Line graphs containing 2-factors with bounded number of components

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Abstract

Let $G$ be a connected simple graph of order $n$. We use $L(G)$ to denote the line graph of $G$, where $L(G)$ has the edge set of $G$ as its vertex set and two vertices in $L(G)$ are adjacent if and only if the corresponding two edges in $G$ share a common endvertex. A 2-factor of $G$ is a spanning subgraph $H$ of $G$ such that every vertex in $H$ has degree 2. A lot of results on the components of a 2-factor in $G$ have appeared by studying the conditions on the

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minimum degree of $G$. In this paper, instead of studying the minimum degree, we use a different approach and obtain the following: if
\[ \max\{d(x), d(y)\} \geq \frac{n - \mu}{p} - 1 \]
holds whenever $xy \notin E(G)$ and $|U| \geq 3$, where $U = \{u : d(u) < \frac{n - \mu}{p} - 1\}$, $p > 0$ and $\mu \geq 0$ are integers, then for $n$ sufficiently large relative to $p$ and $\mu$, $L(G)$ has a 2-factor with at most $p + 1$ components. Moreover, $L(G)$ has a 2-factor with at most $p$ components if $|U| \leq 1$. Especially, it extends a result of [10] saying that if $\delta(G) \geq \frac{n}{p} - 1$, then $L(G)$ has a 2-factor with at most $p$ components. We also show the graphs satisfying the conditions mentioned above are $(p+2)$-supereularian, i.e., they have a spanning even subgraph with at most $p+2$ components. All results are best possible.

Keywords: 2-factor; reduced graph; line graph; dominating Eulerian subgraph; $k$-supereularian graph

1 Introduction

We follow [1] for terminology and notation not defined here, and consider loopless finite graphs in which multiple edges are allowed. Let $G$ be a graph and let $O(G)$ denote the set of all vertices in $G$ with odd degrees. An Eulerian graph is a connected graph $G$ with $O(G) = \emptyset$. The graph $K_1$ is an Eulerian graph. If a graph contains a spanning Eulerian subgraph, then it is called supereulerian. For literatures on supereulerian graphs, see the survey of Catlin [4] and its complement by Chen and Lai [5].

An Eulerian subgraph $H$ of a graph $G$ is dominating if $G - V(H)$ is edgeless, and in this case we call $H$ a dominating Eulerian subgraph (DES).

We use $L(G)$ to denote the line graph of $G$, where $L(G)$ has $E(G)$ as its vertex set and two vertices in $L(G)$ are adjacent if and only if the corresponding two edges in $G$ share a common endvertex. The following theorem explains the relationship between dominating Eulerian subgraphs in graph $G$ and Hamiltonian cycles in the line graph $L(G)$.

Theorem 1. (Harary and Nash-Williams, [7]) Let $G$ be a graph with $|E(G)| \geq 3$. Then $L(G)$ is Hamiltonian if and only if $G$ has a DES.

A 2-factor is a 2-regular spanning subgraph of $G$. A Hamiltonian cycle is then a 2-factor, and in one sense, it is the simplest 2-factor as it is composed of a single cycle. A circuit is an Eulerian subgraph with at least three vertices. Let $F$ be a vertex subset of $V(G)$. An Eulerian subgraph $H$ of $G$ is called $F$-Eulerian if $F \subseteq V(H)$.

A star is the complete bipartite graph $K_{1,m}$. For a given graph $G$, we say that $G$ has a $p$-system that dominates if there is a family $S$ of edge-disjoint circuits and stars with at least three edges in $G$ such that every
edge of $G$ is either in one of the circuits or stars, or is incident to a circuit in $S$, where $p = |S|$. The following result gives a characterization of graphs $G$ such that $L(G)$ contains a 2-factor with exactly $p$ components.

Theorem 2. (Gould and Hynds, [6]) Let $G$ be a graph without isolated vertices. The line graph $L(G)$ contains a 2-factor with $p$ components if and only if $G$ has a $p$-system that dominates.

There have been efforts using minimum degree or Ore-type degree sums to study the existence of 2-factors with a bounded number of components. Niu and Xiong applied this approach to line graphs and obtained the following result.

Theorem 3. (Niu and Xiong, [10]) Let $G$ be a connected simple graph of order $n$ and $p$ a positive integer such that $\delta(G) \geq \lceil n/p \rceil - 1$. If $n$ is sufficiently large relative to $p$, then $G$ has an even factor with at most $p$ components, and then $L(G)$ has a 2-factor with at most $p$ components.

Catlin [3] showed the following:

Theorem 4. (Catlin, [3]) Let $G$ be a connected simple graph of order $n$, and let $p \geq 2$ be an integer. If $d(u) + d(v) > \frac{2n}{p} - 2$ whenever $uv \notin E(G)$, and if $n \geq 4p^2$, then exactly one of the following conclusions holds:

(1) $G$ has a spanning Eulerian subgraph;

(2) $G$ is contractible to a graph $G_1$ of order less than $p$ and containing no spanning Eulerian subgraph;

(3) $p = 2$, and $G - x = K_{n-1}$ for some $x \in V(G)$ with $d(x) = 1$.

Motivated by the theorems above, in this paper, we are going to show the following main result, of which Theorem 3 is a special case.

Theorem 5. Let $G$ be a connected simple graph of order $n$, and let $p$ be a positive integer, $\mu$ a nonnegative integer, $U = \{v : d(v) < \frac{n-\mu}{p} - 1\}$, where $G[U]$ is a clique. If $n > p^3 + 6p^2 + 6p + \mu + \mu$ and $|U| \geq 3$, and if $\max\{d(x), d(y)\} \geq \frac{n-\mu}{p} - 1$ whenever $xy \notin E(G)$, then $L(G)$ has a 2-factor with at most $p + 1$ components. Moreover, if $|U| \leq 1$, then $L(G)$ has a 2-factor with at most $p$ components. Especially, if $\delta(G) \geq \frac{n}{p} - 1$ (this implies $U = \emptyset$), then $L(G)$ has a 2-factor with at most $p$ components.

We organize the paper as follows. In Section 2, we present Catlin's reduction method which will be used in the proof of Theorem 5 (in Section 3); Section 4 is devoted to a corollary; the sharpness of Theorem 5 is presented in the last section.
2 Introduction to Catlin's reduction method

In 1988, Catlin defined collapsible graphs in [2]. Let $G$ be a graph. For $R \subseteq V(G)$, a subgraph $\Gamma$ of $G$ is called an $R$-subgraph if both $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph is collapsible if $G$ has an $R$-subgraph for every even set $R \subseteq V(G)$. Apparently, $K_1$ is a collapsible graph. Let $H$ be a connected subgraph of $G$. We use $G/H$ to denote the graph obtained from $G$ by contracting $H$, that is to say, we replace $H$ with a vertex $v_H$ such that the number of edges in $G/H$ joining any $v \in V(G) - V(H)$ to $v_H$ in $G/H$ equals the number of edges joining $v$ in $G$ to $H$. We say $G$ is contractible to $G'$ if $G$ contains pairwise vertex-disjoint connected subgraphs $H_1, H_2, \ldots, H_k$ with $\bigcup_{i=1}^{k} V(H_i) = V(G)$ such that $G'$ is obtained from $G$ by successively contracting $H_1, H_2, \ldots, H_k$. Each subgraph $H_i$ of $G$ is called the preimage of the vertex $v_{H_i}$ in $G'$, and $v_{H_i}$ is called the image of $H_i$. If $H_i$ is not a single vertex in $G$, then we call $v_{H_i}$ a nontrivial vertex in $G'$. For any vertex $v \in V(H_1)$, we also say that $v_{H_1}$ is the image of the vertex $v$. Catlin [2] showed that every graph $G$ has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$. The reduction of $G$ is the graph obtained from $G$ by successively contracting $H_1, H_2, \ldots, H_k$. A nontrivial vertex in the reduction of $G$ is a vertex which is the image of a nontrivial connected subgraph of $G$. If a graph is the reduction of some graph, then we say the graph is reduced.

Theorem 6. (Catlin, [2]) Let $G$ be a connected graph and $G'$ the reduction of $G$. Then each of the following holds.

(a) $G$ is supereulerian if and only if $G'$ is supereulerian;

(b) $G'$ is triangle-free with $\delta(G') \leq 3$;

(c) If $G$ is reduced, then $G$ is a simple graph with $\delta(G') \leq 3$ and with either $G \in \{K_1, K_2\}$, or $|E(G)| \leq 2|V(G)| - 4$;

(d) If $G$ is collapsible, then $G$ is supereulerian, i.e., $G$ has a spanning Eulerian subgraph;

(e) Let $L$ be a collapsible subgraph of $G$, $v_L$ the vertex in $G/L$ to which $L$ is contracted, and $M \subseteq V(G) - V(L)$. Then $G$ has an Eulerian subgraph $H$ such that $M \cup V(L) \subseteq V(H)$ if and only if $G/L$ has an Eulerian subgraph $H'$ such that $M \cup \{v_L\} \subseteq V(H')$. 

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3 Proof of Theorem 5

Define

\[ J_p(G) = \{ v \in V(G) : d(v) \geq \frac{n - \mu}{p} - 1 \}. \]

Let \( G' \) be the reduction of \( G \) and \( n' = |V(G')| \). Let \( p \) be a positive integer, and consider the condition that for any \( xy \notin E(G) \),

\[ \max\{d(x), d(y)\} \geq \frac{n - \mu}{p} - 1. \tag{3.1} \]

We shall assume that (3.1) holds and that \( n \) is sufficiently large (say \( n > p^3 + 6p^2 + 6p + \mu p + \mu \)). Let \( U = \{ v : d(v) < \frac{n - \mu}{p} - 1 \} \), where \( G[U] \) is clearly a complete subgraph (clique) of \( G \). Moreover, \( d(v) \geq \frac{n - \mu}{p} - 1 \) for any \( v \in V(G) \setminus U \).

Let \( c = p + 5 \) and let

\[ W = \{ v \in V(G') : d_{G'}(v) \leq c \} \text{ and } W' = \{ v \in W : v \text{ is nontrivial} \}. \]

We shall prove several claims to help us establish the conclusion in our main result.

Claim 1. \( |W \setminus W'| \leq 1. \)

Proof: Since every vertex \( v \) of \( W \setminus W' \) is trivial, \( d_G(v) = d_{G'}(v) \leq c < \frac{n - \mu}{p} - 1 \) when \( n > p^3 + 6p + \mu \). Hence \( W \setminus W' \subseteq U \). Recall that \( G[U] \) is a complete subgraph of \( G \). Since every complete graph with order at least 3 is collapsible, \( G[U] \) is contracted to one vertex in \( G' \) if \( |U| \geq 3 \). Hence \( W' \setminus W' \) contains at most one vertex in \( G' \). This proves Claim 1. \( \Box \)

Furthermore, Claim 1 implies the stronger statement that if \( W \neq W' \), then it forces \( |U| = 1 \) and \( W - W' = U \).

Claim 2. For any \( v \in W' \), if \( H_v \) denotes the preimage of \( v \) in \( G \) and either \( |U| \leq 1 \) or \( G[U] \not\subseteq H_v \), then

\[ |V(H_v)| \geq \frac{n - \mu}{p} - d_{G'}(v). \tag{3.2} \]

Proof: Since \( v \) is nontrivial, \( |V(H_v)| \geq 3 \). Hence we can take a vertex \( x \in V(H_v) - U \) because either \( |U| \leq 1 \) or \( G[U] \not\subseteq H_v \). Then we have

\[ \frac{n - \mu}{p} - 1 \leq d_G(x) \leq d_{H_v}(x) + d_{G'}(v), \]

which implies

\[ d_{H_v}(x) \geq \frac{n - \mu}{p} - 1 - d_{G'}(v). \]
\[ d_{\mathcal{C}^*}(v). \text{ So } |V(H_u)| \geq d_{H_u}(x) + 1 \geq \frac{n - \mu}{p} - d_{\mathcal{C}^*}(v). \text{ This proves Claim 2.} \]

\[ \square \]

Claim 3. If \(|U| \leq 1\), then \(|W'| \leq p\); if \(|U| \geq 3\), then \(|W'| \leq p + 1\).

**Proof:** Suppose first that \(|U| \leq 1\). For any \(v \in W'\), by Claim 2 we have
\[ |V(H_u)| \geq \frac{n - \mu}{p} - d_{\mathcal{C}^*}(v), \text{ so } n \geq |W'| \left( \frac{n - \mu}{p} - c \right). \]
This is equivalent to
\[ |W'| \leq \frac{np}{n - \mu - pc}. \]
Since \(|W'|\) is an integer, we have \(|W'| \leq p\) when \(n > p^3 + 6p^2 + 5p + \mu p + \mu\).

Suppose next that \(|U| \geq 3\). Since the collapsible complete subgraph \(G[U]\) is contracted to one vertex in \(G'\), there must be exactly one vertex \(u \in W'\) such that \(G[U] \subseteq H_u\). By Claim 2, \(n - 1 \geq (|W'| - 1) \left( \frac{n - \mu}{p} - c \right)\).

This is equivalent to \(|W'| \leq \frac{(n - 1)p}{n - \mu - pc} + 1\). Since \(|W'|\) is an integer, we have \(|W'| \leq p + 1\) when \(n > p^3 + 6p^2 + 4p + \mu p + \mu\). \(\square\)

Claim 4. \(V(G') = W\).

**Proof:** By contradiction, we assume that \(V(G') \setminus W \neq \emptyset\). Note that every vertex in \(V(G') \setminus W\) has degree at least \(c + 1\) in \(G'\). Since \(G'\) is simple (\(G'\) is reduced), this means
\[ n' \geq c + 2. \quad (3.3) \]

We count the adjacencies to get \(c|V(G') \setminus W| \leq 2|E(G')| \leq 4n' - 8\) by Theorem 6 (c), which means \(|V(G') \setminus W| \leq \frac{4n' - 8}{c}\). So it follows that
\[ |W| = n' - |V(G') \setminus W| \geq \left( 1 - \frac{4}{c} \right) n' + \frac{8}{c}. \quad (3.4) \]

By Claims 1 and 3, \(|W| \leq |W'| + 1 \leq p + 2\). Hence by (3.3) and (3.4),
\[ \left( 1 - \frac{4}{c} \right) (c + 2) + \frac{8}{c} \leq \left( 1 - \frac{4}{c} \right) n' + \frac{8}{c} \leq p + 2. \]

It follows that \(p + 5 = c \leq p + 4\), a contradiction. Therefore, we must have \(V(G') = W\). \(\square\)

Claim 5. Every vertex in \(J_p(G)\) is contained in the preimage of some vertex in \(W'\).
Proof: Since $n > p^2 + 6p + \mu$, the degree of vertices in $J_p(G)$ will exceed $c$, and so Claim 5 follows from Claim 4. \(\square\)

Note that by Claim 5 and by Theorem 6 (e), if $G'$ has a $W'$-Eulerian subgraph, then $G$ has a $J_p(G)$-Eulerian subgraph. Here we do not distinguish whether $G'$ has a $W'$-Eulerian subgraph or not. Since all vertices in $W'$ are nontrivial, we can suppose that $W' = \{v_1, v_2, \ldots, v_m\}$. For any $v_i \in W'$, the pre-image of $v_i$ in $G$ denoted by $H_i$ is collapsible and hence has a spanning Eulerian subgraph by Theorem 6. Moreover, $|V(H_i)| \ge 3$ since every vertex in $W'$ is nontrivial. By Claim 1, we divide $G'$ into $m$ parts $P_1, P_2, \ldots, P_m$, where each $P_i$ is an induced subgraph of $v_i$ and its neighbors (trivial) from $W \setminus W'$. Therefore, $G$ is divided into $m$ parts which are the corresponding pre-images of $P_1, P_2, \ldots, P_m$ and each part has a dominating Eulerian subgraph $H_i$, thus $\bigcup_{i=1}^m H_i$ is an $m$-system of $G$ that dominates. By Theorem 2, $L(G)$ has a 2-factor with $m$ components. Since $m = |W'| \le p + 1$ (by Claim 3), it follows that $L(G)$ has a 2-factor with at most $p + 1$ components. Moreover, if $|U| \le 1$, then $m = |W'| \le p$ by Claim 3, so $L(G)$ has a 2-factor with at most $p$ components. Therefore, the proof of Theorem 5 is completed. \(\square\)

4 A Corollary

A graph is called $k$-supereulerian, if $G$ has a spanning even subgraph with at most $k$ components. The following result was proved recently.

Theorem 7. (Niu, Lai and Xiong, [9]) Let $G$ be a connected graph and $G'$ be the reduction of $G$. Then $G$ is $k$-supereulerian if and only if $G'$ is $k$-supereulerian.

By Theorem 7 and by the proof of Theorem 5, we obtain the following consequence.

Corollary 8. Let $G$ be a connected simple graph of order $n$, and let $p$ be a positive integer and $\mu$ a nonnegative integer. If $\max\{d(x), d(y)\} \ge \frac{n - \mu - 1}{p}$ holds for any $xy \notin E(G)$, then for $n > p^2 + 6p^2 + 6p + \mu p + \mu$, $G$ is $(p + 2)$-supereulerian if $|U| \ge 3$, where $U = \{v : d(v) < \frac{n - \mu - 1}{p}\}$, and $G$ is $(p + 1)$-supereulerian if $|U| \le 1$. Moreover, $G$ is $p$-supereulerian if $U = \emptyset$.

5 Sharpness

In this section, we shall give an example to show the sharpness of Theorem 5 and Corollary 8. Let $G_1, G_2, \ldots, G_p$ be $p$ vertex-disjoint complete graphs
of order \( \frac{n-\mu}{p} \) and \( G_\mu \) an additional complete graph of order \( \mu \). Obtain \( G \) by joining exactly one vertex of \( G_i \) to exactly one vertex of \( G_\mu \) for \( i = 1, 2, \ldots, p \). Note that \( \max\{d(x), d(y)\} \geq \frac{n-\mu}{p} - 1 \) whenever \( xy \notin E(G) \) (where \( \mu = |\{v : d(v) < \frac{n-\mu}{p} - 1\}| \) and the equation can be achieved for some pairs of nonadjacent vertices \( x, y \). If \( \mu > 2 \) and \( n \) is sufficient large, then \( L(G) \) does not have a 2-factor with at most \( p \) components; if \( \mu = 1 \), then \( L(G) \) has no 2-factor with \( p - 1 \) components; if \( \mu = 2 \) and \( p \geq 3 \), then \( L(G) \) has no 2-factor with at most \( p \) components by the fact that \( G \) has no \( p \)-system that dominates. Especially, when \( \mu = 2 \), if \( p \leq 2 \) and any one of the two vertices in \( G_p \) is adjacent to at most one of \( G_i \), then \( L(G) \) has no 2-factors at all since \( G \) has no system that dominates. For \( \mu = 0 \), let \( G' \) be the graph of order \( n \) obtained from \( G_1, G_2, \ldots, G_p \) such that \( G/G_1, G_2, \ldots, G_p \) is a tree. Then \( G' \) satisfies the condition of Theorem 5 and \( L(G') \) has a 2-factor with \( p \) components. This shows Theorem 5 is best possible.

If \( \mu = 2 \), \( G \) is \((p + 2)\)-supereulerian but not \((p + 1)\)-supereulerian; if \( \mu \neq 2 \), then \( G \) is \((p + 1)\)-supereulerian but not \( p \)-supereulerian; if \( \mu = 0 \) then \( G' \) is \( p \)-supereulerian but not \((p - 1)\)-supereulerian. This shows that Corollary 8 is best possible.

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References


