leq j \leq n - 2$. Let $d'_j = d_j - c_j$ for each $1 \leq j \leq n - 2$. Then the degree sequence $(d'_1, \ldots, d'_{n-2}) \in F_{n-2}$
\( \sum_{j=1}^{n-2} d'_j = \sum_{j=1}^{n-2} d_j - \sum_{j=1}^{n-2} c_j = \sum_{j=1}^{n-2} d_j - k = \sum_{j=1}^{n} d_j - 16 = 8(n-2). \)

and \( d'_j \geq 4 \) for each \( 1 \leq j \leq n - 2 \). Moreover, Eq. (1) is satisfied as well.

(ii) The proof is similar to (i). Denote \( t = 24 - d_{n-2} - d_{n-1} - d_n \). If \( n \geq 5 \), then by \( \sum_{i=1}^{n} d_i = 8n - 8 \), we obtain

\[
\sum_{i=1}^{n} d_i = 8n - 8 \geq 4(n - 3) + 24 = 4(n - 3) + (d_n + d_{n-1} + d_{n-2}) + t.
\]

Thus there exists a minimal integer \( i_0 \in [n - 3] \) such that \( \sum_{j=1}^{i_0} d_j \geq 4i_0 + t \). Let \( c_j = d_j - 4 \) for \( 1 \leq j \leq i_0 - 1 \), \( c_{i_0} = t - \sum_{j=1}^{i_0-1} d_j \) and \( c_j = 0 \) if \( i_0 + 1 \leq j \leq n - 3 \). Let \( d'_j = d_j - c_j \) for \( 1 \leq j \leq n - 3 \). Then \( (d'_1, \ldots, d'_{n-3}) \in \mathcal{F}_{n-3} \) as

\[
\sum_{j=1}^{n-3} d'_j = \sum_{j=1}^{n-3} d_j - \sum_{j=1}^{n-3} c_j = \sum_{j=1}^{n-3} d_j - t = \sum_{j=1}^{n} d_j - 24 = 8(n - 3),
\]

and \( d'_j \geq 4 \) for each \( 1 \leq j \leq n - 3 \). Furthermore, Eq. (2) holds as well.

To prove Theorem 1.4, we verify the following key Lemma first.

**Lemma 3.2.** For any nonincreasing multigraphic sequence \( d = (d_1, d_2, \ldots, d_n) \) with \( \sum_{i=1}^{n} d_i = 8n - 8 \) and \( d_n \geq 4 \), \( d \) has a strongly \( \mathbb{Z}_5 \)-connected realization.

**Proof.** We apply induction on \( n \). If \( 2 \leq n \leq 3 \), then all the degree sequences satisfying the assumption \( \sum_{i=1}^{n} d_i = 8n - 8 \) and \( d_n \geq 4 \) are depicted below in Fig. 2.

It follows from Lemma 2.1(iii)(v) and Lemma 2.7 that each graph above is strongly \( \mathbb{Z}_5 \)-connected, and so Lemma 3.2 holds if \( 2 \leq n \leq 3 \). Thus we assume that \( n \geq 4 \) and Lemma 3.2 holds for integers smaller than \( n \). Notice that \( 4 \leq d_n \leq 7 \), since \( \sum_{i=1}^{n} d_i = 8n - 8 \).

**Case 1:** \( d_n = 4 \).

Since \( \sum_{i=1}^{n} d_i = 8n - 12 \geq 4(n - 1) + 4 \), similar to the proof of Lemma 3.1, there exist a sequence \( d' = (d'_1, \ldots, d'_{n-1}) \) and nonnegative integer \( c_i \) for each \( i \in [n-1] \) such that \( \sum_{j=1}^{n-1} c_j = 4 \), \( d_i = d'_i + c_i \) and \( d'_i \geq 4 \). Then \( \sum_{i=1}^{n-1} d'_i = 8(n - 1) - d_n - \sum_{i=1}^{n-1} c_i = 8n - 2 \). By Lemma 3.1, \( d' \) is multigraphic and \( d' \) has a strongly \( \mathbb{Z}_5 \)-connected realization \( G' \) by induction on \( n \). Let \( G \) be the graph obtained from \( G' \) by adding one new vertex \( v_0 \) and \( c_i \) edges joining the vertex \( v_0 \) with \( d'_i \)-vertex for each \( 1 \leq i \leq n - 1 \). As \( G/G' = 4K_2 \in \langle S\mathbb{Z}_5 \rangle \) and \( G' \in \langle S\mathbb{Z}_5 \rangle \), \( G \) is a strongly \( \mathbb{Z}_5 \)-connected realization of \( d \) by Lemma 2.1(iii)(v).

**Case 2:** \( d_n = 5 \) or \( d_n = 6 \).

In this case, we shall divide our discussion according to \( (d_{n-1}, d_n) \) or \( (d_{n-2}, d_{n-1}, d_n) \).

If \( (d_{n-1}, d_n) \in \{ (5, 5), (6, 5) \} \), by Lemma 3.1(i), there exists \( d' = (d'_1, d'_2, \ldots, d'_{n-2}) \in \mathcal{F}_{n-2} \) such that \( d_i = d'_i + c_i \) where \( \sum_{i=1}^{n-2} c_i = 6 \) if \( (d_{n-1}, d_n) = (5, 5) \) and \( \sum_{i=1}^{n-2} c_i = 5 \) if \( (d_{n-1}, d_n) = (6, 5) \). By Lemma 3.1, \( d' \) is multigraphic. By induction on \( n \), \( d' \) has a strongly \( \mathbb{Z}_5 \)-connected realization \( G' \). Construct the graph \( G \) from \( G' \) by adding two new vertices \( v_{n-1}, v_n \) with parallel edges \( v_n v_{n-1} \) and for each \( i \in [n - 2] \), joining \( c_i \) edges from the \( d'_i \)-vertex to \( v_{n-1}, v_n \) to obtain a new graph \( G \) as a \( d \)-realization. Since \( G/G' = J_1 \) (see Fig. 1), \( G' \in \langle S\mathbb{Z}_5 \rangle \) and \( J_1 \in \langle S\mathbb{Z}_5 \rangle \) by Lemma 2.7, we conclude that \( G \) is a strongly \( \mathbb{Z}_5 \)-connected realization of \( d \) by Lemma 2.1(v).

If \( n \geq 5 \) and \( (d_{n-2}, d_{n-1}, d_n) \in \{ (7, 7, 5), (6, 6, 6), (7, 6, 6), (7, 7, 6) \} \), by Lemma 3.1(ii), there exists \( d' = (d'_1, d'_2, \ldots, d'_{n-3}) \in \mathcal{F}_{n-3} \) satisfying \( d_i = d'_i + c_i \) and Eq. (2). Since \( \sum_{i=1}^{n-3} d'_i = 8(n-4) \) and \( \min_{i \in [n-3]} d'_i \geq 4 \) and by Lemma 3.1, \( d' \) is multigraphic. Then \( d' \) has a strongly \( \mathbb{Z}_5 \)-connected realization \( G' \) by induction on \( n \).

If \( (d_{n-2}, d_{n-1}, d_n) = (7, 7, 5) \), let \( A = \{ v \in V(G') : v \text{ is a } d'_i \text{-vertex with } c_i > 0 \text{ and } i \in [n - 3] \} \). We construct a graph \( G \) from \( G' \) by adding three new vertices \( v_{n-2}, v_{n-1}, v_n \) and 12 edges such that \( |v_{n-2}, v_{n-1}, v_n| = 3 \), \( |v_{n-2}, v_{n-1}, v_n| = 4 \), \( |v_n, A| = 2 \), \( |v_{n-2}, A| = 3 \) to obtain a new graph \( G \) so that \( G \) is a \( d \)-realization. By Lemmas 2.1 and 2.7(iii)(v), as \( G' \in \langle S\mathbb{Z}_5 \rangle \),
and $G/G'/(v_{n-1}, v_{n-2}) = J_1 \in \langle SZ_5 \rangle$, we have $G \in \langle SZ_5 \rangle$, which provides a strongly $\mathbb{Z}_5$-connected realization of $d$. Similarly, if $(d_{n-2}, d_{n-1}, d_n) \in \{(6, 6, 6), (7, 6, 6), (7, 7, 6)\}$, we accordingly construct a graph $G$ such that $G/G' \in \langle J_2, J_3, J_4 \rangle$, respectively, and $x_0 \in V(J)$ with $j \in \{J_2, J_3, J_4\}$ (see Fig. 1) is the vertex onto which $G'$ is contracted in $G/G'$. Thus $d$ has a realization $G$. By Lemma 2.1(v) and Lemma 2.7, $G$ is a strongly $\mathbb{Z}_5$-connected realization of $d$.

The remaining case is $n = 4$ and $\sum_{i=1}^n d_i = 24$, and then $(d_1, d_2, d_3, d_4) = (6, 6, 6, 6)$. By Lemma 2.7, the graph $J_2$ (see Fig. 1) is the desired graph.

Case 3: $d_n = 7$.

If $d_n = 7$, by $\sum_{i=1}^n d_i = 8n - 8$, then $d_n = d_{n-1} = \cdots = d_{n-6} = 7$, which implies that $n \geq 7$. Thus

$$\sum_{i=1}^{n-4} d_i = 8n - 8 - 28 \geq 4(n - 4) + 4.$$

By a similar argument as in Lemma 3.1, there exist a degree sequence $d' = (d'_1, \cdots, d'_{n-4})$ and nonnegative integer $c_i$ such that $d_i = d'_i + c_i$ and $d'_i \geq 4$ for $1 \leq i \leq n - 4$, where $\sum_{i=1}^{n-4} c_i = 4$. Thus

$$\sum_{i=1}^{n-4} d'_i = \sum_{i=1}^n d_i - \sum_{i=n-3}^n d_i - \sum_{i=1}^{n-4} c_i = 8(n - 1) - 28 - 4 = 8(n - 5).$$

By Lemma 3.1, $d'$ is multigraphic. By induction on $n$, $d'$ has a strongly $\mathbb{Z}_5$-connected realization $G'$. We construct the graph $G$ from $G'$ and $3C_4$ by adding $c_i$ edges between $d'_i$-vertex and vertices of $3C_4$ such that $d'_i(x) = 7$ for any $x \in V(3C_4)$ so that $G$ is a $d$-realization. By Lemma 2.1(ii) and Theorem 2.2(i), $3C_4 \in \langle SZ_5 \rangle$. By Lemma 2.1(iii) (v) and $(G/G')/3C_4 = 4K_2 \in \langle SZ_5 \rangle$, $G$ is a strongly $\mathbb{Z}_5$-connected $d$-realization. This completes the proof. ■

Now we are ready to prove Theorem 1.4.

**Theorem 1.4.** For any nonincreasing multigraphic sequence $d = (d_1, d_2, \ldots, d_n)$, $d$ has a strongly $\mathbb{Z}_5$-connected realization if and only if $\sum_{i=1}^n d_i \geq 8n - 8$ and $d_n \geq 4$.

**Proof.** To prove the necessity, by Theorem 2.2(ii) and Lemma 2.1(iii), if $G \in \langle SZ_5 \rangle$ with degree sequence $(d_1, d_2, \ldots, d_n)$, then $\sum_{i=1}^n d_i \geq 8n - 8$ and $d_n \geq 4$.

For sufficiency, suppose the contrary that the nonincreasing multigraphic sequence $(d_1, d_2, \ldots, d_n)$ is a counterexample with $\sum_{i=1}^n d_i$ minimized. By Lemma 3.2, $\sum_{i=1}^n d_i > 8n - 8$ and $d_n \geq 4$. If $d_2 = 4$, then by Theorem 2.3, we have $\sum_{i=1}^n d_i \leq 2\sum_{i=2}^n d_i = 8n - 8$, a contradiction. Thus we assume that $d_2 \geq 5$. Let $(d'_1, d'_2, d'_3, \ldots, d'_n) = (d_1 - 1, d_2 - 1, d_3, \ldots, d_n)$. By Theorem 2.4, $(d'_1, d'_2, d'_3, \ldots, d'_n)$ is multigraphic, and so by the minimality of $(d_i, d'_i, \ldots, d'_n)$, $(d'_1, d'_2, d'_3, \ldots, d'_n)$ has a strongly $\mathbb{Z}_5$-connected realization $G'$. Then we obtain the graph $G$ as a $d$-realization from $G'$ by adding one edge between the $d'_1$-vertex and the $d'_2$-vertex. Since $G' \in \langle SZ_5 \rangle$, it follows from Lemma 2.1(v) that $G \in \langle SZ_5 \rangle$, a contradiction. ■

3.2. Proof of Theorem 1.3

**Theorem 3.** For any nonincreasing multigraphic sequence $d = (d_1, d_2, \ldots, d_n)$, $d$ has a modulo 5-orientation realization if and only if $d_i \notin \{1, 3\}$ for every $1 \leq i \leq n$.

**Proof.** To prove the necessity, let $(d_1, \ldots, d_n)$ be any multigraphic sequence, by the definition of modulo 5-orientation, we achieve $d_i \notin \{1, 3\}$ for every $1 \leq i \leq n$.

For sufficiency, suppose the contrary that the nonincreasing multigraphic sequence $d = (d_1, \ldots, d_n)$ is a counterexample with $m = \sum_{i=1}^n d_i$ minimized. By Theorem 2.3, $d_1 \leq \sum_{i=2}^n d_i$.

**Claim A.** $d_1 \leq \sum_{i=2}^n d_i - 4$.

By contradiction, we assume that $d_1 \geq \sum_{i=2}^n d_i - 2$, $\sum_{i=2}^n d_i$. If $d_1 = \sum_{i=2}^n d_i$, then $d$ has a unique realization $G$ by setting $v_1$ as the center vertex adjacent to the vertices $v_2, \ldots, v_n$ with $d_2, \ldots, d_n$ multiple edges, respectively. Now we are to prove that $G$ has a modulo 5-orientation $D$. For each $2 \leq i \leq n - 1$, if $d_i$ is even, then we orient one half of the edges from $v_i$ toward $v_1$ and orient rest edges from $v_1$ to $v_i$. If $d_i$ is odd, we assign $\frac{d_i + 5}{2}$ edges with the orientation from $v_1$ into vertex $v_i$ and $\frac{d_i - 5}{2}$ edges with opposite direction. Thus $G$ is a modulo 5-orientation realization of $d$, a contradiction.

Assume that $d_1 = \sum_{i=2}^n d_i - 2$. From the above oriented graph $G$ with degree sequence $(\sum_{i=2}^n d_i, d_2, \ldots, d_n)$, we pick up one directed edge oriented into the vertex $v_1$, denoted by $e_1$, and another edge oriented out from $v_1$, denoted by $e_2$, where $e_1 \cap e_2 = \{v_1\}$. Let $G'$ be the graph obtained from $G$ by lifting two edges $e_1, e_2$ to become a new edge. It is easy to see that $G'$ preserves the modulo 5-orientation and that $G'$ has degree sequence $d = (\sum_{i=2}^n d_i - 2, d_2, \ldots, d_n)$. This contradicts the assumption that $d$ is a counterexample.
3.3. Proof of Theorem 1.5

A graph is called cubic if it is 3-regular. For a cubic graph $G$, a $Y - \Delta$ operation on a vertex $v$ is to replace the vertex $v$ with a triangle, where each edge incident with $v$ in $G$ becomes an edge incident to a vertex of the triangle. It is clear that applying...
Y − Δ operation on a cubic graph still results a cubic graph, and it follows from Lemma 2.1(i)(ii) that any graph obtained from $K_4$ by $Y − Δ$ operation is $Z_5$-connected. We will use this observation (and in fact a stronger property) in the proof of Theorem 1.5. Before presenting the proof, we shall handle some special cases first. If a sequence $d$ consists of the terms $d_1, \ldots, d_t$ having multiplicities $m_1, \ldots, m_t$, we may write $d = (d_1^{m_1}, \ldots, d_t^{m_t})$ for convenience.

**Lemma 3.3.** Each of the integral multigraphic sequences $(17, 9^3), (14, 9^4), (16, 9^4), (16, 9^6)$ has a 9-edge-connected strongly $Z_5$-connected realization.

**Proof.** For $d = (17, 9^3)$, we construct a graph $G$ as $d$-realization from $J_1$ in Fig. 1 by adding a new vertex $x_4$ with 2 parallel edges $x_1x_4$ and 7 multiple edges $x_4x_4$, respectively, then adding 3, 2 multiple edges $x_1x_2, x_2x_2$, respectively. It is routine to check that $G$ is 9-edge-connected, i.e., for any $S \in \mathcal{V}(G)$ with $|S| = 1$ or 2, we have $|[S, \mathcal{V}(G) \setminus S]_c| \geq 9$. By Lemmas 2.7 and 2.1(iii)(v), $G$ is a strongly $Z_5$-connected $d$-realization.

For $d = (16, 9^4)$, we construct the graph $G_0$ from two disjoint copies of $K_4$ with labeled vertices $v', v''$ respectively, by identifying vertices $v', v''$ to a new vertex and lifting the two edges $e_1, e_2$, where $e_1, e_2$ are adjacent to $v', v''$ in each $K_4$. It is easy to check that $G_0$ is 9-edge-connected. Since $G_0$ contains $J_2$ (see Fig. 1) as a subgraph and by Lemmas 2.7 and 2.1(v), $G_0$ is a strongly $Z_5$-connected $d$-realization.

For $d = (16, 9^6)$, we obtain the desired graph $G_1$ gained from $J_1$ in Fig. 1 by adding two new vertices $x_4, x_5$ with edges $x_1x_4, x_1x_5$ and $3, 3, 7$ parallel edges $x_3x_5, x_1x_5, x_3x_3, x_4x_5$, respectively. For any $S \in \mathcal{V}(G_1)$, it is easy to check that $|[S, \mathcal{V}(G_1) \setminus S]_c| \geq 9$. Thus $G_1$ is a 9-edge-connected strongly $Z_5$-connected $d$-realization by Lemma 2.7 and Lemma 2.1(iii)(v).

For $d = (14, 9^4)$, we have the desired graph $G_2$ obtained from above $G_1$ by lifting the two edges $x_1x_5$ and $x_4x_5$. Let $S \in \mathcal{V}(G_2)$. It is routine to verify that $|[S, \mathcal{V}(G_2) \setminus S]_c| \geq 9$ for any $S \in \mathcal{V}(G_2)$. Therefore $G_2$ is a 9-edge-connected strongly $Z_5$-connected $d$-realization by Lemmas 2.7 and 2.1(iii)(v). ■

**Theorem 1.5.** For any nonincreasing multigraphic sequence $d = (d_1, d_2, \ldots, d_n)$ with $\min_{i\in[n]} d_i \geq 9$, $d$ has a 9-edge-connected strongly $Z_5$-connected realization.

**Proof.** Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing multigraphic sequence with $d_n \geq 9$. By Theorem 2.3, we have $d_1 \leq \sum_{i=1}^{n} d_i$. If $n = 2$, then $d_1 = d_2$ and it is obvious to verify this statement by Lemma 2.1(iii). We argue by induction on $m = \sum_{i=1}^{n} d_i$ and assume that $n \geq 3$ and that Theorem 1.5 holds for smaller value of $m$. We are to construct a 9-edge-connected strongly $Z_5$-connected $d$-realization.

**Case 1:** $d_1 = 9$.

Since $d_1 \geq 9$, we have $(d_1, d_2, \ldots, d_n) = (9, 9, \ldots, 9)$. Since $\sum_{i=1}^{n} d_i$ is even and $n \geq 3$, this implies that $n$ is even and $n \geq 4$. We obtain a graph $G'$ by applying $Y − Δ$ operation on the complete graph $K_4$ several times until the cubic graph processes $n$ vertices. By Lemma 2.1(ii)(ii), $G' \in \langle Z_5 \rangle$. Let $G = 3G'$. Then $G \in \langle SZ_5 \rangle$ by Theorem 2.2(i). Since $G'$ is 3-edge-connected, $G$ is a 9-edge-connected strongly $Z_5$-connected $d$-realization.

**Case 2:** $d_2 \geq 10$.

In this case, $d_1 \geq d_2 \geq 10$, and we let $d' = (d_1 - 1, d_2 - 1, d_3, \ldots, d_n)$. By Theorem 2.4, $d'$ is multigraphic. By induction on $m$, $d'$ has a 9-edge-connected strongly $Z_5$-connected realization $G'$. Construct the graph $G$ from $G'$ by adding one edge joining $(d_1 - 1)$-vertex and $(d_2 - 1)$-vertex in graph $G'$. By Lemma 2.1(v), $G$ is also a 9-edge-connected strongly $Z_5$-connected realization of $d$.

Now, we consider the remaining case.

**Case 3:** $d_1 \geq 10$ and $d_2 = \cdots = d_n = 9$.

If $d_1 \geq 18$, we let $d'' = (d_1 - 9, 9, \ldots, d_{n-1})$. Then $d''$ is multigraphic as $d_1 - 9 \leq \sum_{i=2}^{n-1} d_i$ and by Theorem 2.3. By induction on $m$, there exists a 9-edge-connected strongly $Z_5$-connected graph $G''$ as $d''$-realization. Construct the graph $G$ by adding one new vertex $v_n$ and 9 parallel edges joining $v_n$ and $(d_1 - 9)$-vertex in $G''$. By Lemma 2.1(iii)(v), $G$ is the desired graph. Combining Case 1, we assume that $10 \leq d_1 \leq 17$. Below.

**Case 3.1:** $d_1$ is odd.

Since $\sum_{i=1}^{n} d_i$ is even, $n$ is even and $n \geq 4$. If $n = 4$ and $11 \leq d_1 \leq 15$, we let $d_1 - 9 = 2q$, where $1 \leq q \leq 3$. Let $v$ be an arbitrary vertex in $3K_4$ and let $e_1, \ldots, e_q$ be non-parallel edges not adjacent to $v$ in $3K_4$. We obtain the graph $G$ as $d$-realization from $3K_4$ by inserting the edges $e_1, \ldots, e_q$ to the vertex $v$. By Lemma 2.5(i), $G$ is 9-edge-connected. Since $G$ contains $J_2$ as a spanning subgraph, by Lemmas 2.7 and 2.1(v), $G \in \langle S\mathcal{Z}_5 \rangle$. Otherwise, $(d_1, d_2, d_3, d_4) = (17, 9, 9, 9)$, which has already been handled in Lemma 3.3.

If $n \geq 6$, we obtain a graph $G'$ by applying $Y − Δ$ operation on $K_4$ repeatedly until the cubic graph processes $n$ vertices. Denote the last obtained vertex by $v_1$ in $G'$, which is in the last generated triangle. Let $d_1 - 9 = 2q$, where $1 \leq q \leq 4$. We select $q$ edges $e_1, \ldots, e_q$ that are coming from the edges of the basic graph $K_4$, which are not adjacent to $v_1$ in the graph $G'$. Obtain the graph $G$ from $3G'$ by inserting the edges $e_1, \ldots, e_q$ and $v_1$. By Lemma 2.5(i), $G$ is 9-edge-connected. To verify that $G$ is strongly $Z_5$-connected, we first observe that the graph induced by the vertices of the last generated triangle is strongly $Z_5$-connected as it contains $J_1$ as a spanning subgraph. Then we can contract the last generated triangle and consecutively
contract all the generated triangles, the remaining graph is strongly $\mathbb{Z}_5$-connected as it contains a $J_2$ as a spanning subgraph. By Lemma 2.1(v), $G$ is a strongly $\mathbb{Z}_5$-connected $d$-realization.

**Case 3.2:** $d_1$ is even.

Since $\sum_{i=1}^{n} d_i$ is even, $n$ is odd and $n \geq 3$. When $n = 3$, we have $d = (d_1, d_2, d_3) = (d_1, 9^2)$ and it is straightforward to obtain a 9-edge connected $d$-realization $G$ containing the graph $J_1$. If $n = 5$ and $d_1 = 14$ or $d_1 = 16$ or $n = 7$ and $d_1 = 16$, then the multigraphic sequences are $(14, 9^2)$, $(16, 9^2)$, $(16, 9^2)$, which are all verified by Lemma 3.3.

The remaining cases are as follows: $n \geq 9$, or $n = 7$ and $10 \leq d_1 \leq 14$, or $n = 5$ and $10 \leq d_1 \leq 12$. We construct a graph $G'$ by applying $\Delta - \Delta$ operation on $K_4$ repeatedly until the cubic graph processes $n - 1$ vertices. Let $E' \in E(G')$ consist of the edges of the base graph $K_4$ and one edge in each generated triangle in $G$. Thus $|E'| \geq 8$ if $n \geq 9$; $|E'| = 7$ if $n = 7$; $|E'| = 6$ if $n = 5$. Let $d_1 = 2q$. Note that $|E'| \geq q$. We select the edges $e_1, \ldots, e_q$ in $E'$ and obtain the graph $G$ from $3G'$ by inserting the edges $e_1, \ldots, e_q \in E'$ to a new vertex $v_1$. By Lemma 2.5(ii), $G$ is 9-edge connected. Clearly, $G$ is a $d$-realization. To see that $G$ is strongly $\mathbb{Z}_5$-connected, we first recall that $J_1$ and $J_2$ are strongly $\mathbb{Z}_5$-connected by Lemma 2.7. By contracting $J_1$ and $3K_2$ in the generated triangles of $G$ consecutively, the resulting graph consists of 5 vertices, namely $v_1$ and the remaining 4 vertices induced a graph containing $J_2$. We then contract $J_2$ and the resulting 2q parallel edges to obtain $K_1$. This shows that $G$ is a strongly $\mathbb{Z}_5$-connected $d$-realization by Lemma 2.1(v). The proof is completed.

**Proof of Corollary 1.6.** We assume that $d = (d_1, \ldots, d_n)$ is a nonincreasing multigraphic sequence with $d_1 \geq 8$. By Theorem 2.3, $d_1 \leq \sum_{i=1}^{n} d_i$. The case of $n = 2$ is trivial. Assume that $n \geq 3$. Suppose to the contrary that $(d_1, \ldots, d_n)$ is a counterexample with $m = \sum_{i=1}^{n} d_i$ minimized.

If $d_1 \geq 10$, let $d' = (d_1', d_2', \ldots, d_n') = (d_1 - 2, d_2, \ldots, d_n)$. If $d_1 - 2 = d_2' \geq d_3' = d_2$, then $d_1' \leq d_1 \leq \sum_{i=2}^{n} d_i$. Otherwise, $d_1' = d_1 - 2 < d_2'$. If $d_1 = 8$, we let $d_1 = 1$.

If $d_1 = 8$, then $d_1 = \cdots = d_n = 8$. Hence $G = 4C_n$ is a 8-edge-connected modulo 5-orientation $d$-realization, a contradiction. Assume that $d_1 = 9$ in the following. As $\sum_{i=1}^{n} d_i$ is even, we have $d_2 = 9$. If $d_n = 8$, we let $d_i = (d_1, d_2, \ldots, d_{n-1}) = (d_1, d_2, \ldots, d_{n-1})$. Then $d_1' \leq d' \leq d_i$ is a contradiction.

By applying $\Delta - \Delta$ operation on $K_4$ repeatedly until the cubic graph processes $n - 1$ vertices. Let $E' \in E(G')$ consist of the edges of the base graph $K_4$ and one edge in each generated triangle in $G$. Thus $|E'| \geq 8$ if $n \geq 9$; $|E'| = 7$ if $n = 7$; $|E'| = 6$ if $n = 5$. Let $d_1 = 2q$. Note that $|E'| \geq q$. We select the edges $e_1, \ldots, e_q$ in $E'$ and obtain the graph $G$ from $3G'$ by inserting the edges $e_1, \ldots, e_q \in E'$ to a new vertex $v_1$. By Lemma 2.5(ii), $G$ is a $d$-realization. Clearly, $G$ is a $d$-realization. To see that $G$ is strongly $\mathbb{Z}_5$-connected, we first recall that $J_1$ and $J_2$ are strongly $\mathbb{Z}_5$-connected by Lemma 2.7. By contracting $J_1$ and $3K_2$ in the generated triangles of $G$ consecutively, the resulting graph consists of 5 vertices, namely $v_1$ and the remaining 4 vertices induced a graph containing $J_2$. We then contract $J_2$ and the resulting 2q parallel edges to obtain $K_1$. This shows that $G$ is a strongly $\mathbb{Z}_5$-connected $d$-realization by Lemma 2.1(v). The proof is completed.

**Acknowledgments**

The authors would like to thank two anonymous referees for their careful reading of the manuscript and helpful comments. The research of Hong-Jian Lai is partially supported by Chinese National Natural Science Foundation grants CNNSF 11771039 and CNNSF 11771443.

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