Spanning trails in essentially 4-edge-connected graphs

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A connected graph $G$ is essentially 4-edge-connected if for any edge cut $X$ of $G$ with $|X| < 4$, either $G - X$ is connected or at most one component of $G - X$ has edges. In this paper, we introduce a reduction method and investigate the existence of spanning trails in essentially 4-edge-connected graphs. As an application, we prove that if $G$ is 4-edge-connected, then for any edge subset $X_0 \subseteq E(G)$ with $|X_0| \leq 3$ and any distinct edges $e$, $e' \in E(G)$, $G$ has a spanning $(e, e')$-trail containing all edges in $X_0$, which solves a conjecture posed in [W. Luo, Z.-H. Chen, W.-G. Chen, Spanning trails containing given edges, Discrete Math. 306 (2006) 87–98].

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1. Introduction

Graphs considered in this paper are finite and loopless. Undefined terms will follow [2]. A trail is a finite sequence $T = u_0e_1u_1e_2u_2 \cdots e_
u u_r$, whose terms are alternately vertices and edges, with $e_i = u_{i-1}u_i$ ($1 \leq i \leq r$), where the edges are distinct. A trail $T$ is a closed trail if $u_0 = u_r$ and is called a $(u, v)$-trail if $u_0 = u$ and $u_r = v$, and is called an $(e, e')$-trail if $e = e_1$ and $e' = e_r$. A closed trail is also called an Eulerian subgraph. A trail $T$ is called a spanning trail if $V(T) = V(G)$. A graph is called supereulerian if it has a spanning closed trail.

A graph $G$ is nontrivial if $E(G) \neq \emptyset$. An edge cut $X$ of a graph $G$ is essential if both components of $G - X$ are nontrivial; and $G$ is essentially $k$-edge-connected if $G$ is connected and does not have an essential edge cut of size less than $k$. It follows from the definition, we have the following proposition:

Proposition 1.1. Let $G$ be an essentially $k$-edge-connected graph with the minimum degree $\delta(G)$ and the edge-connectivity $\kappa'(G)$. Then $\kappa'(G) = \min(\delta(G), k)$.

For a graph $G$, the line graph of $G$, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ are adjacent in $L(G)$. It follows from the definitions that a line graph $L(G)$ is $k$-connected if and only if $G$ is essentially $k$-edge-connected. For line graphs, Thomassen has a well known conjecture [12]: “every 4-connected line graph is Hamiltonian”. By a theorem of Harary and Nash-Williams [6], to prove Thomassen’s conjecture, one can prove the equivalent version: every essentially 4-edge-connected graph has a closed trail that contains at least one vertex of every edge in $G$.
On the other hand, motivated by the Chinese postman problem, Boesch et al. [1] introduced the supereulerian problem, that is to determine if a graph \( G \) has a spanning closed trail. Pulleyblank [11] showed that this is an NP-complete problem. Catlin [3] and Jaeger [7] proved the following:

**Theorem 1.2** (Catlin [3] and Jaeger [7]). A 4-edge-connected graph has a spanning closed trail.

As shown in [10], Theorem 1.2 can be improved in the sense that a 4-edge-connected graph can have spanning closed trail containing some fixed edges. In [10], Luo et al. defined a graph \( G \) to be \( r \)-edge-Eulerian-connected if for any edge subset \( X \subseteq E(G) \) with \( |X| \leq r \) and any distinct edges \( e, e' \in E(G) \), \( G \) has a spanning \((e, e')\)-trail containing all edges in \( X \). Define \( \xi(r) \) to be the smallest integer \( k \) such that every \( k \)-edge-connected graph is \( r \)-edge-Eulerian-connected. They proved the following:

**Theorem 1.3** (Luo, Chen and Chen [10]). Let \( r \geq 0 \) be an integer. Then

\[
\xi(r) = \begin{cases} 4, & 0 \leq r \leq 2, \\ r + 1, & r \geq 4. \end{cases}
\]

For \( r = 3 \), Luo et al. [10] indicated that \( 4 \leq \xi(3) \leq 5 \), and conjectured \( \xi(3) = 4 \).

In this paper, we introduce a reduction method on essentially 4-edge-connected graphs and investigate spanning trails in essentially 4-edge-connected graphs. As an application, we prove the following:

**Theorem 1.4.** If \( G \) is a 4-edge-connected graph, then for any \( X_0 \subseteq E(G) \) with \( |X_0| \leq 3 \) and any distinct edges \( e, e' \in E(G) \), \( G \) has a spanning \((e, e')\)-trail \( T \) such that \( X_0 \subseteq E(T) \). Thus, \( G \) is 3-edge-Eulerian-connected and so \( \xi(3) = 4 \).

Theorem 1.4 confirmed the conjecture above, and so all the values of \( \xi(r) \) are determined for all integer \( r \geq 0 \).

In the rest of the paper, we provide the theory of Catlin’s reduction method which is an important tool to solve problems related to spanning trails, and introduce a new reduction method on essentially 4-edge-connected graphs in Section 2. The results of spanning trails in essentially 4-edge-connected graphs are given in Section 3. We will discuss 3-edge-Eulerian-connected graphs and give the proof of the conjecture \( \xi(3) = 4 \) in Section 4.

2. Reductions of essentially 4-edge-connected graphs

In this section, we shall develop a reduction method for essentially 4-edge-connected graphs and prove some associate results on spanning trails that will be needed in the proof of Theorem 1.4.

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). For vertex disjoint subsets \( V_1, V_2 \subseteq V(G) \), let \([V_1, V_2]_G\) denotes the set of all edges in \( G \) with one end in \( V_1 \) and the other in \( V_2 \). For vertex disjoint subgraphs \( H, L \) of \( G \), we write \([H, L] = [V(H), V(L)]_G\), and define \( \partial_G(H) = [V(H), V(G) - V(H)]_G \), called the boundary of \( H \) in \( G \). When \( H = K_1 \) is a single vertex \( v \), we denote \( \partial_G(v) \) as \( \partial_G[H] \) and \( |\partial_G[H]| = \partial_G(v) \).

For a graph \( G \) and \( X \subseteq E(G) \), the contraction \( G/X \) is obtained from \( G \) by identifying the two ends of each edge in \( X \) and then by deleting the resulting loops. If \( H \) is a subgraph of \( G \), then we write \( G/H \) for \( G/E(H) \). When \( H \) is connected, we use \( v_H \) to denote the vertex in \( G/H \) onto which \( H \) is contracted. Note that \( E(G/H) = E(G) - E(H) \) and \( V(G/H) = (V(G) - V(H)) \cup \{v_H\} \).

For an edge \( xy \in E(G) \), we let \( \theta(xy) \) be the vertex in \( G/xy \) onto which the edge \( xy \) is contracted.

A graph \( G \) is collapsible [3] if for any subset \( S \subseteq V(G) \) with \(|S| \equiv 0 \pmod{2} \), \( G \) has a spanning connected subgraph \( L_S \) such that the set of odd degree vertices in \( L_S \) is precisely \( S \). As shown in [3], if \( G \) is a simple graph and \( H \) is a maximal collapsible subgraph of \( G \), then \( G/H \) is also a simple graph. Furthermore, Catlin [3] showed that any graph \( G \) has a unique collection of vertex disjoint collapsible subgraphs \( H_1, H_2, \ldots, H_c \), and \( G/(H_1 \cup H_2 \cup \cdots \cup H_c) \) obtained by contracting each \( H_i \) into a single vertex \( v_{H_i} \), is called the reduction of \( G \). As always, \( K_1 \) is considered both supereulerian and collapsible, and has infinite edge-connectivity. It was shown in [3] if \( G' \) is the reduction of \( G \), then \( G' \) is simple and \( K_3 \)-free and \( \kappa'(G') \geq \kappa'(G) \).

A graph \( G \) is reduced if its reduction is \( G \) itself. The theory on collapsible graphs is useful for both simple graphs and multigraphs. Let \( F(G) \) be the minimum number of additional edges that must be added to \( G \) to result in a graph \( G^* \) with at least two edge-disjoint spanning trees. The following are some useful theorems which will be needed.

**Theorem 2.1.** Let \( G \) be a graph and let \( H \) be a collapsible subgraph of \( G \). Let \( v_H \) be the vertex in \( G/H \) onto which \( H \) is contracted.

(i) \([3]\) Suppose that \( u \neq v_H \) and \( G/H \) has a \((u, v)\)-trail \( T \) containing \( v_H \). If \( v \neq v_H \), then \( G \) has a \((u, v)\)-trail \( T \subseteq E(T) \) and \( V(T) = (V(T') - \{v_H\}) \cup V(H) \). If \( v = v_H \), then for any \( v' \in V(H) \), \( G \) has a \((u, v')\)-trail \( T \subseteq E(T') \) and \( V(T) = (V(T') - \{v_H\}) \cup V(H) \).

(ii) \((\text{Theorem } 1.3 \text{ of } [4])\) If \( \kappa'(G) \geq 2 \) and \( F(G) \leq 2 \), then the reduction of \( G \) is in \( \{K_1, K_2, t \} \) for some integer \( t \geq 2 \).

(iii) \([3]\) If \( G \) is reduced, then \( F(G) = 2|V(G)| - |E(G)| = 2 \).

(iv) \((\text{Theorem } 2.3 \text{ of } [9])\) If \( G \) is collapsible, then for any \( u, v \in V(G) \), \( G \) has a spanning \((u, v)\)-trail.

(v) \([3]\) \( G \) is supereulerian if and only if \( G/H \) is supereulerian. In particular, \( G \) is supereulerian if and only if the reduction of \( G \) is supereulerian.

Next, we introduce a new reduction method for preserving essentially 4-edge-connected property of graphs, which develops the ideas deployed in the proof of Theorem 3.1 in [8].
For a graph \( G \) and for each integer \( i > 0 \), define

\[ D_i(G) = \{ v \in V(G) : d_G(v) = i \}. \]

Let \( z \in D_2(G) \) with \( N_G(z) = \{ z_1, z_2 \} \) such that \( z_1 \in D_4(G) \) and \( N_G(z_1) = \{ z, w_1, w_2, w_3 \} \). For \( i \in \{ 1, 2, 3 \} \), if \( w_i \in D_2(G) \), then let \( N_G(w_i) = \{ z_1, w'_i \} \). For \( j \in \{ 1, 2 \} \), let \( G'_j = (G - \{ z_1 \}) + \{ z w_j, w_{3-j}w_3 \} \), and \( W(G'_j) = \{ e = xy \in E(G'_j) : x, y \in D_2(G'_j) \} \). Define

\[ G_i = G'_i / W(G'_i). \]  

(1)

For an essentially 4-edge-connected graph \( G \), if \( w_i \in D_2(G) \), then \( N_G(w_i) = \{ z_1, w'_i \} \cap D_2(G) = \emptyset \). Thus, if an edge \( e \in W(G'_i) \), then \( e \in \{ z w_j, w_{3-j}w_3 \} \) (see Fig. 1).

**Theorem 2.2.** Let \( G \) be an essentially 4-edge-connected graph with \( \delta(G) \geq 2 \) and \( D_2(G) = \emptyset \). Let \( z \in D_2(G) \) with \( N_G(z) = \{ z_1, z_2 \} \) such that \( z_1 \in D_4(G) \) and \( N_G(z_1) = \{ z, w_1, w_2, w_3 \} \). For \( i \in \{ 1, 2, 3 \} \), if \( w_i \in D_2(G) \), then let \( N_G(w_i) = \{ z_1, w'_i \} \). Let \( G_1 \) and \( G_2 \) be the graphs defined by (1) above. Then either \( G_1 \) or \( G_2 \) is also essentially 4-edge-connected and \( \delta(G_j) \geq 2 \) and \( D_3(G_j) = \emptyset \) (\( j = 1, 2 \)).

**Proof.** Since \( G \) is essentially 4-edge-connected with \( \delta(G) \geq 2 \), by Proposition 1.1, \( G \) is 2-edge-connected. Then by the definition of \( G_j \) (\( j = 1, 2 \)), \( G_j \) is connected with \( \delta(G_j) \geq 2 \) and \( D_3(G_j) = \emptyset \). It suffices to show that either \( G_1 \) or \( G_2 \) is essentially 4-edge-connected. For \( j \in \{ 1, 2 \} \), by (1), when \( w_{3-j}w_3 \in W(G'_j) \), we shall use \( w_{3-j} \) to denote the vertex \( \theta(w_{3-j}w_3) \) in \( G_j \); and when \( w_j \in D_2(G) \), use \( z \) to denote the vertex \( \theta(z w_j) \) in \( G_j \). Let \( x_1, x_2 \) and \( x_3 \) denote the vertices in \( G_1 \) and \( G_2 \) such that

\[ x_1 = \begin{cases} 
  w_1 & \text{if } w_1 \notin D_2(G), \\
  w'_1 & \text{if } w_1 \in D_2(G),
\end{cases} \quad x_2 = \begin{cases} 
  w_2 & \text{if } w_2 \notin D_2(G), \\
  w'_2 & \text{if } w_2 \in D_2(G),
\end{cases} \quad x_3 = \begin{cases} 
  w_3 & \text{if } w_{3-j} \notin D_2(G) \text{ in } G_j, j \in \{ 1, 2 \}, \\
  w'_2 & \text{if } w_2 \in D_2(G) \text{ in } G_1, \\
  w'_1 & \text{if } w_1 \in D_2(G) \text{ in } G_2.
\end{cases} \]  

(2)

(3)

The notation \( x_3 \) in (3) is for the convenience in our discussion below for \( G_1 \) and \( G_2 \), respectively. In \( G_1 \), if \( w_2 \in D_2(G) \), then (3) defines \( x_3 = w_3 \) in \( G_1 \); if \( w_2 \notin D_2(G) \), then (3) defines \( x_3 = w_3 \) (see Fig. 2 for \( G_1 \)). Similarly, one can find what \( x_3 \) is in \( G_2 \) from (3).

Since \( G \) is essentially 4-edge-connected, by \( D_3(G) = \emptyset \) and by (2),

\[ d_G(x_i) \geq 4, \quad \text{if } 1 \leq i \leq 2. \]  

(4)

By way of contradiction, suppose both \( G_1 \) and \( G_2 \) are not essentially 4-edge-connected. Then \( G_1 \) and \( G_2 \) have minimum essential edge cuts \( X \) and \( Y \), respectively, such that \( 2 \leq |X| \leq 3 \) and \( 2 \leq |Y| \leq 3 \).

**Claim 1.** For any essential edge cuts \( X \) in \( G_1 \) and \( Y \) in \( G_2 \) with \( 2 \leq |X| \leq 3 \) and \( 2 \leq |Y| \leq 3 \), \( X \cap \{ x_1, x_2, x_3 \} = \emptyset \) and \( Y \cap \{ x_2, x_1, x_3 \} = \emptyset \).

We will prove the case for \( X \) only. The proof for \( Y \) is similar and hence omitted. By way of contradiction, suppose \( X \) contains either \( z x_1 \) or \( x_2 x_3 \). (we may, without lose of generality, assume that \( z \) and \( x_2 \) are in the same component of \( G_1 - X \)), then define

\[ X' = \begin{cases} 
  (X - z x_1) \cup \{ z_1 w_1 \} & \text{if } z x_1 \in X \text{ and } z x_2 x_3 \notin X, \\
  (X - x_2 x_3) \cup \{ z_1 w_3 \} & \text{if } z x_2 x_3 \in X \text{ and } z x_1 \notin X, \\
  (X - \{ z x_1, x_2 x_3 \}) \cup \{ z_1 w_1, z_1 w_3 \} & \text{if } z x_2 x_3 \in X \text{ and } z x_1 \in X.
\end{cases} \]

Thus, \( X' \) is an essential edge cut of \( G \) with \( |X'| = |X| \), contrary to the assumption that \( G \) is essentially 4-edge-connected. Claim 1 is proved.

Since \( X \cap \{ x_1, x_2, x_3 \} = \emptyset \), \( z x_1 \) and \( x_2 x_3 \) must be in distinct components of \( G_1 - X \). Let \( A_1 \) and \( A_2 \) be the two components of \( G_1 - X \) with \( z x_1 \in E(A_1) \) and \( x_2 x_3 \in E(A_2) \).
By(7)and(6),

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This contradiction establishes the theorem. □

### 3. Spanning trails in essentially 4-edge-connected graphs

For a reduced graph $G$ with $\delta(G) \geq 2$, let $d_i = |\mathcal{D}_i(G)|$. Then $|V(G)| = \sum_{i \geq 2} d_i$ and $2|E(G)| = \sum_{i \geq 2} id_i$, by Theorem 2.1(iii),

$$2F(G) = 4 \sum_{i \geq 2} d_i - \sum_{i \geq 2} id_i - 4. \quad (8)$$
Theorem 3.1. Let $G$ be an essentially 4-edge-connected graph with $\delta(G) \geq 2$ and $|D_2(G) \cup D_3(G)| \leq 5$. Then each of the following holds.

(i) If $|D_2(G)| \leq 3$, then $G$ is collapsible.

(ii) Either $G$ is supereulerian or the reduction of $G$ is $K_{2,5}$ such that all the vertices of degree 2 in the reduction are trivial.

(iii) If $|D_2(G)| \geq 2$, then for any pair of distinct vertices $u, v \in D_2(G)$, $G$ has a spanning $(u, v)$-trail.

Proof. Since $G$ is an essentially 4-edge-connected graph with $\delta(G) \geq 2$, by Proposition 1.1, $\kappa'(G) \geq 2$. We argue by contradiction and assume that $G$ is a counterexample with $|V(G)|$ minimized. \hfill (10)

If $G$ is collapsible, then Theorem 3.1(i) holds. Hence we may assume that $G$ is not collapsible. Let $G'$ be the reduction of $G$. Then $G' \neq K_1$ and $\kappa'(G) \geq 2$. If $F(G') \leq 2$, then by Theorem 2.2 $G'$ is a $K_{2,2}$, for some $t \geq 2$. Since $G$ is essentially 4-edge-connected, we must have $t \in \{4, 5\}$ and any vertex in $D_2(G')$ must be a trivial contraction, and so we can view $D_2(G') \subseteq D_2(G)$. Thus, $|D_2(G)| \geq |D_2(G')| = t \geq 4$. If $t = 4$, then $K_{2,4}$ is Eulerian and so by Theorem 2.1(ii) $G$ is supereulerian. If $G$ is not supereulerian, then the reduction of $G$ must be $K_{2,5}$, and so Theorem 3.1(ii) must hold. Moreover, by inspection, if $u \in D_2(K_{2,1})$ and $v \in V(K_{2,1} - u)$, then $K_{2,1}$ always has a spanning $(u, v)$-trail, and so by Theorem 2.1(i), Theorem 3.1(iii) must hold. Hence we may assume that $D_2(G) = \{u, v\}$ and $G$ has spanning $(u, v)$-trails. \hfill (13)

We proceed our proof by verifying the following claims and let $D_2(G) = \{a, b, c, u, v\}$.

Claim 1. For any $z \in \{a, b, c\} = D_2(G) - \{u, v\}$, $|M(z) \cap D_2(G)| \geq 2$ and $|M^*(z) \cap D_2(G)| \geq 2$.

Proof of Claim 1(a). By symmetry, it suffices to show that $|M(z) \cap D_2(G)| \geq 2$. By contradiction, suppose $|M(z) \cap D_2(G)| \leq 1$. Then we may assume $M(z) \cap D_2(G) \subseteq \{w_3\}$.

Using the reduction method and the same notations in Theorem 2.2, we obtain two graphs $G_1$ and $G_2$ from $G$ with $\delta(G_1) \geq 2$ and $D_2(G_1) = \emptyset (i = 1, 2). By Theorem 2.2, we may assume that $G_1$ is essentially 4-edge-connected. Since $M(z) \cap D_2(G) \subseteq \{w_3\}$, $w_1, w_2 \not\in D_2(G)$, and by (1), we have $G_1 = (G - \{z_1\}) + \{z, w_1, w_2, w_3\}$, $x_1 = w_1, x_2 = w_2$ and $x_3 = w_3$. Thus we may view $D_2(G_1) = D_2(G)$. By (10), $G_1$ has a spanning $(u, v)$-trail $H_1'$. Since $z$ has degree 2 in $G_1$ and $z \not\in \{u, v\}$, $z_1 \in E(H_1')$. Define

$$H_1 = \begin{cases} G[E(H_1' \ast \{x_1\}) \cup \{z_1, z_2, w_1\}] & \text{if } x_2 x_3 \not\in E(H_1') \\ G[E(H_1' \ast \{x_1, z_2, w_3\}) \cup \{z_1, z_2, w_1, w_2 z_1, z_1 w_3\}] & \text{if } x_2 x_3 \in E(H_1'). \end{cases}$$

Then $H_1$ is a spanning $(u, v)$-trail of $G$, contrary to (13). This proves Claim 1(a).
Thus, by (10), $c \in M(z)$ and $M(z) \cap \{u, v\} = \emptyset$. We may assume that $z = a$. By Claim 1(a), $|M(z) \cap D_2(G)| \geq 2$. Since $z = a \notin M(z)$ and $M(z) \cap \{u, v\} = \emptyset$, $M(z) \cap D_2(G) = D_2(G) - \{a, u, v\} = \{b, c\}$. We may assume that $w_1 = b$ and $w_2 = c$, and so $d_G(w_1) = d_G(w_2) = 2$ and $d_G(w_3) = 4$. Let $N_G(w_i) = \{x, w_i'\} (i = 1, 2)$. Again using the reduction method on $G$ as in Theorem 2.2, we obtained two graphs $G_1$ and $G_2$ with $\delta(G_i) \geq 2$ and $D_3(G_i) = \emptyset (i = 1, 2)$. By Theorem 2.2, we may assume that $G_1$ is essentially 4-edge-connected. Then since $d_G(z) = d_G(w_1) = 2$, and $d_G(w_3) = 4$, $G_1 = (G - \{z_1\}) + \{z w_1, w_2 w_3\}$ with $W(G_1) = \{z w_1\} = \{b\}$, and so $G_1 = G_1^+ / z w_1$. Since $G_1 = G_1^+ / z w_1$ and $z w_1^+ \in E(G_1)$, and with $x_1 = w_1', x_2 = w_2$ and $x_3 = w_3 = c$ (see Fig. 2(I) for $G_1$). Thus, by (10), $G_1$ has a spanning $(u, v)$-trail $H_0$.

Since $\{z, x_3\} = \{a, c\} \subseteq D_2(G_1) - \{u, v\}$, $x_1 z = x_3 w_1', x_2 w_3 = x_3 w_2$ are both in $E(H_0)$. Since $d_G(w_2) = d_G(c) = d_G(c) = 2$ and $c \notin \{u, v\}$, $w_2 w_3$ is also in $E(H_0)$. Define

$$H_1 = (H_0 - \{x_1 z, w_2 w_3\}) + \{z_1, z_1 w_1, w_1 w_1', z_1 w_2, z_1 w_3\}.$$  

Then $H_1$ is a spanning $(u, v)$-trail in $G$, a contradiction. Thus, Claim 1(b) is proved.

**Claim 2.** For any $z \in D_2(G)$, $|D_2(G) \cap M(z) \cap M^*(z)| \leq 1$.

By the definition of $M(z)$ and $M^*(z)$, $|D_2(G) \cap (M(z) \cap M^*(z))| \leq 3$, where equality holds if and only if $G = K_2 4$. Since $|D_2(G)| = d_2 = 5$, $G \neq K_2 4$, and so $|D_2(G) \cap M(z) \cap M^*(z)| \leq 2$. If $|D_2(G) \cap M(z) \cap M^*(z)| = 2$, then we may assume that $w_1 = w_1'$ and $w_2 = w_2'$ in $D_2(G)$. Then $\{z_1 z, z_2^+ w_2\}$ is an essential edge cut of $G$, contrary to that $G$ is essentially 4-edge-connected. This proves Claim 2.

**Claim 3.** For all $y \in \{u, v\}$, $M(y) \cap M^*(y) \cap \{a, b, c\} = \emptyset$.

Without loss of generality, we may assume $y = u$. By way of contradiction, suppose there is a vertex $z$ in $\{a, b, c\}$ such that $z \in M(u) \cap M^*(u)$. Let $N_G(u) = \{u_1, u_2\}$. Then $z u_1$ and $z u_2$ are the two edges incident with $z$. Let $G_0 = G / z u_2$ with $u_2 = \theta(z u_2)$. Then $u_1 u_2 \in E(G_0)$. Note $G_0$ has the same essentially edge-connectivity as $G$ and $\delta(G_0) \geq 2$ with $|V(G_0)| < |V(G)|$. Therefore, by (10), $G_0$ has a spanning $(u, v)$-trail $H_0$.

If $u_1 u_2 \in E(H_0)$, then $H = H_0 - u_1 u_2 + \{z u_1, z u_2\}$ is a spanning $(u, v)$-trail in $G$, contrary to (13). If $u_1 u_2 \notin E(H_0)$, then $H_0$ is a spanning $(u, v)$-trail in $G_0$, and one only one of $u_1 u_2$ or $u_2 u_1$ (say $u u_1$) is in $H_0$, then $H = H_0 - u_1 u_2 + \{u_1 u_2, z_1 z_2\}$ is a spanning $(u, v)$-trail in $G$, a contradiction again. Claim 3 is proved.

For $\{a, b, c\} = D_2(G) - \{u, v\}$, let $N_G(a) = \{a_1, a_2\}, N_G(b) = \{b_1, b_2\}$, and $N_G(c) = \{c_1, c_2\}$. Then since $G$ is essentially 4-edge-connected and by (12), $d(a_1) = d(a_2) = d(c_1) = d(c_2) = 4$ where $i = 1, 2$. Let $S = N_G(a) \cup N_G(b) \cap N_G(c)$. If $|S| = 2$, then $S = N_G(a) = N_G(b) = N_G(c)$, contrary to Claim 2. Thus, $|S| \geq 3$. In the following, we assume $N_G(a) = \{a_1, a_2\} \subseteq S$ and let $x \in S - \{a_1, a_2\}$. Thus,

$$S = \{a_1, a_2, x, \ldots\}.$$  

By Claim 1(a) and (b), $|M(a) \cap M^*(a)| \geq 2$, $|M(a) \cap D_2(G)| \geq 2$, $|M(a) \cap \{u, v\}| \geq 1$ and $|M^*(a) \cup \{u, v\}| \geq 1$. We may assume that $b \in M(a) = M(a) - \{a\}, u \in M(a) = M(a) - \{a\}$, and by Claim 3 $M(a) \cap \{u, v\} = \emptyset$. And so we may assume $a \in N_G(b) - \{a\}$ (see Fig. 4(A)).

Case 1. $b \in M^*(a)$ (see Fig. 4(B)).

Then $N_G(b) = \{a_1, a_2\} = N_G(a)$. Since $N_G(c) \subseteq S$, $c$ must be adjacent to $x$, and so $x \in N_G(c)$. We may assume that $x = c_1$ and $M(c) = N_G(c_1) - \{c\}$. By Claim 1(a), $|M(c) \cap D_2(G)| \geq 2$, $c_1$ must be adjacent to another two degree 2 vertices in addition to $c$. Hence, since $N_G(c) = N_G(b), u$ and $v$ must be the two vertices adjacent to $c_1$ and so $N_G(u) = \{a_1, c_1\}$ and $N_G(v) = \{a_2, c_1\}$. Therefore, the another vertex $c_2 \in N_G(c)$ is not in $\{a_1, a_2\}$. Otherwise, $c \in M(u) \cap M^*(u) \cap \{a, b, u, v\}$, contrary to Claim 3. Note $M^*(c) = N_G(c_2) - \{c\}$. Thus,

$$D_2(G) \cap M^*(c) = \{a, b, u, v, c\} \cap M^*(c) = \emptyset,$$

counter to Claim 1(a) that $|M^*(c) \cap D_2(G)| \geq 2$.

Case 2. $b \notin M^*(a)$ (see Fig. 4(C)).

Then by Claim 1(a), $M^*(a) = N_G(a_2) - \{a\}$ must have at least two degree 2 vertices, and so $c \in M^*(a) = N_G(a_2) - \{a\}$. Since $b \notin M^*(a), N_G(b) \cap S \neq \emptyset$, and so we may assume $x \in N_G(b) - \{a\}$ (see Fig. 4(C)). Then since both $u$ and $b$ are adjacent to $a_1$, by Claim 3 $u$ is not adjacent to $x$. By Claim 1(a), $M^*(b) = N_G(x) - \{b\}$ must have at least two degree 2 vertices other than $b$ and $u$. Thus, $u$ and $c$ must be in $M^*(b) = N_G(x) - \{b\}$. Therefore, $N_G(v) = \{a_2, x\} = N_G(c)$, contrary to Claim 3.

We have a contradiction for each case above, and so the statement (13) is false. The theorem is proved.  \qed
In Theorem 3.12 of [5], Catlin and Lai proved that if a 3-edge-connected graph $G$ has at most 9 edge cuts of size 3, then $G$ is supereulerian. For an essentially 4-edge-connected graph $G$ with $\delta(G) \geq 3$, we have the following:

**Theorem 3.2.** If $G$ is an essentially 4-edge-connected graph with $\delta(G) \geq 3$ and $|D_3(G)| < 10$, then $G$ is collapsible and has a spanning $(u, v)$-trail for any $u, v \in V(G)$.

**Proof.** Since $G$ is essentially 4-edge-connected with $\delta(G) \geq 3$, by Proposition 1.1, $\kappa'(G) \geq 3$. Let $G'$ be the reduction of $G$. By way of contradiction, suppose $G$ is not collapsible. Then $G' \not\cong K_1$ and $\kappa'(G') \geq 3$. Let $d_1 = |D_1(G')|$. Then since $\kappa'(G') \geq 3$, $d_1 = d_2 = 0$. Since $G$ is essentially 4-edge-connected, $G$ does not have an essential edge cut of size 3, and so $d_3 = |D_3(G')| \leq |D_3(G)| < 10$. If $F(G') \leq 2$, then by Theorem 2.1(ii), $G' \in \{K_1, K_{2,1}\}$ ($t \geq 2$), contrary to $G' \not\cong K_1$ and $\kappa'(G') \geq 3$. Hence, $F(G') \geq 3$, then by (9) and $d_2 = 0$,

$$\sum_{i=5} (i - 4)d_i + 10 \leq 2d_2 + d_3;$$

$$10 \leq d_3 < 10,$$

a contradiction. Thus, $G$ must be collapsible. By Theorem 2.1(iv), for any $u, v \in V(G)$, $G$ has a spanning $(u, v)$-trail. The theorem is proved. \(\square\)

**Remark.** The Petersen Graph shows that Theorem 3.2 is best possible in the sense that $|D_3(G)| < 10$ is necessary.

### 4. Graphs that are 3-edge-Eulerian-connected

In this section, we shall investigate what graphs are 3-edge-Eulerian-connected. First, we prove the following theorem, as stated in Theorem 1.4, which proves the conjecture posed in [10].

**Theorem 4.1.** If $G$ is a 4-edge-connected graph, then $G$ is 3-edge-Eulerian-connected and so $\xi(3) = 4$.

**Proof.** Let $G$ be a graph with $\kappa'(G) \geq 4$, and let $X \subseteq E(G)$ be an edge set with $|X| = 3$. Pick any pair of edges $e', e'' \in E(G) - X$. Let $L$ be the graph obtained from $G$ by subdividing each edge $e \in X \cup \{e', e''\}$ exactly once. (That is, for each edge $e = a_1b_1 \in X \cup \{e', e''\}$, we replace $e$ by a path $a_1\nu_1b_1$, by inserting a new vertex $\nu_1$.) Then $D_2(L)$ is the set of the five degree 2 vertices generated by the subdivision, and $L$ is 2-edge-connected and essentially 4-edge-connected. By Theorem 3.1(iii), $L$ has a spanning $(\nu_1', \nu_2')$-trail. This implies that $G$ has a spanning $(e', e'')$-trail containing $X$, and so by definition, $G$ is 3-edge-Eulerian-connected. \(\square\)

As we know many 3-edge-connected graphs such as the Petersen graph have no spanning closed trail, the edge-connectivity in Theorem 4.1 cannot be lowered to 3-edge-connected. However, a 3-edge-Eulerian-connected graph is not necessarily 4-edge-connected. For example, let $G$ be a graph obtained from $K_5$ ($n \geq 8$) and a vertex $v$ by joining $v$ to $v_1$ and $v_2$ with two edges $v_1v_2$ and $v_2v_3$, where $v_1, v_2 \in V(K_5)$ and $v \not\in V(K_5)$. Then $G$ is a 3-edge-Eulerian-connected graph with $d(v) = 2$. We have the following necessary conditions for 3-edge-Eulerian-connected graphs.

**Proposition 4.2.** Let $G$ be a 3-edge-Eulerian-connected graph with $|E(G)| \geq 6$. Then $G$ must be essentially 4-edge-connected with $D_1(G) = \emptyset$.

**Proof.** We shall first show that $G$ does not have an edge cut of size 3. By contradiction, assume that $G$ an edge cut of $G$ with $|X| = 3$. Let $H_1$ and $H_2$ be the two components of $G - X$ with $|E(H_1)| \leq |E(H_2)|$. Since $G$ is 3-edge-Eulerian-connected with $|E(G)| \geq 6$ and $|X| = 3$, we may assume that $|E(H_2)| \geq 2$. Let $e_1$ and $e_2$ be two distinct edges in $E(H_2)$. Then $G$ has a spanning $(e_1, e_2)$-trail $T$ with $X \subseteq E(T)$. Since both $e_1, e_2 \in E(H_2)$, $T' = T/(H_2 \cap T)$ is a spanning closed trail of $H_2$ that contains $X$. Since $T'$ is a spanning closed trail and $X$ is an edge cut, $|X| = |E(T') \cap X| \equiv 0 \pmod{2}$, contrary to that $|X| = 3$. Hence $G$ does not have an edge cut of size 3 and so $D_3(G) = \emptyset$.

To show $G$ is essentially 4-edge-connected, it suffices to show that $G$ does not have an essential edge cut $X'$ with $|X'| = 2$. By way of contradiction, suppose that such an edge cut $X'$ exists and $G - X'$ has two components $H_1'$ and $H_2'$. Since $X'$ is an essential edge cut, we can pick an edge $e'_i \in E(H'_i)$, ($1 \leq i \leq 2$). Since $|X'| = 2 < 3$ and $G$ is 3-edge-Eulerian-connected, $G$ has a spanning $(e'_1, e'_2)$-trail $T'$ such that $X' \subseteq E(T')$. Let $e''$ be an edge not in $G$ joining the two end vertices of $T'$. Then $T' + e''$ is a spanning closed trail of $G + e''$, which contains a 3-edge-cut $X' \cup \{e''\}$ of $G + e''$. This yields a contradiction as the intersection of any close trail and any edge cut must have an even number of edges. \(\square\)

Let $G$ be the graph shown in Fig. 5 with $s \geq 6$, where $v$ is a vertex of degree 2, and $e' \in E(H_1)$ and $e'' \in E(H_2)$. Let $X = \{e_1, e_2, e_3\}$ be the set of the three edges shown in Fig. 5. As we can see that a trail started from $e_1$ in $H_1$ must ended in $H_1$ after tracing through the three edges in $X$ and vertex $v$. Hence, there is no spanning $(e', e'')$-trail $T$ in $G$ such that $X \subseteq E(T)$ and $V(T) = V(G)$. Thus, an essentially 4-edge-connected graph $G$ with $D_3(G) = \emptyset$ may not be 3-edge-Eulerian connected. It remains a problem to completely characterize the structures of 3-edge-Eulerian connected graphs.

Let $G_0 = G - \{v\} + v_1v_2$, then $G_0$ is 4-edge-connected and $X_0 = \{e_1, e_2, e_3, v_1v_2\}$ is an edge-cut of $G_0$. And $G_0$ has no spanning $(e', e'')$-trails containing $X_0$. This shows that Theorem 4.1 is best possible in the sense that 4-edge-connected graph cannot be 4-edge-Eulerian-connected.
Fig. 5. $G$ which is not 3-edge-Eulerian connected.