Note

A property on reinforcing edge-disjoint spanning hypertrees in uniform hypergraphs

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ABSTRACT

Suppose that H is a simple uniform hypergraph satisfying |E(H)| = k|V(H)| − 1. A k-partition \( \pi = \{X_1, X_2, \ldots, X_k\} \) of \( E(H) \) such that \( |X_i| = |V(H)| - 1 \) for 1 ≤ i ≤ k is a uniform k-partition. Let \( \pi_k(H) \) be the collection of all uniform k-partitions of \( E(H) \) and define \( \varepsilon(\pi) = \sum_{i=1}^{k} c(H[X_i]) - k \), where \( c(H) \) denotes the number of maximal partition-connected sub-hypergraphs of \( H \). Let \( \varepsilon(H) = \min_{\pi \in \pi_k(H)} \varepsilon(\pi) \). Then \( \varepsilon(H) \geq 0 \) with equality holds if and only if \( H \) is a union of \( k \) edge-disjoint spanning hypertrees. The parameter \( \varepsilon(H) \) is used to measure how close \( H \) is being from a union of \( k \) edge-disjoint spanning hypertrees.

We prove that if \( H \) is a simple uniform hypergraph with \( |E(H)| = k|V(H)| - 1 \) and \( \varepsilon(H) > 0 \), then there exist \( e \in E(H) \) and \( e' \in E(H^c) \) such that \( \varepsilon(H - e + e') < \varepsilon(H) \). This generalizes a former result, which settles a conjecture of Payan. The result iteratively defines a finite \( \varepsilon \)-decreasing sequence of uniform hypergraphs \( H_0, H_1, H_2, \ldots, H_m \) such that \( H_0 = H \), \( H_m \) is the union of \( k \) edge-disjoint spanning hypertrees, and such that two consecutive hypergraphs in the sequence differ by exactly one hyperedge.

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1. Introduction

We consider finite graphs and finite hypergraphs. Definitions will be introduced in Section 2. Throughout the paper, let \( k \geq 1 \) be an integer, \( H \) denotes a hypergraph, \( c(H) \) denotes the number of maximal partition-connected sub-hypergraphs of \( H \), and \( \omega(H) \) denotes the number of connected components of \( H \). By definition, for a graph \( G \), partition-connectedness is equivalent to connectedness, and so \( c(G) = \omega(G) \). For a hypergraph \( H \), as mentioned in [2], partition-connectedness is a stronger property than connectedness and so \( \omega(H) \) and \( c(H) \) are different in general. For \( X \subseteq E(H) \), \( H(X) \) denotes the spanning sub-hypergraph of \( H \) with edge set \( X \), whereas \( H[X] \) denotes the sub-hypergraph of \( H \) induced by \( X \). If \( H = (V, E) \) is an \( r \)-uniform hypergraph, then the complement of \( H \), denoted by \( H^c \), is an \( r \)-uniform hypergraph with \( V(H^c) = V(H) \) and \( E(H^c) = V^{[r]} - E(H) \).

In [7], Payan considered the following problem. Let \( G \) be a connected simple graph on \( n \geq 2 \) vertices and \( k(n-1) \) edges. Payan introduced an integral function \( \varepsilon(G) \) to measure how the graph \( G \) is closed to having \( k \) edge-disjoint spanning trees in such a way that \( G \) has \( k \) edge-disjoint spanning tree if and only if \( \varepsilon(G) = 0 \). Payan asked the question whether it is always possible to make a finite number of edge exchanges between edges in \( G \) and edges not in \( G \) so that the corresponding values of \( \varepsilon \) will be strictly decreasing until it becomes zero. Payan [7] conjectured that the problem has an affirmative answer (confirmed in [4]). In this paper, we study the corresponding problem in hypergraphs.

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Suppose that $H$ is a simple uniform hypergraph satisfying $|E(H)| = k(|V(H)| - 1). A k$-partition $\pi = (X_1, X_2, \ldots, X_k)$ of $E(H)$ such that $|X_i| = |V(H)| - 1$ for $1 \leq i \leq k$ is called a uniform $k$-partition. Let $P_k(H)$ be the collection of all uniform $k$-partitions of $E(H).$ We define
\[
e(\pi) = \sum_{i=1}^{k} c(H(X_i)) - k,
\]
and
\[
e(H) = \min_{\pi \in P_k(H)} e(\pi).
\]

By definition, $e(H) \geq 0.$ By Corollary 2.6 of [2] or Theorem 2.2(i), $e(H) = 0$ if and only if for every $1 \leq i \leq k$, $H(X_i)$ is a spanning hypertree of $H.$ Thus $e(H) = 0$ if and only if $H$ has $k$ edge-disjoint spanning hypertrees.

The following result was conjectured by Payan [7] and proved in [4].

**Theorem 1.1 ([4]).** If $G$ is a simple graph with $|E(G)| = k(|V(G)| - 1)$ and $e(G) > 0$, then there exist $e \in E(G)$ and $e' \in E(G')$ such that $e(G - e + e') < e(G)$.

Note that a simple graph is a 2-uniform hypergraph. The main purpose of this note is to extend Theorem 1.1 to all uniform hypergraphs.

**Theorem 1.2.** If $H$ is a simple uniform hypergraph with $|E(H)| = k(|V(H)| - 1)$ and $e(H) > 0$, then there exist $e \in E(H)$ and $e' \in E(H')$ such that $e(H - e + e') < e(H)$.

**Remark.** (1) The parameter $e(H)$, first introduced by Payan in [7] for graphs, can be considered as a measurement that how close $H$ is from being an edge-disjoint union of $k$ spanning hypertrees. Theorem 1.2 iteratively defines a finite $e$-decreasing sequence of uniform hypergraphs $H_0, H_1, H_2, \ldots, H_m$ such that $H_0 = H, H_m$ is the union of $k$ edge-disjoint spanning hypertrees, and such that any two consecutive hypergraphs in the sequence differ by exactly one hyperedge.

(2) This problem is related to connectivity augmentation problems for a network (modeled as a graph or hypergraph). The traditional connectivity augmentation problem is, adding some edges to increase the connectivity (or edge connectivity, partition connectivity, etc.) of a network. Here a kind of “dynamic augmentation” is considered, i.e.,

- The number of edges in the network does not change.
- In each stage, one edge is deleted and another edge is added from outside, where the two edges are called an edge pair.
- In each stage, partition connectivity augmentation happens, which is so-called “dynamic augmentation”.

In this paper, the existence of such edge pairs to augment partition connectivity of a uniform hypergraph is confirmed. It is still interesting to design algorithms to locate those edge pairs.

2. Preliminaries

A hypergraph $H$ is a pair $(V, E)$ where $V$ is the vertex set of $H$ and $E$ is a collection of not necessarily distinct nonempty subsets of $V,$ called hyperedges or simply edges of $H.$ A loop is a hyperedge that consists of a single vertex. A hypergraph $H$ is **nontrivial** if $E(H) \neq \emptyset.$ A hypergraph $H$ is simple if for any $e_1, e_2 \in E(H),$ $e_1 \not\subseteq e_2.$ For an integer $r > 0,$ and a set $V,$ let $V^{(r)}$ denote the family of all $r$-subsets of $V.$ A simple hypergraph $H = (V, E)$ is $r$-uniform if $E \subseteq V^{(r)}.$ If $H = (V, E)$ is an $r$-uniform hypergraph, then the complement of $H,$ denoted by $H^c$, is an $r$-uniform hypergraph with $V(H^c) = V \setminus E.$

If $W \subseteq V(H),$ the hypergraph $(W, E_W),$ where $E_W = \{e \in E(H) : e \subseteq W\}$ is a sub-hypergraph induced by the vertex subset $W,$ and is denoted by $H[W].$ If $X \subseteq E(H)$ and $V_{X} = \bigcup_{e \in X} e,$ then $(V_{X}, X)$ is defined as the sub-hypergraph induced by the edge subset $X$ and is denoted by $H[X].$

A hypergraph $H$ is connected if there is a hyperedge intersecting both $W$ and $V - W$ for every non-empty proper subset $W$ of $V(H).$ A connected component of a hypergraph $H$ is a maximal connected sub-hypergraph of $H.$ A hypergraph $H$ is $k$-partition-connected if $\|P\| \geq k(\|P\| - 1)$ for every partition $P$ of $V(H),$ where $\|P\|$ denotes the number of classes in $P$ and $\|P\|$ denotes the number of edges intersecting at least two classes of $P.$ Equivalently, $H$ is $k$-partition-connected if, for any subset $X \subseteq E(H), |X| \geq k(\alpha(H - X) - 1).$ A 1-partition-connected hypergraph is also referred as a partition-connected hypergraph. It follows from definition that a graph is partition-connected if and only if it is connected. In general, partition-connected hypergraphs must be connected, but a connected hypergraph may not be partition-connected. The partition connectivity of $H$ is the maximum $k$ such that $H$ is $k$-partition-connected.

A hypergraph $H$ is a hyperforest if for every nonempty subset $U \subseteq V(H),$ $|E[H[U]]| \leq |U| - 1.$ A hyperforest $T$ is called a hypertree if $|E(T)| = |V(T)| - 1.$ By a hypercircuit, we mean a hypergraph $C$ with $|E(C)| = |V(C)|$ but $|X| < |V(C[X])|$ for any proper subset $X \subset E(C).$ For a hypergraph $H,$ let $\tau(H)$ be the maximum number of edge-disjoint spanning hypertrees in $H$ and $\alpha(H)$ is the minimum number of edge-disjoint hyperforests whose union is $E(H).$ For a graph $G,$ $\tau(G)$ is the spanning tree packing number of $G$ and $\alpha(G)$ is the arboricity of $G.$
The following theorem of Nash-Williams and Tutte shows that the $k$-partition-connectedness of a graph $G$ is equivalent to the property that $G$ has $k$ edge-disjoint spanning trees. Nash-Williams then published a dual theorem, characterizing graphs that can be decomposed to at most $k$ forests.

**Theorem 2.1.** Let $G$ be a graph.

(i) (Nash-Williams [5], Tutte [8]). $\tau(G) \geq k$ if and only if for any $X \subseteq E(G)$, $|X| \geq k(\omega(G - X) - 1)$.

(ii) (Nash-Williams [6]). $\alpha(G) \leq k$ if and only if for any subgraph $S$, $|E(S)| \leq k(|V(S)| - 1)$.

Frank, Király and Kriesell [2] extended both results to hypergraphs.

**Theorem 2.2** (Frank, Király and Kriesell [2]). Let $H$ be a hypergraph.

(i) $\tau(H) \geq k$ if and only if for every $X \subseteq E(H)$, $|X| \geq k(\omega(H - X) - 1)$ (or, equivalently, $H$ is $k$-partition-connected).

(ii) $\alpha(H) \leq k$ if and only if for any sub-hypergraph $S$, $|E(S)| \leq k(|V(S)| - 1)$.

By Theorem 2.2(i), $\tau(H)$ is the partition connectivity of $H$ and a hypertree is a minimal partition-connected hypergraph.

Let $e$ be a hyperedge in a hypergraph $H$ (notice that $e$ is also a subset of $V(H)$). By $H/e$ we denote the hypergraph obtained from $H$ by contracting the hyperedge $e$ into a new vertex $v_0$ and by removing resulting loops if there are any. That is, $V(H/e) = (V(H) - e) \cup \{v_0\}$ and a hyperedge $e' \in E(H/e)$ if and only if either $e' = e''$ for some $e'' \in E(H)$ with $e'' \cap e = \emptyset$ or $e' = (e'' - e) \cup \{v_0\}$ for some $e'' \in E(H) \setminus \{e\}$ with $e'' \cap e \neq \emptyset$. If $X \subseteq E(H)$, then $H/X$ is a hypergraph obtained from $H$ by contracting all hyperedges in $X$. If $S$ is a sub-hypergraph of $H$, then $H/S$ denotes $H/E(S)$.

For any nonempty subset $X \subseteq E(H)$, the **density** of $X$ is defined to be

$$d_H(X) = \frac{|X|}{|V(H/X)| - \omega(H[X])}.$$ 

We often use $d_H$ for $d_H(E(H))$. Following [1,3], the **strength** $\eta(H)$ and the **fractional arboricity** $\gamma(H)$ of a nontrivial hypergraph $H$ are defined, respectively, as

$$\eta(H) = \min \left\{ \frac{|X|}{\omega(H - X) - \omega(H)} \right\} \quad \text{and} \quad \gamma(H) = \max \{d(H[X])\},$$

where the minimum and maximum are taken over all edge subsets $X \subseteq E(H)$ so that the denominators are nonzero. It is mentioned in [3] that, Theorem 2.2 shows that for a connected hypergraph $H$, $\tau(H) \geq k$ if and only if $\eta(H) \geq k$; and $\alpha(H) \leq k$ if and only if $\gamma(H) \leq k$, which gives

$$\tau(H) = \lceil \eta(H) \rceil \quad \text{and} \quad \alpha(H) = \lfloor \gamma(H) \rfloor$$

for a connected hypergraph $H$. 

**Remark.** There is also an equivalent definition for $\eta(H)$, as below

$$\eta(H) = \min \left\{ \frac{|E(H) - X|}{|V(H/X)| - \omega(H)} \right\}. \quad (2)$$

**Proof of the Remark.** Let $X' = E(H) - X$. Each connected component of $H - X'$ corresponds to a vertex of $H/X$, and thus $\omega(H - X') = |V(H/X)|$. Hence

$$\min \left\{ \frac{|E(H) - X|}{|V(H/X)| - \omega(H)} \right\} = \min \left\{ \frac{|X'|}{\omega(H - X') - \omega(H)} \right\},$$

where the minimums are taken over all edge subsets $X \subseteq E(H)$ (or equivalently $X' \subseteq E(H)$) so that the denominators are nonzero, which finishes the proof of the remark.

It follows by definitions that for any nontrivial hypergraph $H$, $\eta(H) \leq d(H) \leq \gamma(H)$. A hypergraph $H$ is **uniformly dense** if $d(H) = \gamma(H)$. We have the following property.

**Theorem 2.3** ([3]). For a hypergraph $H$, the following are equivalent.

(i) $\eta(H) = \gamma(H)$.

(ii) $\eta(H) = d(H)$.

(iii) $d(H) = \gamma(H)$. 

**Theorem 2.3** generalizes the corresponding results in [1] from graphs to hypergraphs. Let $\tau_k$ be the family of all $k$-partition-connected hypergraphs. By Theorem 2.2(i), $\tau_k$ is the family of all hypergraphs each of which contains $k$ edge-disjoint spanning hypertrees.

**Proposition 2.4** ([3]). Each of the following statements holds.

(C1) $\tau_k \neq \emptyset$.

(C2) If $e \in E(H)$ and $H \in \tau_k$, then $H/e \in \tau_k$.

(C3) If for some $S \subseteq E(H)$, both $S$, $H/S \in \tau_k$, then $H \in \tau_k$. 

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Lemma 2.5. Let $H$ be a hypergraph with $d(H) \geq k > \eta(H)$. Then $H$ has a connected sub-hypergraph $S$ such that $\eta(S) > k$. In particular, $d(S) > k$ and $\tau(S) \geq k$.

**Proof.** As $d(H) \geq k > \eta(H)$, by Theorem 2.3, $\gamma(H) > k$. By the definition of $\gamma(H)$, there exists a connected sub-hypergraph $S$ such that $d(S) = \gamma(H) > k$. (We can always choose $S$ to be connected, for otherwise if $S$ contains $s$ connected components $S_i$, $1 \leq i \leq s$, we claim $d(S_i) = \gamma(H)$. First, by definition of $\gamma(H)$, $d(S_i) \leq d(S) = \gamma(H)$. Thus $\frac{|E(S_i)|}{|V(S_i)|-1} \leq d(S)$, which implies $|E(S_i)| < d(S)(|V(S_i)| - 1)$). If one of $d(S_i)$ is strictly less than $d(S)$, then $\sum_{1 \leq i \leq n} |E(S_i)| < \sum_{1 \leq i \leq n} d(S)(|V(S_i)| - 1) = d(S)(\sum_{1 \leq i \leq n} |V(S_i)| - s)$. Thus $d(S) = \frac{\sum_{1 \leq i \leq n} |E(S_i)|}{\sum_{1 \leq i \leq n} |V(S_i)| - 1} < d(S)$, a contradiction. Thus $d(S) = d(S) = \gamma(H)$. In this case, we can choose $S$ to be any connected component.

Thus $d(S) \leq \gamma(S) \leq \gamma(H) = d(S)$, which implies that $d(S) = \gamma(S)$. By Theorem 2.3, $\eta(S) = d(S) = \gamma(S) > k$. In particular, $\tau(S) \geq k$. \hfill \□

Lemma 2.6. Let $C$ be a hypercircuit and $e \in E(C)$. Then $C - e$ is a hypertree (and thus partition-connected).

**Proof.** By the definition of a hypercircuit, for every nonempty subset $U \subseteq V(C) = V(C - e)$, $|E(C[U])| \leq |U| - 1$. Since $|E(C)| = |V(C)|$, we have $|E(C - e)| = |V(C)| - 1$. By definition, $C - e$ is a hypertree. \hfill \□

3. The proof of Theorem 1.2

Let $P_k(H)$ be the collection of all $k$-partitions of $E(H)$, and define $\varepsilon'(H) = \min_{\pi \in P_k(H)} \varepsilon(\pi)$.

**Lemma 3.1.** For any uniform hypergraph $H$ with $|E(H)| = k(|V(H)| - 1)$, $\varepsilon(H) = \varepsilon'(H)$.

**Proof.** Since $P_k(H) \subseteq P_k(H)$, it follows from definition that $\varepsilon(H) \geq \varepsilon'(H)$. Thus it suffices to show that $\varepsilon(H) \leq \varepsilon'(H)$.

Let $n = |V(H)|$. For each $\pi = (X_1, X_2, \ldots, X_k) \in P_k(H)$, define

$$\varphi(\pi) = \sum_{i=1}^{k} \max\{|X_i| - n + 1, 0\}.$$ 

Thus $\varphi(\pi) \geq 0$, and $\varphi(\pi) = 0$ if and only if $\pi \in P_k(H)$.

Choose a $\pi = (X_1, X_2, \ldots, X_k) \in P_k(H)$ such that $\varepsilon(\pi) = \varepsilon'(H)$ and such that $\varphi(\pi)$ is minimized. We claim that $\pi \in P_k(H)$. Assume that $\varphi(\pi) > 0$, and without loss of generality, we may assume that $|X_1| > n - 1$ and $|X_2| < n - 1$. Thus $H(X_1)$ must contain a hypercircuit $C$, and let $e \in E(C)$. Define $\pi' = (X'_1, X'_2, \ldots, X'_k)$ as below.

$$X'_i := \begin{cases} X_1 - \{e\}, & \text{if } i = 1, \\ X_2 \cup \{e\}, & \text{if } i = 2, \\ X_i, & \text{if } i > 2. \end{cases}$$

Then $\pi' \in P_k(H)$. By Lemma 2.6, the removal of an edge in a hypercircuit does not affect the partition-connectedness, and so we have $c(H(X_1)) = c(H(X'_1))$. We also have $c(H(X_2)) \geq c(H(X'_2))$. Thus $\varepsilon(H) \leq \varepsilon(\pi) = \varepsilon'(H)$, but $\varphi(\pi') = \varepsilon(\pi) - 1$, contrary to the choice of $\pi$. Thus $\varphi(\pi) = 0$, and so $\pi \in P_k(H)$. Hence $\varepsilon(H) \leq \varepsilon(\pi) = \varepsilon'(H)$, which completes the proof. \hfill \□

In the rest of the paper, we prove Theorem 1.2. Lemma 3.1 suggests that by using $\varepsilon'(H)$, we do not have to restrict our discussion to uniform $k$-partitions, and so in the proof arguments below, all $k$-partitions may not be uniform.

**Proof of Theorem 1.2.** Throughout the proof, we assume that $H$ is an $r$-uniform hypergraph for some integer $r \geq 2$. Since $\varepsilon(H) > 0$, $H$ does not have $k$ edge-disjoint spanning hypertrees, and so by (1), $\eta(H) < k$. Since $|E(H)| = k(|V(H)| - 1)$, we have $d(H) = k$. By Lemma 2.5, there exists a maximal connected sub-hypergraph $S$ with $d(S) > k$ and $\tau(S) \geq k$. In other words, $|E(S)| > k(|V(S)| - 1)$ and $S$ has $k$ edge-disjoint spanning hypertrees.

**Claim 1.** For any $v \in V(H) - V(S)$, there exist $w_1, \ldots, w_{r-1} \in V(S)$ such that $(w_1, \ldots, w_{r-1}, v) \notin E(H)$.

**Proof of Claim 1.** If not, then $v$ is adjacent to every $(r-1)$-subset of $V(S)$ in $H$. Let $|V(S)| = s$. Then there are at least $\binom{s}{r-1}$ hyperedges joining $S$ and $v$. Since $S$ is a simple $r$-uniform hypergraph, $\binom{s}{r-1} \geq |E(S)| > k(s - 1)$. Then $\binom{s}{r-1} = \frac{s^r}{r!} > k \cdot \frac{(s-1)^{r-1}}{(r-1)!}$, which is a contradiction. Thus $|E(H[V(S) \cup \{v\}])| > |E(S)| + k$, and so

$$d(H[V(S) \cup \{v\}]) = \frac{|E(H[V(S) \cup \{v\}])|}{|V(S) \cup \{v\}| - 1} > \frac{|E(S)| + k}{|V(S)|} = \frac{k(|V(S)| - 1) + k}{|V(S)|} = k.$$

Since $H[V(S) \cup \{v\}] / S$ is a multigraph with two vertices and at least $k$ edges, $H[V(S) \cup \{v\}] / S$ is $k$-partition-connected. As $S$ is $k$-partition-connected, it follows by Proposition 2.4(C3) that $H[V(S) \cup \{v\}]$ is $k$-partition-connected. Thus $\tau(H[V(S) \cup \{v\}]) \geq k$, contrary to the maximality of $S$. This proves the claim.
Since $S$ has $k$ edge-disjoint spanning hypertrees, $E(S)$ has a $k$-partition $(Y_1, Y_2, \ldots, Y_k)$ such that each $S(Y_i)$ is a spanning partition-connected sub-hypergraph of $S$ for $1 \leq i \leq k$. As $|E(S)| > k(|V(S)| - 1)$, one of these spanning partition-connected sub-hypergraphs, (say $S(Y_1)$), must contain a hypercircuit $C$. Let $e \in E(C)$.

Choose $\pi = (X_1, X_2, \ldots, X_k) \in P_k^r(H)$ such that $\varepsilon(\pi) = \varepsilon'(H)$. Define $X'_i = (X_i - E(S)) \cup Y_i$ for $1 \leq i \leq k$ and $\pi' = (X'_1, X'_2, \ldots, X'_k)$.

**Claim 2.** $c(H(X'_i)) \leq c(H(X_i)))$ for $1 \leq i \leq k$.

**Proof of Claim 2.** It suffices to show that for any maximal partition-connected sub-hypergraph $T$ of $H(X_i)$, the sub-hypergraph $T'$ induced by $(E(T) - E(S)) \cup Y_i$ is also partition-connected. Since $T$ is partition-connected, by Proposition 2.4(C2), $T/S$ is partition-connected. Thus $T'/Y_i = T/S$ is partition-connected. As $S(Y_i)$ is partition-connected, by Proposition 2.4(C3), $T'$ is partition-connected. This proves the claim.

By Claim 2, $\varepsilon'(H) \leq \varepsilon(\pi') \leq \varepsilon(\pi) = \varepsilon'(H)$. By Lemma 3.1, $\varepsilon'(H) = \varepsilon(H) > 0$. Thus we may assume that $c(H(X'_j)) \geq 2$ for some $j$. Since $H(X'_j)$ has a partition-connected sub-hypergraph $S(Y_j)$, $H(X'_j)$ has a maximal partition-connected sub-hypergraph $R$ containing $S(Y_j)$. Furthermore, $H(X'_j)$ must have a vertex $v \in V(H) - V(S)$ such that $v$ is not in $R$ since $c(H(X'_j)) \geq 2$. By Claim 1, there are vertices $w_1, w_2, \ldots, w_{r-1} \in V(S)$ such that $e' = \{w_1, w_2, \ldots, w_{r-1}, v\} \not\in E(H)$. Define the hyperedge subset

$$X''_j := \begin{cases} (X'_i - \{e\}) \cup \{e'\}, & \text{if } i = j, \\ (X'_i - \{e\}), & \text{if } i \neq j. \end{cases}$$

Note that $j = 1$ is possible and the hyperedge $e$ is in $X'_1$.

Let $F = H - e + e'$. Then $\pi'' = (X''_1, X''_2, \ldots, X''_k) \in P_k^r(F)$.

**Claim 3.** $\varepsilon(\pi'') < \varepsilon(\pi')$.

**Proof of Claim 3.** When $i \neq j$, since $e \in E(C) \subseteq X'_i$, $c(F(X''_i)) = c(H(X'_i))$. When $i = j$, let $R'$ be the maximal partition-connected sub-hypergraph of $H(X'_j)$ that contains $v$. As $(R + e')/R = K_2$ (a graph with two vertices and one edge) is partition-connected, and $R$ is partition-connected, by Proposition 2.4(C3), $R + e'$ is partition-connected. Also, $(R + e')/R' = R'$ is partition-connected. Again by Proposition 2.4(C3), $(R + e')/R'$ is partition-connected in $F(X''_i)$. As $R$ and $R'$ are two maximal partition-connected sub-hypergraphs in $H(X'_j)$, it follows that $c(F(X''_j)) < c(H(X'_j))$. By definition, $\varepsilon(\pi'') < \varepsilon(\pi')$, completing the proof of the claim.

By Lemma 3.1 and Claim 3, $\varepsilon(F) = \varepsilon'(F) \leq \varepsilon(\pi'') < \varepsilon(\pi') = \varepsilon'(H) = \varepsilon(H)$. This proves the theorem. □

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