Modulo orientations with bounded independence number

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Abstract

A mod \((2p + 1)\)-orientation \(D\) of a graph \(G\) is an orientation of \(G\) such that \(d^+_G(v) \equiv d^-_G(v) \mod 2p + 1\) for any vertex \(v \in V(G)\). Extending Tutte’s integer flow conjectures, it was conjectured by Jaeger that every \(4p\)-edge-connected graph has a \((2p + 1)\)-orientation. However, this conjecture has been disproved in Han et al. (2018) recently. Infinite families of \(4p\)-edge-connected graphs (for \(p \geq 3\)) and \((4p + 1)\)-edge-connected graphs (for \(p \geq 5\)) with no mod \((2p + 1)\)-orientation are constructed in Han et al. (2018). In this paper, we show that every family of graphs with bounded independence number has only finitely many contraction obstacles for admitting mod \((2p + 1)\)-orientations, contrasting to those infinite families. More precisely, we prove that for any integer \(t \geq 2\), there exists a finite family \(\mathcal{F} = \mathcal{F}(p, t)\) of graphs that do not have a mod \((2p + 1)\)-orientation, such that every graph \(G\) with independence number at most \(t\) either admits a mod \((2p + 1)\)-orientation or is contractible to a member in \(\mathcal{F}\). This indicates that the problem of determining whether every \(k\)-edge-connected graph with independence number at most \(t\) admits a mod \((2p + 1)\)-orientation is computationally solvable for fixed \(k\) and \(t\). In particular, the graph family \(\mathcal{F}(p, 2)\) is determined, and our results imply that every \(8\)-edge-connected graph \(G\) with independence number at most two admits a mod 5-orientation.

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1. Introduction

In this paper, we consider graphs which are finite and loopless, with possible parallel edges. We follow [1] for undefined terms and notation. Let \(\mathbb{Z}\) denote the set of integers. For \(k \in \mathbb{Z}\) with \(k > 1\), let \([k]\) = \{1, 2, ..., \(k\)\} and \(\mathbb{Z}_k\) denote the set of all integers modulo \(k\), as well as the (additive) cyclic group of order \(k\). Following [1], for a graph \(G\), \(\alpha(G)\), \(\kappa(G)\), and \(\delta(G)\) denote the independence number, the edge-connectivity, and the minimum degree, respectively. For each edge \(e \in E(G)\), let \(\mu(e)\) be the maximum number of edges joining the two end vertices of \(e\), and denoted \(\mu(G) = \max(\mu(e) : e \in E(G))\) to be the edge multiplicity of \(G\). For vertex subsets \(U, W \subseteq V(G)\), let \([U, W]_c = \{uw \in E(G) : u \in U, w \in W\}\). When \(U = \{u\}\) or \(W = \{w\}\), we use \([u, W]_c\) or \([U, w]_c\) for \([U, W]_c\), respectively. For notational convenience, we also denote \(E_c(v) = \{v, V(G) - \{v\}\}\) and \(d_c(v) = |S, V(G) - S|\) for \(v \in V(G)\) and \(S \subseteq V(G)\). The subscript \(G\) may be omitted when \(G\) is understood from the context.

For an edge set \(X \subseteq E(G)\), the contraction \(G/X\) is the graph obtained from \(G\) by identifying the two ends of each edge in \(X\), and then deleting the resulting loops. If \(H\) is a subgraph of \(G\), then we use \(G/H\) for \(G/E(H)\).

Let \(D = D(G)\) denote an orientation of \(G\). For each \(v \in V(G)\), let \(E^+_G(v)\) \((E^-_G(v))\), respectively) be the set of all arcs directed out from (into, respectively) \(v\). Following [1], \(d^+_G(v) = |E^+_G(v)|\) and \(d^-_G(v) = |E^-_G(v)|\) denote the out-degree and the in-degree of \(v\) under the orientation \(D\), respectively. If a graph \(G\) has an orientation \(D\) such that \(d^+_G(v) - d^-_G(v) \equiv 0 \mod k\) for every

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vertex \( v \in V(G) \), then we say that \( G \) admits a \textbf{modulo} \( k \)-orientation, or a \( k \)-orientation for short. Let \( \mathcal{M}_k \) denote the family of all graphs admitting a \( k \)-orientation. As a connected graph \( G \) has a modulo 2\( p \)-orientation if and only if \( G \) is Eulerian, we focus on the case when \( k = 2p + 1 \) is odd in this paper. We shall always assume that \( p \) is a positive integer throughout this paper.

The concept of modulo orientation is motivated by the integer flow of graphs introduced by Tutte \cite{Tutte17, Tutte18}. An \textbf{integer flow} of a graph \( G \) is an ordered pair \((D,f)\), where \( D \) is an orientation and \( f \) is a mapping from \( E(G) \) to integers such that \( \sum_{e \in D(v)} f(e) - \sum_{e \in D^-(v)} f(e) = 0 \) for every vertex \( v \in V(G) \). An integer flow \((D,f)\) is called a \textbf{nowhere-zero} \( k \)-flow if \( 1 \leq |f(e)| \leq k - 1 \) for each edge \( e \in E(G) \). Jaeger \cite{Jaeger90} observed that, in a graph \( G \), the existence of a mod \((2p+1)\)-orientation is equivalent to the existence of an integer flow \((D,f)\) with \( |f(e)| \in \{p, p+1\} \) for each \( e \in E(G) \), which is called a \textbf{circular} \( (2 + \frac{1}{p}) \)-flow. In particular, it is well-known that a graph admits a nowhere-zero \( 3 \)-flow if and only if it admits a mod \( 3 \)-orientation (see \cite{Tutte17, Wolfowitz56}). Tutte’s 3-flow conjecture (see \cite{Goddyn01}) can be stated as follows.

**Conjecture 1.1** (Tutte). Every 4-edge-connected graph admits a mod 3-orientation.

In addition, as observed by Jaeger \cite{Jaeger90}, Tutte’s famous 5-flow conjecture \cite{Tutte20}, which asserts that every bridgeless graph admits a nowhere-zero 5-flow, is implied by the following conjecture.

**Conjecture 1.2** (Jaeger, \cite{Jaeger90}). Every 9-edge-connected graph admits a mod 5-orientation.

It was originally conjectured by Jaeger \cite{Jaeger91} that every 4\( p \)-edge-connected graph admits a mod \((2p+1)\)-orientation, known as the \textbf{circular} \( 4 \)-flow conjecture. Thomassen \cite{Thomassen91} settled the weak version of 3-flow conjecture and the weak version of Jaeger’s circular flow conjecture by showing every 8-edge-connected graph admits a mod 3-orientation and every \((2k^2 + k)\)-edge-connected graph admits a mod \( k \)-orientation. Lovász, Thomassen, Wu and Zhang \cite{Lovasz92, Thomassen92} further refined the method to show that every 6\( p \)-edge-connected graph admits a mod \((2p + 1)\)-orientation. Very recently, infinite families of 4\( p \)-edge-connected graphs with no mod \((2p+1)\)-orientation were constructed in \cite{Ding14} for every \( p \geq 3 \). There exist \((4p+1)\)-edge-connected graphs admitting no mod \((2p+1)\)-orientation for every \( p \geq 5 \), as well. A new conjecture on modulo orientation is proposed in \cite{Ding14}, that for every positive integer \( p \), there exists a positive constant \( \varepsilon = \varepsilon(p) < \frac{1}{2} \) such that every \( (4 + \varepsilon)p \)-edge-connected graph admits a mod \((2p + 1)\)-orientation. This suggests that while the connectivity requirement may increase for larger \( p \), the truth of the new conjecture still implies Tutte’s 3-flow conjecture and 5-flow conjecture by results of Kochol \cite{Kochol12} and Jaeger \cite{Jaeger90}. The readers are referred to \cite{Woodall91} or \cite{Chen17} for a comprehensive introduction on integer flows and modulo orientations.

In this paper, we investigate \((2p + 1)\)-orientations of graphs with bounded independence numbers. It is known that the complete graph \( K_{4p} \) does not admit a mod \((2p + 1)\)-orientation. Since the modulo orientation property is preserved under contraction, it is straightforward to construct an infinite family of graphs of independence number two without mod \((2p + 1)\)-orientation by replacing a vertex of \( K_{4p} \) with a large complete graph. On the other hand, all those graphs have the behavior that each of them is contractible to \( K_{4p} \). So we may expect to characterize mod \((2p + 1)\)-orientation in the family of graphs with bounded independence number by excluding a list of graphs such that every graph in the family admits a mod \((2p + 1)\)-orientation if and only if it is not contractible to one of the graphs on the list, such as in Kuratowski’s theorem for planar graphs and characterization of graphs embedded on surface by excluding minors. Our first main result asserts that it is indeed the case and such a list contains finitely many graphs only.

Let \( t \geq 1 \) be an integer, and define a finite graph family \( G_0(t) \) to be

\[ G_0(t) = \{ G : G \not\in \mathcal{M}_{2p+1}, \alpha(G) \leq t, \mu(G) \leq 2p - 1 \text{ and } |V(G)| \leq 6pt - 2p \}. \]

**Theorem 1.3.** For any graph \( G \) with \( \alpha(G) \leq t \), \( G \) admits a mod \((2p + 1)\)-orientation if and only if \( G \) is not contractible to a member in \( G_0(t) \).

As a corollary of Theorem 1.3, for a given integer \( k > 0 \), in order to seek mod \((2p + 1)\)-orientations for all \( k \)-edge-connected graphs with independence number at most \( t \), it suffices to search such graphs on at most \( 6pt - 2p \) vertices, which consist of only finitely many graphs and is computationally solvable.

**Corollary 1.4.** The following are equivalent.

(i) Every \( k \)-edge-connected graph \( G \) with \( \alpha(G) \leq t \) admits a mod \((2p + 1)\)-orientation.

(ii) Every \( k \)-edge-connected graph \( G \) with \( \alpha(G) \leq t \), \( \mu(G) \leq 2p - 1 \) and \( |V(G)| \leq 6pt - 2p \) admits a mod \((2p + 1)\)-orientation.

To obtain Theorem 1.3, we need to introduce orientation with boundaries. For a graph \( G \), a function \( b : V(G) \to \mathbb{Z}_{2p+1} \) is called a \textbf{boundary function} of \( G \), or \textbf{boundary} for short, if \( \sum_{v \in V(G)} b(v) \equiv 0 \pmod{2p + 1} \). Denote \( Z(G, \mathbb{Z}_{2p+1}) \) to be the set of all boundary functions of \( G \). Motivated by the group connectivity property defined by Jaeger et al. \cite{Jaeger87}, the concept of strongly \( \mathbb{Z}_{2p+1} \)-connectedness was introduced in \cite{Ding98} (see also \cite{Chen17}), serving as contractible configurations for mod \((2p + 1)\)-orientations.

**Definition 1.5.** A graph \( G \) is \textbf{strongly} \( \mathbb{Z}_{2p+1} \)-\textbf{connected} if, for every \( b \in Z(G, \mathbb{Z}_{2p+1}) \), there is an orientation \( D \) such that \( d_D^+(v) - d_D^-(v) \equiv b(v) \pmod{2p + 1} \) for every vertex \( v \in V(G) \).
Let $\langle \Sigma Z_{2p+1} \rangle$ denote the family of all strongly $Z_{2p+1}$-connected graphs.

Liang et al. [13] proved that the graph family $\langle \Sigma Z_{2p+1} \rangle$ consists of exactly all mod $(2p+1)$-orientation contractible configurations, that is, all those graphs $G$ such that for every supergraph $\Gamma$ containing $G$ as a subgraph, $\Gamma'/G$ has a mod $(2p+1)$-orientation if and only if $\Gamma$ has a mod $(2p+1)$-orientation.

A subgraph $H$ of $G$ is called a maximal $\langle \Sigma Z_{2p+1} \rangle$-subgraph of $G$ if $H \in \langle \Sigma Z_{2p+1} \rangle$ and for any subgraph $L$ of $G$ containing $H$ as a proper subgraph, $L \notin \langle \Sigma Z_{2p+1} \rangle$. Since $K_1 \in \langle \Sigma Z_{2p+1} \rangle$ by definition, every vertex of a graph $G$ lies in a maximal $\langle \Sigma Z_{2p+1} \rangle$-subgraph of $G$. Let $H_1, H_2, \ldots, H_t$ denote the collection of all maximal $\langle \Sigma Z_{2p+1} \rangle$-subgraphs of $G$. Then $G' = G/(\bigcup_{i=1}^{t} E(H_i))$ is the $\langle \Sigma Z_{2p+1} \rangle$-reduction of $G$, and we also say $G$ is $\langle \Sigma Z_{2p+1} \rangle$-reduced to $G'$. A graph $G$ is $\langle \Sigma Z_{2p+1} \rangle$-reduced if $G$ does not have any nontrivial subgraph in $\langle \Sigma Z_{2p+1} \rangle$. By definition, the $\langle \Sigma Z_{2p+1} \rangle$-reduction of a graph is always $\langle \Sigma Z_{2p+1} \rangle$-reduced. Since contraction may bring in new parallel edges, even when $G$ is a simple graph, its $\langle \Sigma Z_{2p+1} \rangle$-reduction may have multiple edges. As the counterexamples constructed in [3] are indeed $\langle \Sigma Z_{2p+1} \rangle$-reduced graphs, the following is also obtained in [3]: there exists infinitely many $(4p+1)$-edge-connected $\langle \Sigma Z_{2p+1} \rangle$-reduced graphs for every $p \geq 5$. While in this paper, we show that there are finitely many $\langle \Sigma Z_{2p+1} \rangle$-reduced graph in the family of graphs with bounded independence number (see Corollary 2.4).

Theorem 1.3 is an immediate corollary of the following Theorem 1.6. In Theorem 1.6, the $\langle \Sigma Z_{2p+1} \rangle$-reduction operation, a special contraction which preserves mod $(2p+1)$-orientations, would be used to replace a general contraction operation. For any integer $t > 0$, define $F(t)$ and $G(t)$ to be graph families such that

$$F(t) = \{ G : G \text{ is } \langle \Sigma Z_{2p+1} \rangle \text{-reduced with } 2 \leq |V(G)| \leq 6pt - 2p \text{ and } \alpha(G) \leq t \}$$
$$G(t) = F(t) \setminus M_{2p+1}.$$

**Theorem 1.6.** Let $t > 0$ be an integer. Each of the following holds.

(i) A graph $G$ with $\alpha(G) \leq t$ is strongly $Z_{2p+1}$-connected if and only if the $\langle \Sigma Z_{2p+1} \rangle$-reduction of $G$ is not in $F(t)$.

(ii) A graph $G$ with $\alpha(G) \leq t$ admits a modulo $(2p+1)$-orientation if and only if the $\langle \Sigma Z_{2p+1} \rangle$-reduction of $G$ is not in $G(t)$.

More descriptions concerning the graph families $F(t)$ and $G(t)$ will be presented below when $t = 2$. In particular, Theorem 1.7 confirms that simple graphs with independence number 2 and large order admit mod $(2p+1)$-orientations under edge-connectivity $4p$.

Let $K_n$ denote a complete graph with $V(K_n) = \{v_1, \ldots, v_n\}$. For nonnegative integers $s_1, s_2, \ldots, s_{n-1}$, let $K_n(s_1, s_2, \ldots, s_{n-1})$ be the graph obtained from $K_n$ by replacing the edge $v_i v_j$ by $s_i$ parallel edges joining $v_i$ and $v_j$, for each $i \in [n-1]$, and define

$$\kappa(2p+1) = \{ K_n(s_1, s_2, \ldots, s_{n-1}) : 2 \leq n \leq 4p + 1 \text{ and } 0 \leq s_i \leq 2p - 1, \forall i \in [n-1] \},$$

$$\kappa_i(2p+1) = \kappa(2p+1) \setminus M_{2p+1} \text{ and } \kappa_2(2p+1) = \kappa(2p+1) \setminus \langle \Sigma Z_{2p+1} \rangle.$$

**Theorem 1.7.** Let $G$ be a simple graph of order at least $10p + 1$ with $\alpha(G) \leq 2$. Each of the following holds.

(i) $G$ admits a mod $(2p+1)$-orientation if and only if $\langle \Sigma Z_{2p+1} \rangle$-reduction of $G$ is not in $\kappa_1(2p+1)$.

(ii) $G$ is strongly $Z_{2p+1}$-connected if and only if the $\langle \Sigma Z_{2p+1} \rangle$-reduction of $G$ is not in $\kappa_2(2p+1)$.

(iii) If $\kappa_i(G) \geq 2p$ and $\delta(G) \geq 4p$, then $G$ is strongly $Z_{2p+1}$-connected (and therefore, admits a mod $(2p+1)$-orientation).

As mod $5$-orientation of graphs with multiple edges is related to $5$-flow conjecture (see [5,10]), we also show the corresponding Theorem 1.8 for all graphs with independence number two in the mod $5$-orientation case. Note that this verifies Conjecture 1.2 for all graphs with order at least 21 and independence number at most two.

Let $\kappa^*(5)$ be the family of graphs such that $H \in \kappa^*(5)$ if and only if $H / M_5$, $H$ is $\langle \Sigma Z_5 \rangle$-reduced, and $H$ contains a subgraph isomorphic to $K_{V(H) - 1}$ with $2 \leq |V(H)| \leq 9$ and $\kappa^*(H) \leq 7$.

**Theorem 1.8.** Let $G$ be a graph of order at least 21 with $\alpha(G) \leq 2$. Each of the following holds.

(i) $G$ admits a mod $5$-orientation if and only if the $\langle \Sigma Z_{2p+1} \rangle$-reduction of $G$ is not in $\kappa^*(5)$.

(ii) $G$ admits a mod $5$-orientation provided it is $8$-edge-connected.

Luo et al. [15] characterized mod $3$-orientations of graphs with independence number at most 2, and thus verifies Tutte’s $3$-flow conjecture for graphs with independence number at most 2. In a consequence paper [11], Li, Luo and Wang adopt a similar idea as in this paper and develop some new reduction method to obtain analogous results for mod $3$-orientations. The results in paper [11] further confirm Tutte’s $3$-flow conjecture for graphs with independence number at most 4.

The remainder of this paper is organized as follows: In Section 2, we introduce some tools and give the proofs of Theorems 1.6 and 1.7. The proof of Theorem 1.8 is presented in Section 3, and we conclude this paper with a few remarks in the last section.

### 2. Reductions on mod $(2p+1)$-orientations

#### 2.1. Some tools

We first display the needed tools in our proofs of the main results. Lemma 2.1 is a brief summary of certain basic properties from [8,9,12].
Lemma 2.1 ([8,9] and [12]). Let G be a graph and let m, p > 0 be integers. Each of the following holds.
(i) If \( G \in \langle S\mathbb{Z}_{2p+1} \rangle \) and \( e \in E(G) \), then \( G/e \in \langle S\mathbb{Z}_{2p+1} \rangle \).
(ii) If \( H \subseteq G \), and if both \( H \in \langle S\mathbb{Z}_{2p+1} \rangle \) and \( G/H \in \langle S\mathbb{Z}_{2p+1} \rangle \), then \( G \in \langle S\mathbb{Z}_{2p+1} \rangle \).
(iii) Let \( mk_2 \) denote the loopless graph with two vertices and \( m \) parallel edges. Then \( mk_2 \in \langle S\mathbb{Z}_{2p+1} \rangle \) if and only if \( m \geq 2p \).
(iv) The complete graph \( k_2 \in \langle S\mathbb{Z}_{2p+1} \rangle \) if and only if \( n = 1 \) or \( n \geq 4p + 1 \).
(v) \( G \in \mathcal{M}_{2p+1} \) if and only if its \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduction \( G' \in \mathcal{M}_{2p+1} \).
(vi) \( G \in \langle S\mathbb{Z}_{2p+1} \rangle \) if and only if its \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduction \( G' = k_1 \).

Let \( G \) be a graph and \( b \in Z(G, \mathbb{Z}_{2p+1}) \) be a boundary function. Define an integer valued mapping \( \tau : 2^V(G) \mapsto \{0, \pm 1, \ldots, \pm(2p + 1)\} \) as follows: for each vertex \( v \in V(G) \),
\[
\tau(v) = \begin{cases} 
\frac{d(v)}{2} & \text{mod } 2 \\
\frac{b(v)}{2} & \text{mod } 2p + 1.
\end{cases}
\]

For a vertex set \( A \subseteq V(G) \), let \( b(A) = \sum_{v \in A} b(v) \) (mod \( 2p + 1 \)), \( d(A) = |\{v \in V(G) \setminus A | b(v)|} \) and define \( \tau(A) \) to be
\[
\tau(A) = \begin{cases} 
\frac{d(A)}{2} & \text{mod } 2 \\
\frac{b(A)}{2} & \text{mod } 2p + 1.
\end{cases}
\]

Theorem 2.2 (Lovász, Thomassen, Wu and Zhang, Theorem 3.1 of [14]). Let \( G \) be a graph and \( b \in Z(G, \mathbb{Z}_{2p+1}) \). Let \( z_0 \) be a vertex of \( V(G) \) and let \( D_{z_0} \) be a pre-orientation of \( E(z_0) \). Assume that
(i) \( |V(G)| \geq 3 \),
(ii) \( d(z_0) \leq 4p + |\tau(z_0)| \), and the edges incident with \( z_0 \) are pre-directed such that \( d^+(z_0) - d^-(z_0) \equiv b(z_0) \) (mod \( 2p + 1 \)).
(iii) \( d(A) \geq 4p + |\tau(A)| \) for each nonempty \( A \subseteq V(G) \setminus \{z_0\} \) with \( |V(G) \setminus A| \geq 2 \).

Then \( D_{z_0} \) can be extended to an orientation \( D \) of the entire graph \( G \) such that, for each vertex \( v \in V(G) \),
\[
d^+_D(v) - d^-_D(v) = b(v) \pmod{2p + 1}.
\]

Theorem 2.2 implies that every \( 6p \)-edge-connected graph is strongly \( \mathbb{Z}_{2p+1} \)-connected. We would further explore more properties concerning \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduced graphs below by utilizing Theorem 2.2.

2.2. Proof of Theorem 1.6
Recall that \( G \in F(t) \) if and only if \( G \) is \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduced with \( 2 \leq |V(G)| \leq 6pt - 2p \) and \( \alpha(G) \leq t \). By Lemma 2.1(iii), every graph in \( F(t) \) has edge multiplicity at most \( 2p - 1 \), and so \( F(t) \) contains finitely many graphs. Note that, by Lemma 2.1(v), Theorem 1.3 is a weak version of Theorem 1.6(ii), and Theorem 1.6(ii) follows from Theorem 1.6(i). We will show a variation of Theorem 1.6(i), as stated in Theorem 2.3.

Theorem 2.3. For any graph \( G \) with \( \alpha(G) \leq t \), \( G \) is strongly \( \mathbb{Z}_{2p+1} \)-connected if and only if the \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduction of \( G \) is not in \( F(t) \).

Proof. By Lemma 2.1(vi), a graph \( G \) is strongly \( \mathbb{Z}_{2p+1} \)-connected if and only if its \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduction is \( K_1 \), which is not in \( F(t) \) by definition. So it remains to show that
\[
\text{if } \langle S\mathbb{Z}_{2p+1} \rangle \text{-reduction of } G \text{ is not in } F(t), \text{ then } G \in \langle S\mathbb{Z}_{2p+1} \rangle.
\]

We shall prove (4) by induction on \( t \). When \( t = 1 \), (4) follows from Lemma 2.1(iv). Assume that \( t \geq 2 \) and (4) holds for smaller values of \( t \).

Let \( G' \) be a counterexample to (4) such that \( |V(G')| \) is minimal. Then \( G'' \), the \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduction of \( G' \), satisfies \( |V(G'')| \geq 6pt - 2p + 1 \) by the definition of \( F(t) \). Hence \( G'' \) itself is a counterexample to (4), and so \( |V(G'')| = |V(G')| \) by the minimality of \( |V(G')| \). Therefore, \( G' = G'' \) is a \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduced graph.

Claim A. \( \delta(G') \geq 6p \).
Suppose that \( G' \) has minimal degree at most \( 6p - 1 \) and let \( z \in V(G') \) be a vertex with \( d_{G'}(z) = \delta(G') \leq 6p - 1 \). Denote \( H = G' - (N_{G'}(z) \cup \{z\}) \). Then \( \alpha(H) \leq \alpha(G') - 1 \leq t - 1 \). As \( H \) is \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduced, we have \( |V(H)| \leq 6pt - 2p + 1 \) by (4) with induction hypothesis on \( t - 1 \). It follows that \( 6pt - 2p + 1 \leq |V(G')| = |V(H)| + |N_{G'}(z) \cup \{z\}| \leq 6pt - 2p + 6p = 6pt - 2p \).

This contradiction justifies Claim A.

Now assume \( \delta(G') \geq 6p \). By Theorem 2.2, \( \kappa'(G') < 6p \), and so \( G' \) must have an edge cut of size less than \( 6p \). For a vertex subset \( W \subseteq V(G') \), let \( W^c = V(G') - W \). Among all edge-cuts \( [W, W^c] \) of size at most \( 6p - 1 \) in \( G' \), choose one with \( |W| \) minimized. As \( \delta(G') \geq 6p \), we have \( |W| \geq 2 \). Let \( G_1 = G'/[W^c] \) and \( z_0 \) be the vertex in \( G_1 \) onto which \( W^c \) is contracted. Thus \( d_{G_1}(z_0) = |W| \) and \( W \subseteq G_1 \). Let \( W \subseteq G_1 \) form a new graph \( G \). Note that \( G'[W] = G_1[W] \equiv G[W] = G - z_0 \). We will apply Theorem 2.2 to show the following Claim B, leading a contradiction to the fact that \( G' \) is a \( \langle S\mathbb{Z}_{2p+1} \rangle \)-reduced graph.
Claim B. \( I'[W] = G - z_0 \) is strongly \( \mathbb{Z}_{2p+1} \)-connected.

Let \( D_{z_0} \) be a fixed orientation of \( E_G(z_0) \) such that
\[
4p + 1 \text{ edges are oriented out of } z_0 \text{ and the rest } 2p \text{ edges are oriented into } z_0.
\]

We also use \( D_{z_0} \) to denote the digraph induced by the oriented edges of \( D_{z_0} \). Define \( b_1(v) = d_{D_{z_0}}^+(v) - d_{D_{z_0}}^-(v) \) for each vertex \( v \in N_G(z_0) \cup \{z_0\} \).

For any \( b' \in Z(G - z_0, \mathbb{Z}_{2p+1}) \), we are to show that there exists an orientation \( D' \) of \( G - z_0 \) such that \( d_{D'}^+(v) - d_{D'}^-(v) = b'(v) \) (mod \( 2p + 1 \)) for any vertex \( v \in V(G - z_0) \). Define a mapping \( m : V(G) \to \mathbb{Z}_{2p+1} \) as follows. For any \( x \in V(G) \),
\[
b(x) \equiv \begin{cases} 
  b'(x) + b_1(x) \pmod{2p+1} & \text{if } x \in N_G(z_0); \\
  b_1(z_0) \pmod{2p+1} & \text{if } x = z_0; \\
  b'(x) \pmod{2p+1} & \text{otherwise.}
\end{cases}
\]

We are going to show that \( \text{Theorem 2.2} \) is applicable to this graph \( G \).

As \( b_1(z_0) + \sum_{v \in N_G(z_0)} b_1(v) = 0 \) and \( b' \in Z(G - z_0, \mathbb{Z}_{2p+1}) \), we have \( \sum_{v \in V(G)} b(x) = b_1(z_0) + \sum_{v \in N_G(z_0)} b_1(v) + \sum_{v \in V(G)} b'(v) \equiv 0 \) (mod \( 2p + 1 \)), and so \( b \in Z(G, \mathbb{Z}_{2p+1}) \). Let \( |W| \geq 2 \) and \( |V(G)| \geq 3 \). By (5), both \( d(z_0) = 6p + 1 \) and \( d(z_0) = d_{D_{z_0}}(z_0) - d_{D_{z_0}}(z_0) = 0 \) (mod \( 2p + 1 \)). This, together with (2), implies that \( |\tau(z_0)| = 2p + 1 \), and so \( \text{Theorem 2.2(i)} \) and (ii) are satisfied.

By (3) and the minimality of \( W \), for any \( A \subseteq W \), let \( |A| < |W| \), we have \( d(A) \geq 6p \), or \( d(A) - 4p \geq 2p \). As \( d(A) = \tau(A) \) (mod \( 2p + 1 \)), it follows by a parity argument that \( d(A) \geq 4p + |\tau(A)| \). Thus \( \text{Theorem 2.2(iii)} \) holds, and hence it holds also for the graph \( G \).

By \( \text{Theorem 2.2} \), there exists an orientation \( D \) of \( G \) such that \( d_D^+(v) - d_D^-(v) = b(x) \) (mod \( 2p + 1 \)) for each vertex \( x \in V(G) \).

Moreover, by definition of \( b \), we have \( d_D^+(v) - d_D^-(v) \equiv b'(v) \) (mod \( 2p + 1 \)) for each vertex \( v \in V(G - z_0) \). It follows by definition that \( I'[W] = G - z_0 \) is strongly \( \mathbb{Z}_{2p+1} \)-connected, and thus \( \text{Claim B} \) holds.

Since \( |W| \geq 2 \), \( \text{Claim B} \) is contrary to the assumption that \( \omega \) is \( \mathbb{Z}_{2p+1} \)-reduced. This proves \( \text{Theorem 2.3} \).

Theorem 2.3 immediately leads the following corollary, which reveals that there are finitely many \( \mathbb{Z}_{2p+1} \)-reduced graph in the family of graphs with independence number at most \( t \).

Corollary 2.4. Every \( \mathbb{Z}_{2p+1} \)-reduced graph \( G \) with \( \omega(G) \leq t \) has order at most \( 6pt - 2p \).

2.3. Proof of Theorem 1.7

We need one more lemma before presenting the proof of \( \text{Theorem 1.7} \). For a graph \( G \), let \( \xi(G) \) be the number of nontrivial maximal \( \mathbb{Z}_{2p+1} \)-subgraphs of \( G \).

Lemma 2.5. If \( G \) is a simple graph with \( \omega(G) \leq 2 \), then \( \xi(G) \leq 2 \). Furthermore, \( \xi(G) = 2 \) if and only if \( V(G) \) consists of vertex sets of exactly two maximal \( \mathbb{Z}_{2p+1} \)-subgraphs.

Proof. Assume that \( c = \xi(G) \geq 2 \) and let \( H_1, H_2, \ldots, H_c \) be the nontrivial maximal \( \mathbb{Z}_{2p+1} \)-subgraphs of \( G \). By \( \text{Lemma 2.1(iv)} \), every strongly \( \mathbb{Z}_{2p+1} \)-connected simple graph other than \( K_1 \) has order at least \( 4p + 1 \), and so \( |V(H_i)| \geq 4p + 1 \) for each \( 1 \leq i \leq c \).

By contradiction, we assume that \( c \geq 3 \), and so there exists a vertex \( v \in V(G) \setminus (\{H_1 \cup H_2\}) \). By \( \text{Lemma 2.1(ii)(ii)} \), both \(|[v, V(H_1)]| \leq 2p - 1 \) and \(|[v, V(H_2)]| \leq 2p - 1 \). Since \( |V(H_i)| \geq 4p + 1 \), there exists \( u_i \in V(H_i) \) such that \( u_1 v \notin E(G) \) and \( |[u_1, V(H_2)]| = 0 \). Similarly, there exists \( u_2 \in V(H_2) \) such that \( u_2 v \notin E(G) \) and \( |[u_2, V(H_1)]| = 0 \). Then it follows that \( \{u_1, u_2, v\} \) is an independent set of size 3, contradicting to \( \omega(G) \leq 2 \). This proves that we must have \( \xi(G) \leq 2 \), and when \( \xi(G) = 2 \), \( G = \mathbb{Z} = (H_1 \cup H_2) \).

Proof of Theorem 1.7. Since \( \mathbb{Z}_{2p+1} \subseteq \mathbb{M}_{2p+1} \), we have \( \mathbb{K}_2(2p + 1) = \mathbb{K}_2(2p + 1) \subseteq \mathbb{M}_{2p+1} \) by (1). Thus by \( \text{Lemma 2.1(iv)} \), \( \mathbb{Z}_{2p+1} \)-reduction of \( G \) is a member in \( \mathbb{K}_2(2p + 1) \) with \( s_1 = \cdots = s_{n-1} = 0 \). Hence we assume that \( G \) is not connected and not strongly \( \mathbb{Z}_{2p+1} \)-connected. By \( \text{Lemma 2.1(iv)} \) and \( \text{Corollary 2.4} \), \( |V(H_1)| \geq 4p + 1 \). By \( \text{Lemma 2.5} \), either \( |V(H_2)| > 1 \) and \( V(G) = V(H_1) \cup V(H_2) \) or \( |V(H_2)| = 1 \). If \( V(G) = V(H_1) \cup V(H_2) \), let \( m = |[V(H_1), V(H_2)]| \). If \( m \geq 2 \), then as \( G = \mathbb{K}_2(2p + 1) \) is an \( m \) is \( \mathbb{Z}_{2p+1} \)-reduction of \( G \), it follows by \( \text{Lemma 2.1(ii)} \) that \( G \in \mathbb{Z}_{2p+1} \), contrary to the assumption that \( G \) is not strongly \( \mathbb{Z}_{2p+1} \)-connected. Hence \( m \leq 2p - 1 \), and so \( G = \mathbb{K}_2(2p + 1) \).

Assume that \( |V(H_2)| = 1 \). Then \( H_1 \) is the only non-trivial maximal strongly \( \mathbb{Z}_{2p+1} \)-connected subgraph of \( G \). Let \( V' = V(G) \setminus V(H_1) \). We claim that \( G[V'] \) is a complete graph. Suppose to the contrary that there exist vertices \( v_1, v_2 \in V' \)
such that $v_1, v_2 \notin E(G[V'])$. By Lemma 2.1(ii, iii), $|V_1, V(H_1)| \leq 2p - 1$ and $|V_2, V(H_1)| \leq 2p - 1$. Thus there exists $u \in V(H_1)$ such that $u_1 \notin E(G)$ and $u_2 \notin E(G)$ by $|V(H_1)| \geq 4p + 1$. It follows that $(u, v_1, v_2)$ is an independent set, contrary to the assumption of $\alpha(G) \leq 2$. Therefore, $G[V']$ is a complete graph. By Lemma 2.1(iv), we have $|V'| \leq 4p$. Thus the $(S_{2p+1})$-reduction of $G$ is in $K_{(2p+1)}$. This proves (ii).

Proof of (iii). If $k' \geq 2p$ and $\delta(G) \geq 4p$, we show that the $(S_{2p+1})$-reduction $G'$ is not in $K_{(2p+1)}$, and so $G \in (S_{2p+1})$ follows from (ii). By Lemma 2.5, if $G$ has two nontrivial maximal strongly $Z_{2p+1}$-connected subgraphs $H_1$ and $H_2$, then $V(G) = V(H_1) \cup V(H_2)$, and so $G'(H_1 \cup H_2)$ is a $2k_2$, where $m = ||V(H_2)\cup V(H_1)||$. If $m \leq 2p - 1$, then $G' = m_k_2 \in K_{(2p+1)}$, contrary to the assumption that $k' \geq k' \geq 2p$. Thus $m \geq 2p$ and so by Lemma 2.1(ii) that $G \in (S_{2p+1})$. Hence we assume that $G$ does not have two nontrivial maximal strongly $Z_{2p+1}$-connected subgraphs. By Corollary 2.4 and Lemma 2.5, $G$ has exactly one nontrivial maximal strongly $Z_{2p+1}$-connected subgraph $H_1$. Moreover, $G - V(H_1)$ is a complete graph as shown above in the proof of (ii). Let $u'$ be the vertex in $G'$ onto which $H_1$ is contracted. Since $\delta(G) \geq 4p$, for any vertex $v \in V(G' - u')$, we have $|u', v|_{G'} \geq 4p + 1 - |V'|$, and so $G'$ contains a spanning subgraph isomorphic to $K_{4p+1}/K_5$ when $2p - 1 | V' |$. By Lemma 2.1(iv), $K_{4p+1}/K_5 \ni |V' | \in (S_{2p+1})$, and so $G' \in (S_{2p+1})$. This contradicts that $G' \in (S_{2p+1})$-reduced, unless $|V(G')| = 1$. Therefore, $G \in (S_{2p+1})$ by Lemma 2.1(vi).

3. on mod 5-orientations

The odd-edge-connectivity of a graph is defined as the size of a smallest edge-cut of odd size. A 6p-edge-connected graph must be odd-(6p + 1)-edge-connected, but not vice versa. Tutte’s 3-Flow Conjecture was originally proposed for odd-5-edge-connected graphs [see (1)]. Lovász, Thomassen, Wu and Zhang [14] proved the following result for mod (2p + 1)-orientations concerning odd-edge-connectivity, which strengthens their theorem on modulo orientations.

**Theorem 3.1** (Lovász et al. [14]). Every odd-(6p + 1)-edge-connected graph admits a mod (2p + 1)-orientation.

The main result of this section is Theorem 3.2. For the class of graphs with independence number at most 2, Theorem 3.2 improves Theorem 3.1 for $p = 2$ and verifies Conjecture 1.2 for those values.

**Theorem 3.2.** Every odd-9-edge-connected graph $G$ of order at least 21 and with $\alpha(G) \leq 2$ has a mod 5-orientation.

We need a few more tools for the proof of Theorem 3.2.

**Theorem 3.3** (Hakimi [2]). Let $G$ be a graph and $\ell : V(G) \mapsto \mathbb{Z}$ be a function such that $\sum_{v \in V(G)} \ell(v) = 0$ and $\ell(v) \equiv d_c(v) \pmod{2}$, $\forall v \in V(G)$. Then the following are equivalent.

(i) $G$ has an orientation $D$ such that $d_D^+(v) - d_D^-(v) = \ell(v)$, $\forall v \in V(G)$.

(ii) $|\sum_{v \in V}\ell(v)| \leq |d_c(S)|$, $\forall S \subset V(G)$.

Let $u_1v$ and $u_2v$ be two distinct edges in $G$. We define $G[\{u_1, u_2\}]$ to be the graph obtained from $G$ by deleting the edges $u_1v, u_2v$ and adding a new edge $u_1u_2$, which is called the lifting operation (see [16, 14]). The following lemma of Zhang [22] shows that the odd-edge-connectivity is preserved under certain lifting operation.

**Lemma 3.4** (Zhang [22]). Let $G$ be a graph with odd edge-connectivity $k$. Assume there is a vertex $v \in V(G)$ with $d(v) \neq k$ and $d(v) \neq 2$. Then there exists a pair of edges $u_1v, u_2v$ in $E(G)$ such that $G[\{u_1, u_2\}]$, the graph obtained from $G$ by lifting $u_1v, u_2v$, remains odd edge-connectivity.

**Lemma 3.5.** Let $J_0, J_1$ and $J_2$ be the graphs depicted in Fig. 1. Each of the following holds.

(i) $J_0$ is strongly $Z_5$-connected.

(ii) If $G'$ is a $(S_{Z_5})$-reduced graph on 3 vertices, then $|E(G')| \leq 7$, where $|E(G')| = 7$ if and only if $G'$ is isomorphic to either $J_1$ or $J_2$.

**Proof.** Proof of (i). Let $b \in Z(J_0, Z_5)$. If $b(v_1) \neq 0$, lift two edges $v_1v_2, v_1v_3$ to obtain the graph $G_{v_1, v_2, v_3}$. Since $|v_1, v_2, v_3|\in Z_5|v_2, v_3| = 3$ and $b(v_1) \neq 0$, we can modify the boundary $b(v_1)$ with the three edges in $\{v_1, v_2, v_3\}$. Specifically, orient 1, 3, 0, 2 edges towards $v_1$ when $b(v_1) = 1, 2, 3, 4$, respectively. As $|v_1, v_2, v_3|\in Z_5|v_2, v_3| = 4$ and by Lemma 2.1(iii), we can also modify the boundaries $b(v_2), b(v_3)$ with those four edges. By symmetry, we assume $b(v_1) = b(v_2) = b(v_3) = 0$, then $b(v_1) = 0$ since $b \in Z(J_0, Z_5)$. Orient all the edges in $E(v_1)$ towards $v_1$ and orient all the edges in $E(v_2)$ from $v_2$ to obtain an orientation $D_0$. Then $D$ is a mod 5-orientation of $G$, which agrees with the boundary $b(v_1) = b(v_2) = b(v_3) = 0$. Therefore, (i) must hold.

Proof of (ii). Set $b(v_1) = b(v_2) = b(v_3) = 4$. Then $b \in Z(J_1, Z_5)$. It is routine to check that there is no orientation agreeing with the boundary $b_1$ in $J_1$. Set $b_2(v_1) = b_2(v_2) = 4$ and $b_2(v_3) = 2$. Then $b_2 \in Z(J_2, Z_5)$. It is easy to see that there is no orientation agreeing with the boundary $b_2$ in $J_2$. Notice that $J_1$ and $J_2$ are the only two nonisomorphic graphs on 3 vertices and 7 edges with edge multiplicity at most 3. Now, Lemma 3.5 follows by Lemma 2.1(ii) and the fact that $J_0 \in (S_{Z_5})$, $J_1, J_2 \notin (S_{Z_5})$.

**Lemma 3.6.** Let $G$ be an odd-9-edge-connected graph of order $n \geq 2$. If $G$ contains a subgraph isomorphic to $K_{n-1}$, then $G$ admits a mod 5-orientation.
Proof. It is straightforward to verify the statement when \( n = 2 \) and \( n \geq 10 \) by Lemma 2.1(iv). Let \( G \) be a counterexample with \(|V(G)| + |E(G)|\) minimized. The minimality of \( G \) implies that \( G \) is \((\mathcal{S}\mathcal{Z}_5)\)-reduced. Let \( x \) be a vertex of \( G \) such that \( G - x \) contains a subgraph isomorphic to \( K_{n-1} \) whose vertex set is denoted by \( \{y_1, \ldots, y_{n-1}\} \). We may further assume \(|x, y_i| \geq |x, y_{i+1}| \), \( \forall i \in [n-2] \). If \( G \) contains an even degree vertex, say \( v \), then, by Lemma 3.4, there exist \( \frac{d_G(v)}{2} \) pairs of edges incident with \( v \) such that lifting them results a graph, which contains a subgraph isomorphic to \( K_{n-2} \), is still odd-9-edge-connected and has a mod 5-orientation, a contradiction. This implies every vertex has an odd degree, \( \delta(G) \geq 9 \) and \( n \) is even. Moreover, again by Lemma 3.4 and the minimality of \(|V(G)| + |E(G)|\), we have \( d_G(x) = 9 \).

If \( n = 4 \), then \(|E(G)| \geq 18 \). Since \(|u, v| \leq 3 \) for any \( u, v \in V(G) \) by Lemma 2.1(iii), we have \(|E(G)| = 18 \), and this, in addition, implies that \( G \) is isomorphic to \( 3K_4 \). By Lemma 3.5, \( 3K_3 \in (\mathcal{S}\mathcal{Z}_5) \), and so \( G \cong 3K_4 \) is not \((\mathcal{S}\mathcal{Z}_5)\)-reduced, contrary to the assumption that \( G \) is \((\mathcal{S}\mathcal{Z}_5)\)-reduced. Hence we assume that \( n > 4 \).

As every vertex of \( G \) has an odd degree, the following observations, stated as Claims 1 and 2, follow from Theorem 3.3 and Lemma 3.5.

Claim 1. Let \( \ell : V(G) \mapsto \{5, -5\} \) be a function such that \( \sum_{x \in V(G)} \ell(x) = 0 \). Then

\[
\text{there exists } S \subset V(G) \text{ such that } \sum_{x \in S} \ell(x) > |\partial_C(S)|. \quad (6)
\]

In fact, if (6) fails, then by Theorem 3.3, \( G \) has a mod 5-orientation, contrary to the assumption that \( G \) is a counterexample. As \( n \leq 9 \), by the symmetry between \( S \) and \( V(G) - S \), we may assume that there exists \( S \subset V(G) \) satisfying (6) with \( |S| \leq 4 \) for any given \( \ell \).

Claim 2. Let \( S \) be a vertex subset of \( G \). Each of the following holds.

(i) \( |\partial_C(S)| \geq 12 \) if \( |S| = 1 \).
(ii) \( |\partial_C(S)| \geq 13 \) if \( |S| = 2 \).
(iii) \( |\partial_C(S)| \geq 12 \) if \( |S| = 3 \).

When \( n = 6 \), denote \( X = \{x, y_4, y_5\} \) and \( Y = \{y_1, y_2, y_3\} \). As \( d_C(x) = 9 \), we have \(|x, y_5|, |x, y_4| \leq 1 \) and \(|x, y_4| \leq 2 \). These, together with \(|x, y_5| \leq 3 \), imply that

\[
|x, y_4| = d_C(x) + d_C(y_4) + d_C(y_5) - 2(|x, y_4| + |x, y_5| + |y_4, y_5|) \\
\geq 21 - 2(2 + 1 + 3) = 15. \quad (7)
\]

Set \( \ell(x) = \ell(y_4) = \ell(y_5) = 5 \) and \( \ell(y_1) = \ell(y_2) = \ell(y_3) = -5 \). We will obtain a contradiction by showing that \( \ell \) violates Claim 1. Choose an \( S \subset V(G) \) satisfying (6) with \( |S| \) minimized. Then \( |S| \leq 3 \). By Claim 2(i), \( |S| \neq 1, 2 \), and so \( |S| = 3 \). Thus \( \sum_{x \in S} \ell(x) \leq 15 \). By Claim 2, \( \sum_{x \in S} \ell(x) = 15 \) implying \( S \in \{X, Y\} \), contrary to (7).

Therefore, we assume \( n = 8 \) in the following. Since \( d_C(x) = 9 \) and \(|x, y_i| \geq |x, y_{i+1}| \), \( \forall i \in [7] \), we have

\[
|x, y_7| \leq |x, y_6| \leq |x, y_5| \leq 1, \quad (8)
\]

and

\[
|x, y_7| \leq |x, y_6| \leq 3. \quad (9)
\]

Let \( X_1 = \{x, y_5, y_6, y_7\} \), \( Y_1 = \{y_1, y_2, y_3, y_4\} \), \( X_2 = \{x, y_4, y_5, y_6\} \), and \( Y_2 = \{y_1, y_2, y_3, y_5\} \). Define two functions \( \ell_1 \) and \( \ell_2 \) to be as follows.

\[
\ell_1(v) = \begin{cases} 5, & \text{if } v \in X_1; \\
-5, & \text{if } v \in Y_1. \end{cases}
\]
\[
\ell_2(v) = \begin{cases} 5, & \text{if } v \in X_2; \\
-5, & \text{if } v \in Y_2. \end{cases}
\]

We are to show that either \( \ell_1 \) or \( \ell_2 \) violates Claim 1, leading to a contradiction.

For \( i = 1, 2 \), choose \( S_i \subset V(G) \) satisfying (6) with \( |S_i| \) minimized. By Claim 2(i), we have \( 3 \leq |S_i| \leq 4 \).
Claim 3. If $|S| = 3$, then $|\partial C(S)| = 13$ and $S_1 = X_i \setminus \{x\}$.

As $|S| = 3$, $|\sum_{v \in S} \ell_i(v)| \in [5, 15]$. By (6) and Claim 2(ii), we must have $15 = |\sum_{v \in S} \ell_i(v)| > |\partial C(S)| = 13$. Thus $S_i \subseteq X_i$ or $S_i \subseteq Y_i$. Moreover, $G[S_i]$ is isomorphic to $J_1$ or $J_2$ as $|\partial C(S_i)| = 13$ and by Claim 2(ii).

If $x \in S_i$, then by Claim 2(ii), $S_i \subseteq X_i$ and $|\{x, S_i \setminus \{x\}\}| \geq 4$ as $G[S_i]$ is isomorphic to $J_1$ or $J_2$, contradicting to (8). If $S_i \subseteq Y_i$, then we have $13 = |\partial C(S_i)| = |\{S_i \setminus \{x\}\}| + |\{V(G) \setminus (S_i \setminus \{x\})\}| \geq |\{S_i \setminus \{x\}\}| + 12$. Thus $|\{S_i \setminus \{x\}\}| \leq 1$, and so $|\{y, y_4, y_5, y_6, y_7\}| = 0$. Denote $\{y\} = Y \setminus S_i$. Then $|\{y_i, y_6, y_7\}| \geq 9 - |\{S_i \setminus \{x\}\}| - |\{y, y_4, y_5, y_6, y_7\}| \geq 8$. So, by Lemma 2.1(iii), $G$ is not $\langle S\subseteq Z \rangle$--reduced, a contradiction to the assumption on $G$. Therefore, we conclude that $S_i = X_i \setminus \{x\}$ if $|S| = 3$.

Claim 4. If $|S| = 3$, then $|S_{3-i}| \notin \{3, 4\}$.

Assume $|S_1| = |S_2| = 3$ first. We claim that there exists $s \in S_1 \cup S_2 = \{y_4, y_5, y_6, y_7\}$ such that $d_{G[S_1 \cup S_2]}(s) \geq 7$. If one of $G[S_1], G[S_2]$ is isomorphic to $J_2$, it is routine to verify that the vertex $s$ corresponding to $v_3$ in $J_2$ has degree at least $7$ in $G[S_1 \cup S_2]$. Otherwise, we have $G[S_1] \cong G[S_2] \cong J_1$ by Claim 2(ii), and so one of the vertices $y_6, y_7$ has degree at least $7$ in $G[S_1 \cup S_2]$. Since $d_{G[S_1 \cup S_2]}(s) \geq 7$, it follows by $|s, \{y_1, y_2, y_3\}| \geq 3$ that $d_{G}(s) \geq 10$, contradicting to $d_{G}(s) = 9$ by Claim 2(ii).

We assume $|S| = 3$ and $|S_{3-i}| = 4$. By Claim 3, we have $y_6 \in S_1 \subseteq X_i$, and it follows by Claim 2(ii) and Claim 3 that

$$
|\{y_6, y_7\}| \geq 4.
$$

(10)

Since $|S_{3-i}| = 4$ and by Claim 2(iii), we have $20 = |\sum_{v \in S} \ell_i(v)| > |\partial C(S_{3-i})| = |\{X_{3-i}, Y_{3-i}\}|$ from (6). However, it follows from (9), (10) and $y_6 \in X_i$ that

$$
|\{X_{3-i}, Y_{3-i}\}| = d_{G}(x) - |\{x, y, y_6, y_7\}| + |\{y_4, y_6, y_7\}| \geq 9 - 3 + 10 + |\{y_6, y_7\}| \geq 20 = |\sum_{v \in S_{3-i}} \ell_i(v)|,
$$

a contradiction to (6). Hence Claim 4 holds.

The final step. By Claim 4, we may assume that $|S_1| = |S_2| = 4$. Thus, for $i \in \{1, 2\}, 20 = |\sum_{v \in S} \ell_i(v)| > |\partial C(S_i)| = |\{X_i, Y_i\}|$ by (6) and Claim 2(iii). Then $|\partial C(S_i)| = |\{X_i, Y_i\}| \leq 18$, since $|X_i|$ is even. However, it follows from (8) and (9) that

$$
36 \geq |\{X_i, Y_i\}| + |\{X_2, Y_2\}| = 2d_{G}(x) - |\{x, y_4, y_6, y_7\}| - |\{x, y_6, y_7\}| + 2|\{y_4, y_6, y_7\} - |\{x, y_4, y_6, y_7\}| + d_{C}(y_4) - d_{C}(y_6) - d_{C}(y_7) - |\{x, y_5\}|
$$

$$
\geq 18 - 3 - 3 + 12 + 6 + 8 = 38,
$$

a contradiction. The proof is completed. □

Proof of Theorem 3.2. Let $G$ be an odd-$9$--edge-connected graph with $\alpha(G) \leq 2$ and $G'$ be the $\langle S\subseteq Z \rangle$--reduction of $G$. We shall show that $|V(G')| \leq 9$ and $G'$ contains a subgraph isomorphic to $K_{|V(G')|-1}$. Then $G'$ admits a mod $5$--orientation by Lemma 3.6, and so Theorem 3.2 follows from Lemma 2.1(v).

Denote $G_1$ to be the underline simple graph of $G$. Since $|V(G_1)| \geq 21$, $G_1$ is not $\langle S\subseteq Z \rangle$--reduced by Corollary 2.4, and hence $\xi(G_1) \neq 0$. By Lemma 2.5, we have $1 \leq \xi(G_1) \leq 2$. If $\xi(G_1) = 2$, again by Lemma 2.5, $G_1$ is the $\langle S\subseteq Z \rangle$--reduction of $G_1$, is a graph with at most two vertices, so does $G$. Notice that $|V(G')| \leq |V(G)|$. Assume $\xi(G_1) = 1$ and let $H_1$ be the corresponding nontrivial maximal $\langle S\subseteq Z \rangle$--subgraphs of $G_1$. Clearly, $|V(H_1)| \geq 9$ by Lemma 2.1(iv). Let $H$ be a nontrivial maximal $\langle S\subseteq Z \rangle$--subgraphs of $G$ with $|V(H)|$ maximized. As $G[V(H_1)] \subseteq \langle S\subseteq Z \rangle$, we have $|V(H)| \geq |V(H_1)| \geq 9$. We claim that $\alpha(G - V(H)) = 1$. In fact, suppose that $u, v$ are two non-adjacent vertices in $G - V(H)$. Then, by Lemma 2.1(ii)(iii), we have $|\{u, v\} \setminus V(H)| \leq 3$ and $|\{u, v\} \setminus V(H)| \leq 3$. Since $|V(H)| \geq 9$, there exists $w \in V(H)$ such that $\{u, v, w\}$ forms a independent set of size $3$, a contradiction to $\alpha(G) \leq 2$. Hence $\alpha(G - V(H)) = 1$. Now, by Lemma 2.1(iv), the $\langle S\subseteq Z \rangle$--reduction of $G - V(H)$ has size at most $8$ and independence number. Hence $G'$ has order at most $9$ and contains a subgraph isomorphic to $K_{|V(G')|-1}$. Therefore, Theorem 3.2 follows from Lemma 2.1(v) and Lemma 3.6. □

Note that Theorem 1.8 follows from Theorems 3.2 and 1.6.

4. Concluding remarks

As already mentioned in Section 1, we have proved that there are finitely many $\langle S\subseteq 2p+1 \rangle$--reduced graphs, which are contraction obstacles for admitting a mod $(2p + 1)$--orientation, in the family of graphs with bounded independence number. However, there are infinitely many $(4p + 1)$--edge-connected $\langle S\subseteq 2p+1 \rangle$--reduced graphs without mod $(2p + 1)$--orientation for every $p \geq 5$ as proved in [3]. We ask a meta question that what kind of graph family may have only finitely many contraction obstacles for admitting a mod $(2p + 1)$--orientation. Some dense conditions or degree conditions may work, and certain edge connectivity condition may not work well. The corresponding question on planar graphs is of particular interest, which is open for every $p \geq 2$. 

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Problem 4.1. For each integer $p \geq 2$, are there finitely many $(4p + 1)$-edge-connected $\langle S\mathbb{Z}_{2p+1} \rangle$-reduced planar graphs?

Problem 4.1 can be viewed as a relaxed version of Jaeger's conjecture on planar graphs, and it can be also generalized to graphs embedded on surface.

Problem 4.2. For each positive integer $p$, are there finitely many $(4p + 1)$-edge-connected $\langle S\mathbb{Z}_{2p+1} \rangle$-reduced graphs for the family of graphs embedded on a fixed surface?

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