Note

Connectivity keeping stars or double-stars in 2-connected graphs

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Abstract

In Mader (2010), Mader conjectured that for every positive integer \(k\) and every finite tree \(T\) with order \(m\), every \(k\)-connected, finite graph \(G\) with \(\delta(G) \geq \left\lfloor \frac{3}{2}k \right\rfloor + m - 1\) contains a subtree \(T'\) isomorphic to \(T\) such that \(G - V(T')\) is \(k\)-connected. In the same paper, Mader proved that the conjecture is true when \(T\) is a path. Diwan and Tholiya (2009) verified the conjecture when \(k = 1\). In this paper, we will prove that Mader’s conjecture is true when \(T\) is a star or double-star and \(k = 2\).

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1. Introduction

In this paper, graph always means a finite, undirected graph without multiple edges and without loops. For graph-theoretical terminologies and notation not defined here, we follow [1]. For a graph \(G\), the vertex set, the edge set, the minimum degree and the connectivity number of \(G\) are denoted by \(V(G)\), \(E(G)\), \(\delta(G)\) and \(\kappa(G)\), respectively. The order of a graph \(G\) is the cardinality of its vertex set, denoted by \(|G|\). \(k\) and \(m\) always denote positive integers.

In 1972, Chartrand, Kaugars, and Lick proved the following well-known result.

Theorem 1.1 ([2]). Every \(k\)-connected graph \(G\) of minimum degree \(\delta(G) \geq \left\lfloor \frac{3}{2}k \right\rfloor\) has a vertex \(u\) with \(\kappa(G - u) \geq k\).

Fujita and Kawai proved in [4] that every \(k\)-connected graph \(G\) with minimum degree at least \(\left\lfloor \frac{3}{2}k \right\rfloor + 2\) has an edge \(e = uv\) such that \(G - \{u, v\}\) is still \(k\)-connected. They conjectured that there are similar results for the existence of connected subgraphs of prescribed order \(m \geq 3\) keeping the connectivity.

Conjecture 1 ([4]). For all positive integers \(k, \ m\), there is a (least) non-negative integer \(f_k(m)\) such that every \(k\)-connected graph \(G\) with \(\delta(G) \geq \left\lfloor \frac{3}{2}k \right\rfloor - 1 + f_k(m)\) contains a connected subgraph \(W\) of exact order \(m\) such that \(G - V(W)\) is still \(k\)-connected.

They also gave examples in [4] showing that \(f_k(m)\) must be at least \(m\) for all positive integers \(k, \ m\). In [5], Mader proved that \(f_k(m)\) exists and \(f_k(m) = m\) holds for all \(k, \ m\).

Theorem 1.2 ([5]). Every \(k\)-connected graph \(G\) with \(\delta(G) \geq \left\lfloor \frac{3}{2}k \right\rfloor + m - 1\) for positive integers \(k, \ m\) contains a path \(P\) of order \(m\) such that \(G - V(P)\) remains \(k\)-connected.
Let $G$ be a 2-connected graph with minimum degree $\delta(G) \geq \frac{1}{2} m - 1 + t_k(T)$ contains a subgraph $T'$ such that $\kappa(G - V(T')) \geq k$.

Mader showed that $t_k(T)$ exists in [6].

Theorem 1.3 ([6]). Let $G$ be a $k$-connected graph with $\delta(G) \geq 2(k - 1 + m)^2 + m - 1$ and let $T$ be a tree of order $m$ for positive integers $k$, $m$. Then there is a tree $T' \subseteq G$ isomorphic to $T$ such that $G - V(T')$ remains $k$-connected.

Mader further conjectured that $t_k(T) = |T|$.

Conjecture 3 ([5]). For every positive integer $k$ and every tree $T$, $t_k(T) = |T|$ holds.

Theorem 1.2 showed that Conjecture 3 is true when $T$ is a path. Diwan and Tholiya [3] proved that the conjecture holds when $k = 1$. In the next section, we will verify that Conjecture 3 is true when $T$ is a star and $k = 2$. It is proved in the last section that Conjecture 3 is true when $T$ is a double-star and $k = 2$.

A block of a graph $G$ is a maximal connected subgraph of $G$ that has no cut vertex. Note that any block of a connected graph of order at least two is 2-connected or isomorphic to $K_2$.

For a vertex subset $U$ of a graph $G, G[U]$ denotes the subgraph induced by $U$ and $G - U$ is the subgraph induced by $V(G) - U$. The neighborhood $N_G(U)$ of $U$ is the set of vertices in $V(G) - U$ which are adjacent to some vertex in $U$. If $U = \{u\}$, we also use $G - u$ and $N_G(u)$ for $G - \{u\}$ and $N_G(\{u\})$, respectively. The degree $d_G(u)$ of $u$ is $|N_G(u)|$. If $H$ is a subgraph of $G$, we often use $H$ for $V(H)$. For example, $N_G(H), H \cap G$ and $H \cap U$ mean $N_G(V(H))$, $V(H) \cap V(G)$ and $V(H) \cap U$, respectively. If there is no confusion, we always delete the subscript, for example, $d(u)$ for $d_G(u), N(u)$ for $N_G(u), N(U)$ for $N_G(U)$ and so on. A tree is a connected graph without cycles. A star is a tree that has exactly one vertex with degree greater than one. A double-star is a tree that has exactly two vertices with degree greater than one.

2. Connectivity keeping stars in 2-connected graphs

Theorem 2.1. Let $G$ be a 2-connected graph with minimum degree $\delta(G) \geq m + 2$, where $m$ is a positive integer. Then for a star $T$ with order $m$, $G$ contains a star $T'$ isomorphic to $T$ such that $G - V(T')$ is 2-connected.

Proof. If $m \leq 3$, then $T$ is a path, and the theorem holds by Theorem 1.2. Thus we assume $m \geq 4$ in the following.

Since $\delta(G) \geq m + 2$, there is a star $T' \subseteq G$ with $T' \cong T$. Assume $V(T') = \{u, v_1, \ldots, v_{m-1}\}$ and $E(T') = \{uv_i\} 1 \leq i \leq m - 1$.

We say $T'$ is a star rooted at $u$ or with root $u$. Let $G' = G - T'$. Let $B$ be a maximum block in $G'$ and let $l$ be the number of components of $G' - B$. If $l = 0$, then $B = G'$ is 2-connected. We may assume that $l \geq 1$. Let $H_1, \ldots, H_l$ be the components of $G - B$ with $|H_1| \geq \cdots \geq |H_l|$. Take such a star $T'$ so that

(P1) $|B|$ is as large as possible,
(P2) $|H_1|, \ldots, |H_l|$ is as large as possible in lexicographic order, subject to (P1).

We will complete the proof by a series of claims.

Claim 1. $|N(H_i) \cap B| \leq 1$ and $|N(H_i) \cap V(T')| \geq 1$ for each $i \in \{1, \ldots, l\}$.

Since $B$ is a block of $G'$, we have $|N(H_i) \cap B| \leq 1$ for each $i \in \{1, \ldots, l\}$. Since $G$ is 2-connected, $|N(H_i) \cap V(T')| \geq 1$ for each $i \in \{1, \ldots, l\}$.

Claim 2. $l = 1$.

Assume $l \geq 2$. By Claim 1, there is an edge $th$ between $T'$ and $H_1$, where $t \in T'$ and $h \in H_1$. Choose a vertex $x \in H_1$. Since $\delta(G) \geq m + 2$ and $|N(H_i) \cap B| \leq 1$ (by Claim 1), we have $|N(x) \setminus (B \cup \{t\})| \geq m + 2 - 1 - 1 = m$. Thus we can choose a star $T'' \cong T$ with root $x$ such that $V(T'') \cap (B \cup \{t\}) = \emptyset$. But then either there is a larger block than $B$ in $G - T''$, or $G - T'' - B$ contains a larger component than $H_1 (H_1 \cup \{t\})$ is contained in a component of $G - T'' - B$, which contradicts to (P1) or (P2).

Claim 3. $|N(t) \cap B| \leq 1$ and $|N(t) \cap H_1| \geq 2$ for any vertex $t \in V(T')$.

Assume $|N(t) \cap B| \geq 2$. Choose a vertex $x \in H_1$. Since $\delta(G) \geq m + 2$ and $|N(H_1) \cap B| \leq 1$, we have $|N(x) \setminus (B \cup \{t\})| \geq m + 2 - 1 - 1 = m$. Thus we can choose a star $T'' \cong T$ with root $x$ such that $V(T'') \cap (B \cup \{t\}) = \emptyset$. But $G - T''$ has a block containing $B \cup \{t\}$ as a subset, which contradicts to (P1). Thus $|N(t) \cap B| \leq 1$ holds. By $d(t) \geq m + 2$ and $|N(t) \cap B| \leq 1$, we have $|N(t) \cap H_1| = d(t) - |N(t) \cap B| - |N(t) \cap T'| \geq m + 2 - 1 - (m - 1) = 2$.

Claim 4. For any edge $e = (t_1, t_2) \in E(T')$, $|N(t_1, t_2) \cap B| \leq 1$ holds.
By contradiction, assume \(|N(t_1, t_2)| \cap B| \geq 2\). Because \(|N(t_1)| \cap B| \leq 1\) and \(|N(t_2)| \cap B| \leq 1\), we can assume that there are two distinct vertices \(b_1, b_2 \in B\) such that \(t_1, b_1, t_2, b_2 \in E(G)\). Choose a vertex \(x \in H_1\). Since \(\delta(G) \geq m + 2\) and \(|N(H_1)| \cap B| \leq 1\), we have \(|N(x) \setminus (B \cup \{t_1, t_2\})| \geq m + 2 - 1 = m - 1\). Thus we can choose a star \(T'' \cong T\) with root \(x\) such that \(V(T'') \cap (B \cup \{t_1, t_2\}) = \emptyset\). But then \(G - T''\) has a block containing \(B \cup \{t_1, t_2\}\) as a subset, which contradicts to (P1).

Because \(|N(H_1)| \cap B| \leq 1\) and \(G\) is 2-connected, we have \(|N(T') \cap B| \geq 1\). The following claim further shows that \(|N(T') \cap B| = 1\).

Claim 5. \(|N(T') \cap B| = 1\).

By contradiction, assume \(|N(T') \cap B| \geq 2\). If \(N(u) \cap B \neq \emptyset\), say \(N(u) \cap B = \{u', \ldots, u_{m-1}\}\), then we have \(|N(v_1, \ldots, v_{m-1})| \cap B \subseteq \{u'\}\) by Claim 4. That is, \(|N(T') \cap B| = \{u'\}\), a contradiction. Thus \(N(u) \cap B = \emptyset\). Assume, without loss of generality, that there are two distinct vertices \(w\) and \(w'\) in \(B\) such that \(v_1 w, v_2 w' \in E(G)\). If \(N(v_3) \cap B = \emptyset\) or \(|N(v_3) \cap \{v_1, v_2\}| \leq 1\), then we can choose a star \(T''\) with order \(m\) and root \(v_3\) such that \(V(T'') \cap (B \cup \{u, v_1, v_2\}) = \emptyset\). But then \(B \cup \{u, v_1, v_2\}\) is contained in a block of \(G - T''\), contradicting to (P1). Thus we assume \(v_3\) is adjacent to a vertex \(y\) in \(B\) and is adjacent to both \(v_1\) and \(v_2\). Without loss of generality, assume \(y\) is distinct from \(w\). Then we can choose a star \(T''\) with order \(m\) and root \(u\) such that \(V(T'') \cap (B \cup \{v_1, v_2\}) = \emptyset\). But \(B \cup \{v_1, v_2\}\) is contained in a block of \(G - T''\), contradicting to (P1). Thus \(|N(T') \cap B| = 1\).

By Claim 5, \(|N(T') \cap B| = 1\). Assume \(|N(T') \cap B| = \{w\}\). Since \(G\) is 2-connected, we have \(|N(H_1) \cap B| \geq 1\). By Claim 1, \(|N(H_1) \cap B| = 1\). Assume \(|N(H_1) \cap B| = \{z\}\). Let \(P\) be a shortest path from \(z\) to \(w\) going through \(H_1\) and \(T\). Assume \(P := p_1 p_2 \cdots p_{q-1} p_q\), where \(p_1 = z, p_q = w\) and \(p_i \in H_1 \cup T\) for each \(i \in \{2, \ldots, q - 1\}\). Since \(P\) is a shortest path, \(|N(p_i) \cap P| = 2\) for each \(2 \leq i \leq q - 1\). By \(N(T') \cap B = \{w\}\) and \(N(H_1) \cap B = \{z\}\), \(|N(p_i) \cap B| \subseteq \{w, z\} \subseteq V(P)\) for each \(2 \leq i \leq q - 1\). Thus \(|N(p_i) \cap (B \cup P)| = 2\) and \(|N(p_i) \cap (V(G) \setminus (B \cup P))| \geq m\) for each \(2 \leq i \leq q - 1\). This implies \(G - (B \cup P)\) is not 2-connected. For any vertex \(x\) in \(G - (B \cup P)\), we have \(|N(x) \cap P| \leq 3\). For otherwise, we can find a path \(P'\) containing \(x\) from \(z\) to \(w\) going through \(H_1\) and \(T\) shorter than \(P\), a contradiction. By \(\delta(G) \geq m + 2\), \(|N(x) \cap (G - (B \cup P))| \geq m + 2 - 3 = m - 1\). Then we can find a star \(T'' \cong T\) with root \(x\) such that \(T'' \cap (B \cup P) = \emptyset\). But then \(B \cup P\) is contained in a block of \(G - T''\), a contradiction. The proof is thus complete. □

3. Connectivity keeping double-stars in 2-connected graphs

**Lemma 3.1.** Let \(G\) be a graph and \(T\) be a double-star with order \(m\). If there is an edge \(e = uw \in E(G)\) such that \(|N(u) \setminus v| \geq \left\lceil \frac{m}{2} \right\rceil - 1\), \(|N(v) \setminus u| \geq m - 3\) and \(|(N(u) \cup N(v)) \setminus \{u, v\}| \geq m - 2\), then there is a double-star \(T' \subseteq G\) isomorphic to \(T\).

**Proof.** Since \(T\) is a double-star, we have \(m \geq 4\). Assume the double-star \(T\) is constructed from an edge \(e' = u'v'\) by adding \(r\) leaves to \(u'\) and \(s\) leaves to \(v'\), where \(1 \leq r \leq s\) and \(r + s = m - 2\). Then \(1 \leq r \leq \left\lceil \frac{m}{2} \right\rceil - 1\) and \(\left\lceil \frac{m}{2} \right\rceil - 1 \leq s \leq m - 3\). Since \(|N(u) \setminus v| \geq \left\lceil \frac{m}{2} \right\rceil - 1\), \(|N(v) \setminus u| \geq m - 3\) and \(|(N(u) \cup N(v)) \setminus \{u, v\}| \geq m - 2\), we can find a double-star \(T' \cong T\) in \(G\) with center-edge \(e = uw\), where \(u\) is adjacent to \(r\) leaves and \(v\) is adjacent to \(s\) leaves. □

The main idea of the proof of Theorem 3.2 is similar to that of Theorem 2.1, with much more complicated and different details.

**Theorem 3.2.** Let \(T\) be a double-star with order \(m\) and \(G\) be a 2-connected graph with minimum degree \(\delta(G) \geq m + 2\). Then \(G\) contains a double-star \(T' \cong T\) such that \(G - V(T')\) is 2-connected.

**Proof.** Since \(T\) is a double-star, we have \(m \geq 4\). If \(m = 4\), then \(T\) is a path, and the theorem holds by Theorem 1.2. Thus we assume \(m \geq 5\) in the following.

Since \(\delta(G) \geq m + 2\), there is a double-star \(T' \subseteq G\) with \(T' \cong T\). Assume \(V(T') = \{u, v, u_1, \ldots, u_r, v_1, \ldots, v_s\}\) and \(E(T') = \{uw \cup \{uv_i| 1 \leq i \leq r\} \cup \{v_{i'}| 1 \leq j \leq s\}\), where \(1 \leq r \leq s\) and \(r + s = m - 2\). We say \(T'\) is a double-star with center-edge \(uw\). Let \(G' = G - T'.\) Let \(B\) be a maximum block in \(G'\) and let \(l\) be the number of components of \(G' - B\). If \(l = 0\), then \(B = G'\) is 2-connected. So we may assume that \(l \geq 1\). Let \(H_1, \ldots, H_l\) be the components of \(G' - B\) with \(|H_1| \geq \cdots \geq |H_l|\).

Take such a double-star \(T'\) so that

(P1) |\(B|\) is as large as possible,

(P2) |\(|H_1, \ldots, H_l|\) is as large as possible in lexicographic order, subject to (P1).

We will complete the proof by a series of claims.

**Claim 1.** \(|N(H_i) \cap B| \leq 1\) and \(|N(H_i) \cap T'| \geq 1\) for each \(i \in \{1, \ldots, l\}\).

Since \(B\) is a block of \(G'\), we have \(|N(H_i) \cap B| \leq 1\) for each \(i \in \{1, \ldots, l\}\). Since \(G\) is 2-connected, \(|N(H_i) \cap T'| \geq 1\) for each \(i \in \{1, \ldots, l\}\).

**Claim 2.** \(|H_i| \geq 2\) for each \(i \in \{1, \ldots, l\}\).

This claim holds because \(|N(h_i) \cap H_i| = d(h_i) - |N(h_i) \cap T'| - |N(H_i) \cap B| \geq m + 2 - m - 1 = 1\) for any vertex \(h_i \in H_i\), where \(1 \leq i \leq l\).
Claim 3. $l = 1$.

Assume $l \geq 2$. By Claim 1, there is an edge $th$ between $T'$ and $H_1$, where $t \in T'$ and $h \in H_1$. By Claim 2, we can choose an edge $xy \in E(H_1)$. Since $\delta(G) \geq m + 2$ and $|N(H_1) \cap B| \leq 1$ by Claim 1, we have $|N(x) \setminus (B \cup \{y, t\})| \geq m + 2 - 1 - 2 = m - 1$ and $|N(y) \setminus (B \cup \{x, t\})| \geq m + 2 - 1 - 2 = m - 1$. Thus, by Lemma 3.1, we can choose a double-star $T'' \cong T$ with center-edge $xy$ such that $V(T'') \cap (B \cup \{t, v\}) = \emptyset$. But then either there is a larger block than $B$ in $G - T''$, or $G - T'' - B$ contains a larger component than $H_1 (H_1 \cup \{t\})$ is contained in a component of $G - T'' - B$), which contradicts to (P1) or (P2).

Claim 4. $|N(t) \cap B| \leq 1$ and $|N(t) \cap H_1| \geq 2$ for any vertex $t \in V(T')$.

Assume $|N(t) \cap B| \geq 2$. Choose an edge $xy \in E(H_1)$. Since $\delta(G) \geq m + 2$ and $|N(H_1) \cap B| \leq 1$, we have $|N(x) \setminus (B \cup \{y, t\})| \geq m + 2 - 1 - 2 = m - 1$ and $|N(y) \setminus (B \cup \{x, t\})| \geq m + 2 - 1 - 2 = m - 1$. Thus, by Lemma 3.1, we can choose a double-star $T'' \cong T$ with center-edge $xy$ such that $V(T'') \cap (B \cup \{t\}) = \emptyset$. But then $B \cup \{t\}$ is contained in a block of $G - T''$, which contradicts to (P1). Thus $|N(t) \cap B| \leq 1$ holds for any vertex $t \in V(T')$. By $d(t) \geq m + 2$ and $|N(t) \cap B| \leq 1$, we have $|N(t) \cap H_1| = d(t) - |N(t) \cap B| - |N(t) \cap T'| \geq m + 2 - 1 - (m - 1) = 2$.

Claim 5. For any edge $t_1t_2 \in E(T)$, $|N(t_1, t_2) \cap B| \leq 1$ holds.

By contradiction, assume $|N(t_1, t_2) \cap B| \geq 2$. Because $|N(t_1) \cap B| \leq 1$ and $|N(t_2) \cap B| \leq 1$, we can assume that there are two distinct vertices $b_1, b_2 \in B$ such that $t_1b_1, t_2b_2 \in E(G)$. Choose an edge $xy \in E(H_1)$. Since $\delta(G) \geq m + 2$ and $|N(H_1) \cap B| \leq 1$, we have $|N(x) \setminus (B \cup \{y, t_1, t_2\})| \geq m + 2 - 1 - 4 = m - 3$ if $N(t_1) \cap B \leq 1$ (by $m \geq 5$) and $|N(y) \setminus (B \cup \{x, t_1, t_2\})| \geq m + 2 - 1 - 4 = m - 3$.

If $|N(x) \setminus (B \cup \{y, t_1, t_2\})| > m - 3$ or $|N(y) \setminus (B \cup \{x, t_1, t_2\})| > m - 3$, then by Lemma 3.1, we can choose a double-star $T'' \cong T$ with center-edge $xy$ such that $V(T'') \cap (B \cup \{t_1, t_2\}) = \emptyset$. But then $G - T''$ has a block containing $B \cup \{t_1, t_2\}$ as a subset, which contradicts to (P1). Thus we assume $|N(x) \setminus (B \cup \{y, t_1, t_2\})| = m - 3$ and $|N(y) \setminus (B \cup \{x, t_1, t_2\})| = m - 3$, which imply $|N(x) \cap B| = 1$ and $|N(y) \cap B| = 1$. Since $|N(H_1) \cap B| \leq 1$, we can assume $N(x) \cap B = N(y) \cap B = \{x\}$. Without loss of generality, assume $z \notin B_1$.

If $N(x'y \neq N(y') \setminus z$, then $|N(x) \setminus (B \cup \{y, t_1, t_2\})| = m - 2$. So we can choose a double-star $T'' \cong T$ with center-edge $xy$ disjoint from $B \cup \{t_1, t_2, z\}$. But then $G - T''$ contains a larger block than $B$, a contradiction. Thus $N(x') = N(y') = \{x\}$. Because we choose the edge $xy$ in $H_1$ arbitrarily, we conclude that $H_1$ is a complete graph and each vertex not in $H_1$ is adjacent to all vertices in $H_1$ if it is adjacent to one vertex in $H_1$. In particular, every vertex $t$ in $T'$ is adjacent to all vertices in $H_1$ by Claim 4 and the vertex $z$ in $B$ is adjacent to all vertices in $H_1$.

Let $t_4h_4$ be an edge of graph $G$, where $t_4 \in V(T') \setminus \{t_1, t_2, t_3\}$ and $h_4 \in V(H_1)$. Let $h_1$ be a vertex in $H_1$ distinct from $h_4$, then $t_4h_1, h_1z \in E(G)$. Thus we can choose a double-star $T'' \cong T$ with center-edge $t_4h_4$ disjoint from $B \cup \{t_1, h_1\}$. But then $B \cup \{t_1, h_1\}$ is contained in a block of $G - T''$, contradicting to (P1).

Because $|N(H_1) \cap B| \leq 1$ and $G$ is 2-connected, we have $|N'(T') \cap B| \geq 1$. The following claim further shows that $|N'(T') \cap B| = 1$.


By contradiction, assume $|N'(T') \cap B| \geq 2$. If $u \in B \neq 0$, say $N(u) \cap B = \{u\}$, then we have $N((u_1, \ldots, u_r, v)) \cap B \subset \{u\}$ by Claim 5 and $N((v_1, \ldots, v_s)) \cap B \subset \{u\}$ by Claim 6. Thus, $N'(T') \cap B = \{u\}$, a contradiction. Thus $N(u) \cap B = \emptyset$. Similarly, we have $N(v) \cap B \neq 0$. Since $N((u_1, \ldots, u_r)) \cap B \leq 1$ and $N((v_1, \ldots, v_s)) \cap B \leq 1$ (by Claim 6), we have $|N'(T') \cap B| = 2$. Assume, without loss of generality, that there are two distinct vertices $w$ and $w'$ in $B$ such that $u_1w, v_1w' \in E(G)$.

We first show that any vertex $x$ in $\{u_1, \ldots, u_r, v_1, \ldots, v_s\} \setminus \{u_1, v_1\}$ has no neighbors in $B$. By contradiction, assume there is a vertex in $\{u_1, \ldots, u_r, v_1, \ldots, v_s\} \setminus \{u_1, v_1\}$, say $v_i$ for some $i \in \{2, \ldots, s\}$ (the case $u_i$ for some $i \in \{2, \ldots, r\}$ can be proved similarly), such that $N(v_i) \cap B = \{w\}$. If $v_i$ is adjacent to $u$ (or $u_i$), then for any edge $v'u'$ (a neighbor of $v$ in $H_1$), we have $|N(v) \setminus (B \cup \{u_1, v_1, v'_1\})| \geq m + 2 - 4 = m - 2$ (or $|N(v) \setminus (B \cup \{u_1, u_1, v'_1\})| \geq m + 2 - 3 = m - 1$ and $|N(v') \setminus (B \cup \{u, v, u_1, v'_1\})| \geq m + 2 - 1 - 3 = m - 2$). But then $G - T''$ contains a larger block than $B$, a contradiction. Thus neither $u_1$ nor $u_i$ is adjacent to $v_i$. Choose a neighbor $v'_i$ of $v_i$ in $H_1$, since $|N(v'_i) \setminus (B \cup \{u, v, u_1, v'_1\})| \geq m + 2 - 1 - 3 = m - 2$ and $|N(v'_i) \setminus (B \cup \{u, v, u_1, v'_1\})| \geq m + 2 - 1 - 3 = m - 2$, we can find a double-star $T'' \cong T$ with center-edge $v_i'v_i$ such that $T'$ is disjoint from $B \cup \{u, v_1, v'_1\}$ or $B \cup \{u, u_1\}$. But then $G - T''$ contains a larger block than $B$, a contradiction. Thus we have $N((u_1, \ldots, u_r, v_1, \ldots, v_s) \setminus \{u_1, v_1\}) \cap B = \emptyset$.\"
Let $v_2v_i' \in E(G)$, where $v_i'$ is a neighbor of $v_2$ in $H_1$. Since $\delta(G) \geq m + 2$ and $N(v_2) \cap B = \emptyset$, we have $|N(v_2) \setminus (B \cup \{u, v, u_1, v_1, v_2\})| \geq m + 2 - 5 = m - 3$ and $|N(v_i') \setminus (B \cup \{u, v, u_1, v_1, v_2\})| \geq m + 2 - 1 - 5 = m - 4 \geq \lceil \frac{m}{2} \rceil - 1$ (by $m \geq 5$). If $|N(v_2) \setminus (B \cup \{u, v, u_1, v_1, v_2\})| \geq m - 2$, then, by Lemma 3.1, we can find a double-star $T'' \cong T$ with center-edge $v_2v_i'$ such that $T''$ avoids $B \cup \{u, v, u_1, v_1\}$. But then $G - T''$ contains a larger block than $B$, a contradiction. Thus assume $|N(v_2) \setminus (B \cup \{u, v, u_1, v_1, v_2\})| = m - 3$, which implies $v_2$ is adjacent to both $u_1$ and $v_1$. For the edge $uv$, we can verify that $|N(u) \setminus (B \cup \{v, u_1, v_1, v_2\})| \geq m + 2 - 4 = m - 2$ and $|N(v) \setminus (B \cup \{u, u_1, v_1, v_2\})| \geq m + 2 - 4 = m - 2$. By Lemma 3.1, we can find a double-star $T'' \cong T$ with center-edge $uv$ such that $T''$ avoids $B \cup \{u_1, v_1, v_2\}$. But then $B \cup \{u_1, v_1, v_2\}$ is contained in a block of $G - T''$, contradicting to (P1). Thus Claim 7 holds.

By Claim 7, $|N(T') \cap B| = 1$. Assume $N(T') \cap B = \{w\}$. Since $G$ is 2-connected, we have $|N(H_1) \cap B| = 1$ by Claim 1. Let $N(H_1) \cap B = \{z\}$. Let $P$ be a shortest path from $z$ to $w$ going through $H_1$ and $T''$. Assume $P := p_1p_2 \cdots p_q$, where $p_1 = z, p_q = w$ and $p_i \in H_1 \cup T'$ for each $i \in \{2, \ldots, q - 1\}$. Since $P$ is a shortest path, $N(p_i) \cap P = \{p_{i-1}, p_{i+1}\}$ for $2 \leq i \leq q - 1$. Because $\delta(G) \geq m + 2$ and $N(p_i) \cap B \subseteq \{w, z\} \subseteq P$ for each $2 \leq i \leq q - 1$, we know $p_i$ has at least $m$ neighbors not in $B \cup P$, that is, $G - (B \cup P)$ is not empty. For any vertex $x$ in $G - (B \cup P)$, we have $|N(x) \cap P| \leq 3$. For otherwise, we can find a path $P'$ containing $x$ from $z$ to $w$ going through $H_1$ and $T''$ shorter than $P$, a contradiction. By $\delta(G) \geq m + 2$, $|N(x) \cap (G - (B \cup P))| \geq m + 2 - 3 = m - 1$. Choose an edge $xy$ in $G - (B \cup P)$. Since $|N(x) \setminus (B \cup P \cup \{y\})| \geq m + 2 - 4 = m - 2$ and $|N(y) \setminus (B \cup P \cup \{x\})| \geq m + 2 - 4 = m - 2$, we can find a double-star $T'' \cong T$ with center-edge $xy$ such that $T'' \cap (B \cup P) = \emptyset$. But then $B \cup P$ is contained in a block of $G - T''$, a contradiction. The proof is thus complete. \hfill \□

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References