On extremal $k$-supereulerian graphs

Zhaohong Niu $^a$, $^*$, Liang Sun $^b$, Liming Xiong $^b$, Hong-Jian Lai $^{c,d}$, Huiya Yan $^e$

$^a$ School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi, 030006, PR China
$^b$ Department of Mathematics, Beijing Institute of Technology, Beijing, 100081, PR China
$^c$ Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA
$^d$ College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China
$^e$ Mathematics Department, University of Wisconsin-LaCrosse, LaCrosse, WI 54601, USA

Abstrac	

A graph $G$ is called $k$-supereulerian if it has a spanning even subgraph with at most $k$ components. In this paper, we prove that any 2-edge-connected loopless graph of order $n$ is $(\lceil (n-2)/3 \rceil)$-supereulerian, with only one exception. This result solves a conjecture in [Z. Niu, L. Xiong, Even factor of a graph with a bounded number of components, Australas. J. Combin. 48 (2010) 269–279]. As applications, we give a best possible size lower bound for a 2-edge-connected simple graph $G$ with $n > 5k + 2$ vertices to be $k$-supereulerian, a best possible minimum degree lower bound for a 2-edge-connected simple graph $G$ such that its line graph $L(G)$ has a 2-factor with at most $k$ components, for any given integer $k > 0$, and a sufficient condition for $k$-supereulerian graphs.

© 2013 Elsevier B.V. All rights reserved.

1. Introduction

Graphs in this paper are finite, undirected, and loopless. Undefined notation and terminology will follow [2]. Let $G$ be a graph, and let $O(G)$ denote the set of all vertices in $G$ with odd degrees. If $O(G) = \emptyset$, then $G$ is called an even graph. An Eulerian graph is a connected graph $G$ with $O(G) = \emptyset$. If a graph contains a spanning Eulerian subgraph, then it is called supereulerian. In particular, $K_1$ is supereulerian.

Boesch, Suffel, and Tindell [1] proposed the supereulerian graph problem: determine when a graph is supereulerian. They indicated that this might be a difficult problem. Pulleyblank [21] showed that such a decision problem, even when restricted to planar graphs, is NP-complete. Jaeger [14] and Catlin [5] independently showed that every 4-edge-connected graph is supereulerian.

Let $G$ be a graph, and let $X \subseteq E(G)$. The contraction $G/X$ is the graph obtained from $G$ by contracting each edge of $X$ and deleting the resulting loops. For $H \subseteq G$, we write $G/H$ for $G/E(H)$. If $H$ is a connected subgraph of $G$, and if $v_H$ denotes the vertex in $G/H$ to which $H$ is contracted, then $H$ is called the preimage of $v_H$. A vertex $v$ in a contraction of $G$ is nontrivial if $v$ has a nontrivial preimage.

On extremal supereulerian graph problems, Cai [4] proved the following result.

Theorem 1 (Cai, [4]). Let $G$ be a 2-edge-connected simple graph of order $n$. If

$$|E(G)| \geq \left( \frac{n-4}{2} \right) + 6,$$


* Corresponding author.
E-mail address: zhniu@sxu.edu.cn (Z. Niu).

0012-365X/$– see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.disc.2013.09.003
then exactly one of the following holds.

(a) \( G \) is super eulerian.
(b) Equality holds in (1), and \( G \) has a complete subgraph \( H \) of order \( n - 4 \) such that \( G/H = K_{2,3} \).
(c) \( G \) is either \( K_{2,5} \) or the cube minus a vertex.

For 3-edge-connected graphs, Catlin and Chen proved a similar result, which was conjectured by Cai [4].

**Theorem 2** (Catlin and Chen, [8]). Let \( G \) be a 3-edge-connected simple graph of order \( n \). If \( |E(G)| \geq \left( \frac{n-9}{2} \right) + 16 \), then \( G \) is super eulerian.

A graph \( G \) is called \( k \)-supereulerian if \( G \) has a spanning even subgraph with at most \( k \) components. Hence, a \( k \)-supereulerian graph is also \( (k+1) \)-supereulerian, but not vice versa. Let \( k_1, k_2, k_3 \) be three positive integers, \( u, v \) the vertices of \( K_{2,3} \) with degree 3, and \( K_{2,3}(k_1, k_2, k_3) \) the graph obtained from \( K_{2,3} \) by replacing each \( u-v \) path by a path of length \( k_i + 1 \), as shown in Fig. 1. By definition, \( K_{2,3}(1, 1, 1) = K_{2,3} \), and \( K_{2,3}(k_1, k_2, k_3) \) is \((\min(k_1, k_2, k_3) + 1)\)-supereulerian, but not \((\min(k_1, k_2, k_3))\)-supereulerian.

Motivated by the two results above, we investigate the extremal size of \( k \)-supereulerian graphs, and obtain the following result.

**Theorem 3.** Let \( k > 1 \) be an integer, and \( G \) a 2-edge-connected simple graph of order \( n > 5k + 2 \).

\[
|E(G)| \geq \left( \frac{n-3k+1}{2} \right) + 3k + 3, \tag{2}
\]

then exactly one of the following holds.

(a) \( G \) is \( k \)-supereulerian.
(b) Equality holds in (2), and \( G \) has a complete subgraph \( H \) of order \( n - 3k + 1 \) such that \( G/H = K_{2,3}(k, k, k) \), where \( K_{2,3}(k, k, k) \) is depicted in Fig. 1 when \( k_1 = k_2 = k_3 = k \).

A graph \( H \) is collapsible if, for every subset \( X \subseteq V(H) \) with \( |X| \equiv 0 \) (mod 2), \( H \) has a spanning connected subgraph \( H_X \) with \( O(H_X) = X \). In [5], Catlin showed that any graph \( G \) has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs \( H_1, H_2, \ldots, H_c \) such that \( \bigcup_{i=1}^c V(H_i) = V(G) \). The reduction of \( G \), denoted by \( G' \), is the graph obtained from \( G \) by contracting each \( H_i \) (\( 1 \leq i \leq c \)) to a single vertex. A graph \( G \) is reduced if \( G = G' \). The following result is key in the proof of **Theorem 3**.

**Theorem 4.** Let \( G \) be a 2-edge-connected reduced graph of order \( n \), and \( k \) a positive integer such that \( n \leq 3k+2 \). Then \( G \) is either \( k \)-supereulerian or isomorphic to the graph \( K_{2,3}(k, k, k) \).

**Theorem 4** is indeed a conjecture in [19], which is equivalent to saying that every 2-edge-connected loopless graph \( G \) of order \( n \) is either \([(n - 2)/3]\)-supereulerian or \( n - 2 \equiv 0 \) (mod 3), and \( G \cong K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3}) \); see **Theorem 20** and **Proposition 21** for details. In [19], Niu and Xiong proved a similar result, stating that every 2-edge-connected reduced graph \( G \) of order \( n \leq 3k + 1 \leq 10 \) is \( k \)-supereulerian, which was proved by analyzing the structure of \( G \) according to the different values of the circumference of \( G \), and then by showing that \( G \) has a spanning even subgraph with at most \( k \) components. This proof technique fails when \( n \) is large, as the number of possible cases grows very quickly, and the structure of \( G \) becomes much more complicated. In this paper, we use a completely different approach, which utilizes the splitting lemma of Fleischner [12] and a result on perfect matchings in cubic graphs of Edmonds [11], to prove **Theorem 4**.

By a smallest graph in some collection of graphs we mean a graph with the least order, and with the least size amongst all graphs of that order in the collection. As an example, \( K_{2,3} \) is the smallest 2-edge-connected non-supereulerian graph. As an extension, our result above implies that \( K_{2,3}(k, k, k) \) is the smallest 2-edge-connected non-\( k \)-supereulerian graph.

In Section 2, we will assume the validity of **Theorem 4** to prove **Theorem 3**, and present some other applications of **Theorem 4**, whose proof will be postponed to Section 3.
2. Applications of Theorem 4

2.1. Proof of Theorem 3

In this subsection, we use Theorem 4 to prove Theorem 3. First, we present some necessary results.

**Theorem 5** (Catlin, [5]). If $G$ is reduced, then $G$ is simple and triangle free, and with either $G \in \{K_1, K_2\}$ or $|E(G)| \leq 2|V(G)| - 4$. Catlin [5] proved that a connected graph $G$ is supereulerian if and only if its reduction $G'$ is supereulerian. Niu et al. extended this result to $k$-supereulerian graphs.

**Theorem 6** (Niu, Lai and Xiong, [18]). Let $G$ be a connected graph, and $G'$ the reduction of $G$. Then $G$ is $k$-supereulerian if and only if $G'$ is $k$-supereulerian.

Let $F(G)$ denote the minimum number of edges that must be added to $G$ in order to obtain a supergraph that has two edge-disjoint spanning trees. Catlin [6] showed that, if $G$ is reduced, then

$$F(G) = 2|V(G)| - |E(G)| - 2.$$  \hspace{1cm} (3)

**Corollary 7** (Niu, Lai and Xiong, [18]). Let $G$ be a 2-edge-connected graph. If $F(G) \leq k$, then $G$ is $k$-supereulerian.

**Theorem 8** (Catlin and Chen, [8]). Let $G$ be a 2-edge-connected simple graph of order $n$, and let $p > 1$ be an integer. If

$$|E(G)| \geq \frac{n - p + 1}{2} + 2p - 4,$$ \hspace{1cm} (4)

then one of the following holds.

(a) The reduction of $G$ has order less then $p$.

(b) Equality holds in (4), $G$ has a complete subgraph $H$ of order $n - p + 1$, and the reduction of $G$ is $G' = G/H$, a graph of order $p$ and size $2p - 4$.

(c) $G$ is a reduced graph such that either $|E(G)| \in \{2n - 4, 2n - 5\}$ and $n \in \{p + 1, p + 2\}$, or $|E(G)| = 2n - 4$ and $n = p + 3$.

Now, we prove Theorem 3.

**Proof of Theorem 3.** We need to discuss the following two cases by considering the size of $G$. Let $G'$ be the reduction of $G$.

**Case 1.** $|E(G)| \geq \left(\frac{n - 3k - 1}{2}\right) + 6k$.

Let $p = 3k + 2$. Then $n - p + 1 = n - 3k - 1$ and $2p - 4 = 6k$. Hence, (4) holds. In the following, we check the three cases of Theorem 8, and show that $G$ is $k$-supereulerian in each case.

If (a) of Theorem 8 holds, then $|V(G')| < 3k + 2$. Note that $|V(K_{2,3}(k, k, k))| = 3k + 2$. By Theorem 4, $G'$ is $k$-supereulerian. Then $G$ is $k$-supereulerian by Theorem 6.

If (b) of Theorem 8 holds, then $|E(G)| = \left(\frac{n - 3k - 1}{2}\right) + 6k$. There exists a complete subgraph $H$ of $G$ with $|V(H)| = n - 3k - 1$, and $G' = G/H$. That is to say, $|V(G')| = 3k + 2$, and $|E(G')| = 6k$. Note that $|E(K_{2,3}(k, k, k))| = 3k + 3 < 6k$. By Theorem 4, $G'$ is $k$-supereulerian. Then $G$ is $k$-supereulerian by Theorem 6.

If (c) of Theorem 8 holds, then $G = G'$, $|E(G)| \in \{2n - 4, 2n - 5\}$, and $n \in \{p + 1, p + 2, p + 3\}$. Hence, by (3), $F(G) \in \{2, 3\}$. If $F(G) \leq k$, then, by Corollary 7, $G$ is $k$-supereulerian. So we need to consider the remaining case when $k = 2$ and $F(G) = 3$. Hence, $p = 8$, and then $n \in \{9, 10, 11\}$, contrary to $n > 5k + 2 = 12$.

**Case 2.** $\left(\frac{n - 3k - 1}{2}\right) + 3k + 3 \leq |E(G)| \leq \left(\frac{n - 3k - 1}{2}\right) + 6k - 1$.

As $K_1$ is supereulerian, we may assume that $G'$ is 2-edge-connected and that $|V(G')| \geq 2$.

By (3), $F(G') = 2|V(G')| - |E(G')| - 2$. If $F(G') \leq k$, then, by Corollary 7, $G'$ is $k$-supereulerian, and then $G$ is $k$-supereulerian by Theorem 6. Hence, it suffices to consider $F(G') \geq k + 1$ in the following.

Let $e = |E(G)|$, $n' = |V(G')|$, and $e' = |E(G')|$. Then $\left(\frac{n - 3k - 1}{2}\right) + 3k + 3 \leq e \leq \left(\frac{n - 3k - 1}{2}\right) + 6k - 1$. For any graph $H$, we use $e(H)$ to denote $|E(H)|$. Suppose that $H_1, H_2, \ldots, H_m$ are all the maximal collapsible subgraphs of $G$ such that $G'$ is obtained from $G$ by contracting $H_1, H_2, \ldots, H_m$. Assume that $n_i = |V(H_i)|$ for each $i \in \{1, 2, \ldots, m\}$. Since contracting an induced subgraph $H$ does not affect the validity of $e = e(H) + e(G/H)$, and since all maximal collapsible subgraphs are induced, we can contract $H_1, H_2, \ldots, H_m$ in succession, and then

$$e = e' + e(H_1) + e(H_2) + \cdots + e(H_m)$$

$$\leq e' + \left(\frac{n_1}{2}\right) + \left(\frac{n_2}{2}\right) + \cdots + \left(\frac{n_m}{2}\right)$$
and
\[ n = n' + (n_1 - 1) + (n_2 - 1) + \cdots + (n_m - 1), \]
i.e.,
\[ n + m - n' = n_1 + n_2 + \cdots + n_m. \]
Since \( F(G') \geq k + 1 \), by (3), we have \( 2n' - e' - 2 \geq k + 1 \), i.e., \( e' \leq 2n' - k - 3 \). So
\[ e \leq e' + \left( \frac{n_1}{2} \right) + \left( \frac{n_2}{2} \right) + \cdots + \left( \frac{n_m}{2} \right) \]
\[ \leq 2n' - k - 3 + \left( \frac{n_1}{2} \right) + \left( \frac{n_2}{2} \right) + \cdots + \left( \frac{n_m}{2} \right). \]
Now, we define a function
\[ f(n_1, n_2, \ldots, n_m) = 2n' - k - 3 + \left( \frac{n_1}{2} \right) + \left( \frac{n_2}{2} \right) + \cdots + \left( \frac{n_m}{2} \right) \]
\[ = 2n' - k - 3 + \frac{1}{2}(n_1 - n_1) + \frac{1}{2}(n_2 - n_2) + \cdots + \frac{1}{2}(n_m^2 - n_m) \]
subject to \( n_1 + n_2 + \cdots + n_m = n + m - n' \). By convexity, \( f(n_1, n_2, \ldots, n_m) \) reaches its maximum value when \( m = 1 \), i.e.,
\( n_1 = n + 1 - n' \) and \( n_2 = n_3 = \cdots = n_m = 0 \). So \( e \leq 2n' - k - 3 + \left( \frac{n + 1 - n'}{2} \right). \)
If \( G \) is reduced, then \( e = e' \) and \( n = n' \). Since \( e' \leq 2n' - k - 3 \) and \( k > 1 \), we have \( e \leq 2n - 5 \), contrary to (2) when \( n > 5k + 2 \). Hence, \( G \) has at least one nontrivial collapsible subgraph. Note that \( K_1 \) is the nontrivial collapsible simple graph with the smallest order. We have \( n' \leq n - 2 \). Define a new function
\[ g(n') = 2n' - k - 3 + \left( \frac{n + 1 - n'}{2} \right) \]
\[ = \frac{1}{2}n'^2 + \left( \frac{3}{2} - n \right)n' + \left( \frac{1}{2}n^2 + \frac{1}{2}n - k - 3 \right). \]
The symmetric axis of this parabolic function \( g(n') \) is \( n' = n - 3/2 \). Then \( g(n') \) is decreasing when \( n' \leq n - 3/2 \).
By the definitions of functions \( f \) and \( g(n') \), \( g(n') \) is always an upper bound of \( e \). If \( n' = 3k + 3 \), then
\[ g(3k + 3) = \frac{1}{2}n'^2 - \frac{6k + 5}{2}n + \frac{9k^2 + 25k + 12}{2} \]
\[ = \frac{1}{2}n'^2 - \frac{6k + 3}{2}n + \frac{9k^2 + 15k + 8}{2} - n + 5k + 2 \]
\[ = \left( \frac{n - 3k - 1}{2} \right) + 3k + 3 - (n - 5k - 2) \]
\[ < e, \]
when \( n > 5k + 2 \), contrary to \( e \leq g(n') \).
As \( n' \leq n - 2 \), \( g(n') \) is decreasing. Hence, we have \( n' \leq 3k + 2 \). By Theorem 4, \( G' \) is either \( k \)-supereulerian or the graph \( K_{2,3}(k, k, k) \). In the former case, \( G \) is \( k \)-supereulerian by Theorem 6, so (a) of Theorem 3 holds. In the latter case, \( n' = 3k + 2 \),
\( e' = 3k + 3 \), and then \( e \leq e' + \left( \frac{n - n' + 1}{2} \right) = 3k + 3 + \left( \frac{n - 3k - 1}{2} \right) \). By (2), we have \( e = 3k + 3 + \left( \frac{n - 3k - 1}{2} \right) \), which implies that \( G \) has a complete subgraph \( H \) of order \( n - 3k - 1 \) such that \( G/H = K_{2,3}(k, k, k) \). Hence, (b) of Theorem 3 holds.
This completes the proof of Theorem 3. \( \square \)

2.2. The number of components of an even factor

An even factor of \( G \) is a spanning subgraph of \( G \) in which every vertex has a positive even degree. A 2-factor of \( G \) is a spanning subgraph in which every vertex has degree 2. In this subsection, we use Theorem 4 to prove some sufficient conditions for even factors of a graph and 2-factors of its line graph.

Note that a graph is \( k \)-supereulerian if it has a spanning even subgraph with at most \( k \) components. If \( G \) has an even factor with at most \( k \) components, then \( G \) is \( k \)-supereulerian, whereas the converse is not true in general; see [18].

There exist many minimum degree conditions guaranteeing the existence of certain factors of a graph, such as Hamiltonian cycles and spanning Eulerian subgraphs; see, e.g., [5,7,10]. In [19], Niu and Xiong obtained several minimum degree conditions for a graph to have an even factor with a bounded number of components, one of which is the following.
Theorem 9 (Niü and Xiong, [19]). Let $G$ be a 2-edge-connected simple graph of order $n$, and $k \in \{1, 2, 3\}$ such that $\delta(G) \geq \left\lfloor \frac{n}{3k+2} \right\rfloor - 1$. If $n$ is sufficiently large relative to $k$, then $G$ has an even factor with at most $k$ components.

We extend this result to general cases, and give a bit weaker minimum degree condition, with only one exception.

Theorem 10. Let $G$ be a 2-edge-connected simple graph of order $n$, and $k$ a positive integer such that $\delta(G) \geq \left\lfloor \frac{n}{3k+2} \right\rfloor - 1$. If $n$ is sufficiently large relative to $k$, then exactly one of the following holds.

(a) $G$ has an even factor with at most $k$ components.
(b) $G'$, the reduction of $G$, is $K_{2,3}(k, k, k)$, and $G$ has an even factor with exactly $k + 1$ components.

We first present a necessary result for our proof.

Theorem 11 (Niü and Xiong, [19]). Let $p$ be a positive integer, and $G$ a connected simple graph of order $n$ such that

$$\delta(G) \geq \left\lfloor \frac{n}{p} \right\rfloor - 1.$$  \hspace{1cm} (5)

If $n$ is sufficiently large relative to $p$, then the reduction $G'$ of $G$ satisfies $|V(G')| \leq p$, and each vertex of $G'$ is nontrivial.

Now, we prove Theorem 10.

Proof of Theorem 10. By Theorem 11, $|V(G')| \leq 3k + 2$, and each vertex of $G'$ is nontrivial. Then, by Theorem 4, $G'$ is either $k$-supereulerian or the graph $K_{2,3}(k, k, k)$. In the former case, $G'$ has a spanning even subgraph with at most $k$ components $L_1, L_2, \ldots, L_i$, where $i \leq k$. For each $L_i$, let $L_i^* = G(\bigcup_{x \in V(L_i)} V(H_x))$, where $H_x$ is the preimage of $v \in V(L_i)$. Since each vertex of $G'$ is nontrivial, then, by Theorem 6, each $L_i^*$ is supereulerian and nontrivial. By the definitions of collapsible graphs and contraction, $\bigcup_{1 \leq i \leq k} V(L_i^*) = V(G)$ and $V(L_i^*) \cap V(L_j^*) = \emptyset$ for $i \neq j$. Hence, $G$ has an even factor with $k (\leq k)$ components, so (a) of Theorem 10 holds. In the latter case, $G'$ is $(k + 1)$-supereulerian. Then, by arguing similarly as the above case, $G$ has an even factor with exactly $k + 1$ components, so (b) holds. \text{□}

By Theorem 10, we obtain the following corollary immediately, which extends a theorem (Theorem 9 in [5]) of Catlin and improves a theorem (Theorem 8 in [18]) of Niü et al.

Corollary 12. Let $G$ be a 2-edge-connected simple graph of order $n$, and $k$ a positive integer such that $\delta(G) \geq \left\lfloor \frac{n}{3k+2} \right\rfloor - 1$. If $n$ is sufficiently large relative to $k$, then exactly one of the following holds.

(a) $G$ is $k$-supereulerian.
(b) $G'$, the reduction of $G$, is $K_{2,3}(k, k, k)$.

Let $G = (V(G), E(G))$ be a graph. The line graph $L(G)$ of $G$ is the graph with $V(L(G)) = E(G)$, and $x, y \in V(L(G))$ are adjacent as vertices if and only if they are adjacent as edges in $G$. Let $G$ be a simple graph with $\delta(G) \geq 3$, and let $S$ be a set of mutually edge-disjoint even nontrivial subgraphs and stars ($K_{1,s}$, where $s \geq 3$ is an integer). If each star has at least three edges, and every edge in $E(G) \setminus \bigcup_{x \in L(G)} E(L)$ is incident to an even subgraph in $S$, then $S$ is called a system that dominates $G$.

Theorem 13 (Gould and Hynds, [13]). Let $G$ be a simple graph. Then $L(G)$ has a 2-factor with $c$ components if and only if there is a system that dominates $G$ with $c$ elements.

Theorem 13 shows a close relationship between a system that dominates $G$ with $c$ elements and a 2-factor of $L(G)$ with the same number of components. From Theorems 10 and 13, one can easily obtain the following result.

Corollary 14. Let $G$ be a 2-edge-connected simple graph of order $n$, $L(G)$ the line graph of $G$, and $k$ a positive integer such that $\delta(G) \geq \left\lfloor \frac{n}{3k+2} \right\rfloor - 1$. If $n$ is sufficiently large relative to $k$, then exactly one of the following holds.

(a) $L(G)$ has a 2-factor with at most $k$ components.
(b) $G'$, the reduction of $G$, is $K_{2,3}(k, k, k)$, and $L(G)$ has a 2-factor with exactly $k + 1$ components.

2.3. A sufficient condition for $k$-supereulerian graphs

A bond of $G$ is a minimal nonempty edge cut. Let $l > 0, m \geq 0$ be integers, and let $C(l, m)$ denote the graph family such that a graph $G$ of order $n$ is in $C(l, m)$ if and only if $G$ is 2-edge-connected and such that, for every bond $S \subseteq E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $(n - m)/l$.

Catlin and Li [9] were the first to investigate characterizations of supereulerian graphs in $C_{3}(m, l)$. They proved that a graph $G \in C_{2}(5, 0)$ is supereulerian if and only if $G$ is not contractible to $K_{2,3}$. Since then, a series of characterizations of supereulerian graphs in $C_{3}(m, l)$ has been done; see [3,15–17]. In [20], Niü and Xiong considered a similar problem on $k$-supereulerian graphs, and proved the following theorem.
Theorem 15 (Niu and Xiong, [20]). Let $6 \leq l \leq 10$ be an integer, and $G \in C_2(l, 0)$ be a graph of order $n$. Then $G$ is $(l - 4)$-supereulerian.

In this subsection, we extend this result to general cases.

**Theorem 16.** Let $l \geq 6$ be an integer, and $G \in C_2(l, 0)$ be a graph of order $n$. Then $G$ is $(l - 4)$-supereulerian.

Let $D_i(G) = \{v \in V(G) \mid d(v) = i\}$ and $d_i(G) = |D_i(G)|$.

**Theorem 17** (Catlin, [5]). If $G$ is a nontrivial 2-edge-connected reduced graph, then $d_2(G) + d_3(G) \geq 4$. If $d_2(G) + d_3(G) = 4$, then $G$ is Eulerian, and $G$ has four vertices of degree 2.

**Lemma 18** (Niu and Xiong, [20]). Let $G \in C_2(l, m)$ be a graph with $n = |V(G)| > (l + 1)m$. Then either $G' = K_1$ or $d_2(G') + d_3(G') \leq l$, where $G'$ is the reduction of $G$.

**Lemma 19** (Niu and Xiong, [20]). Let $G$ be a 2-edge-connected reduced graph, and $d_i = d_i(G)$. Then

$$2F(G) + 4 + \sum_{j \geq 5} (j - 4)d_j = 2d_2 + d_3.$$

Now, we prove Theorem 16.

**Proof of Theorem 16.** By Theorem 15, we may assume that $l \geq 11$. Let $G'$ be the reduction of $G$. By Theorem 6, it suffices to show that $G'$ is $(l - 4)$-supereulerian. Since $K_1$ is supereulerian, if $G' = K_1$, then we are done. So we may assume that $G'$ is 2-edge-connected and nontrivial. Let $d_i = |D_i(G')|$.

By Theorem 17, if $d_2 + d_3 = 4$, then $G'$ is Eulerian. By Lemma 18, $d_2 + d_3 \leq l$. Therefore, we only consider the case when $5 \leq d_2 + d_3 \leq l$. We shall assume that $G'$ is not $(l - 4)$-supereulerian,

$$2F(G') + 4 + \sum_{j \geq 5} (j - 4)d_j = 2d_2 + d_3,$$

(6)

to find a contradiction.

**Case 1.** $5 \leq d_2 + d_3 \leq l - 1$.

If $F(G') \leq l - 4$, by Corollary 7, $G'$ is $(l - 4)$-supereulerian, contrary to (6). So we may assume that $F(G') \geq l - 3$. From Lemma 19, and since $d_2 + d_3 \leq l - 1$, we have

$$2(l - 1) + \sum_{j \geq 5} (j - 4)d_j \leq 2F(G') + 4 + \sum_{j \geq 5} (j - 4)d_j = 2d_2 + d_3 \leq 2(d_2 + d_3) \leq 2(l - 1).$$

Hence, inequalities must hold everywhere, implying that $d_2 = l - 1$, $d_3 = 0$, and $d_j = 0$ ($j \geq 5$). Thus $G'$ is Eulerian, contrary to (6).

**Case 2.** $d_2 + d_3 = l$.

Let $H_1, H_2, \ldots, H_l$ denote the subgraphs of $G$ whose contraction images in $G'$ are the vertices of degree at most 3 in $G'$. Since $G \in C_2(l, 0)$, for each $i$ with $1 \leq i \leq l$, $|V(H_i)| \geq n/l$. It follows that

$$n = |V(G)| \geq \sum_{i=1}^{l} |V(H_i)| \geq \frac{ln}{l} = n,$$

and hence $|V(G')| = l$. Denote $l = 3k + j$, where $j \in \{0, 1, 2\}$. By Theorem 4, $G'$ is either $k$-supereulerian or the graph $K_{3,3}(k, k, k)$, which is $(k + 1)$-supereulerian. Since $l \geq 11$, we have $k < k + 1 \leq l - 4$, and then $G'$ is $(l - 4)$-supereulerian, contrary to (6).

This completes the proof of Theorem 16. □

3. **Proof of Theorem 4**

In this section, for presentational convenience, we shall show the validity of Theorem 4 by proving the following equivalence form.

**Theorem 20.** Let $G$ be a 2-edge-connected graph of order $n \geq 3$. Then exactly one of the following holds.

(a) $G$ is $\lceil \frac{n - 2}{3} \rceil$-supereulerian.

(b) $n - 2 \equiv 0 \pmod{3}$, and $G \cong K_{3,3}(\frac{n - 2}{3}, \frac{n - 2}{3}, \frac{n - 2}{3})$.

**Proposition 21.** Theorem 4 is equivalent to Theorem 20.
Proof. First, we show that Theorem 20 implies Theorem 4. Let \( G \) be a graph of order \( n \) satisfying the hypotheses of Theorem 4. If \( n < 3 \), then, since \( G \) is a 2-edge-connected reduced graph, we have \( G \cong K_1 \), which is supereulerian. Hence, we may assume that \( n \geq 3 \). By Theorem 20, \( G \) is either \( \lceil \frac{n-2}{3} \rceil \)-supereulerian or the graph \( K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3}) \). Note that \( n \leq 3k+2 \). In the former case, \( G \) is \( k \)-supereulerian since \( \lceil \frac{n-2}{3} \rceil \leq k \) and by the definition of \( k \)-supereulerian graphs. In the latter case, we have \( n - 2 \equiv 0 \pmod 3 \). If \( \frac{n-2}{3} < k \), then \( G \) is \( k \)-supereulerian; else \( \frac{n-2}{3} = k \), i.e., \( G \cong K_{2,3}(k, k, k) \). So Theorem 4 holds.

Conversely, let \( G \) be a graph satisfying the hypotheses of Theorem 20, let \( n = 3k + j \), where \( k \) is a positive integer and \( j \in \{0, 1, 2\} \), and let \( G' \) be the reduction of \( G \). Then \( n(G') \leq n = 3k + j \leq 3k + 2 \). By Theorem 4, \( G' \) is either \( k \)-supereulerian or the graph \( K_{2,3}(k, k, k) \). In the former case, \( G \) is \( \lceil \frac{n-2}{3} \rceil \)-supereulerian by the fact that \( \lceil \frac{n-2}{3} \rceil = k \) and by Theorem 6. In the latter case, we have \( n(G') = n = 3k + 2 \), and then \( \frac{n-2}{3} = k \). Theorem 20 holds. \( \square \)

Before proving Theorem 20, we present several auxiliary results.

Let \( v \) be a vertex of a graph \( G \), and let \( e_1 = vu_1 \) and \( e_2 = vu_2 \) be two edges of \( G \) incident to \( v \). The operation of splitting off the edges \( e_1 \) and \( e_2 \) from \( v \) consists of deleting \( e_1 \) and \( e_2 \) and then adding a new edge \( e \) joining \( v_1 \) and \( v_2 \), depicted in Fig. 2. The following theorem, due to Fleischner, shows that under certain conditions this operation can be performed without creating cut edges.

**Theorem 22** (Fleischner, [12]). Let \( G \) be a 2-connected graph, and \( v \) a vertex of \( G \) of degree at least four with at least two distinct neighbors. Then some two non-multiple edges incident to \( v \) can be split off so that the resulting graph is connected and has no cut edges.

For \( S \subseteq V(G) \) and \( E \subseteq E(G) \), let \( G - S \) and \( G - E \) denote the subgraph obtained from \( G \) by deleting all the vertices in \( S \) and the subgraph obtained from \( G \) by deleting all the edges in \( E \), respectively. For \( H \subseteq G \), we denote \( G - V(H) \) by \( G - H \), for abbreviation. For \( e = uv \notin E(G) \) with \( u, v \in V(G) \), let \( G + e \) denote the graph obtained by adding \( e \) to \( G \). We present a lemma and a theorem of Edmonds, which are used in the proof of Theorem 20.

**Lemma 23.** Let \( G \) be a 2-edge-connected graph, \( v \) a vertex of \( G \), and \( e \) an edge of \( G \).

(a) If \( G^* \) is a graph obtained from \( G \) by splitting off two edges incident to \( v \), and \( G^* \cong K_{2,3}(k, k, k) \), then \( G \) is \( k \)-supereulerian.
(b) If \( G^* = G - e \) and \( G^* \cong K_{2,3}(k, k, k) \), then \( G \) is \( k \)-supereulerian.

**Proof.** (a) Note that \( G^*(\cong K_{2,3}(k, k, k)) \) is \((k + 1)\)-supereulerian. It is easy to check that the number of supereulerian components of all the graphs obtained from \( G^* \) by deleting any edge \( u_1u_2 \) and adding two edges \( u_1u_3 \) and \( u_2u_3 \), where \( u \in V(G^*) \setminus \{u_1, u_2\} \) (this procedure can be looked upon as the reverse of splitting off two adjacent edges), will reduce by at least 1. Hence, \( G \) is \( k \)-supereulerian.

(b) Note that adding a new edge to \( G^* \) will reduce at least one supereulerian component. \( G \) is \( k \)-supereulerian. \( \square \)

A graph is called \( k \)-regular if all vertices have degree \( k \). A perfect matching in a graph is a spanning 1-regular subgraph.

**Theorem 24** (Edmonds, [11]). For every 2-edge-connected 3-regular graph, there exists a constant \( p \) and 3\( p \) perfect matchings such that each edge is in \( p \) of them.

For a path \( P = x_0x_1 \ldots x_{k-1}x_k \), the vertices \( x_1, \ldots, x_{k-1} \) are called the internal vertices of \( P \). Let \( \hat{P} = x_1 \ldots x_{k-1} \) be the subpath of \( P \) induced by its internal vertices. In the following, let \( n_c(G) \) denote the number of components of \( G \).

Now, we prove Theorem 20.

**Proof of Theorem 20.** We shall assume that Theorem 20 does not hold, to find a contradiction. Let \( G \) be a counterexample of Theorem 20 with \( |E(G)| \) minimized.

First, we prove the following two claims.

**Claim 1.** \( G \) is 2-connected.
Proof of Claim 1. Suppose, to the contrary, that $G$ has a cut vertex $u$. Let $H$ be a component of $G - u$, $G_1 = G[V(H) \cup \{u\}]$, $n_1 = |V(G_1)|$, and $G_2 = G - V(H)$, $n_2 = |V(G_2)|$, depicted in Fig. 3. Then $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \{u\}$, $n = n_1 + n_2 - 1$, both $G_1$ and $G_2$ are 2-edge-connected.

For $i = 1, 2$, by the 2-edge-connectivity of $G$, we have $n_i \geq 3$. Since $|E(G_i)| < |E(G)|$ and by the minimality of $G$, either $G_i$ is $\lceil \frac{n_i - 2}{3} \rceil$-supereulerian or $n_i - 2 \equiv 0 \pmod{3}$, and $G_i \cong K_{2,3}(\frac{n_i - 2}{3}, \frac{n_i - 2}{3}, \frac{n_i - 2}{3})$. Now, we distinguish the following three cases.

Case 1. For $i = 1, 2$, $G_i$ is $\lceil \frac{n_i - 2}{3} \rceil$-supereulerian.

Denote $n_i = 3k_i + j_i$, where $j_i \in \{0, 1, 2\}$. Then $\lceil \frac{n_i - 2}{3} \rceil = k_i$, and hence $G_i$ is $k_i$-supereulerian. Note that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \{u\}$, $G$ is $(k_1 + k_2 - 1)$-supereulerian. Since

$$k_1 + k_2 - 1 = \frac{3k_1 + 3k_2 - 1 - 2}{3} \leq \frac{3k_1 + j_1 + 3k_2 + j_2 - 1 - 2}{3} = \frac{n - 2}{3} \leq \left\lfloor \frac{n - 2}{3} \right\rfloor,$$

$G$ is $\lceil \frac{n - 2}{3} \rceil$-supereulerian, a contradiction.

Case 2. Exactly one of $G_i$ ($i = 1, 2$) $(G_1$, say) is $\lceil \frac{n_i - 2}{3} \rceil$-supereulerian.

Denote $n_1 = 3k_1 + j$, where $j \in \{0, 1, 2\}$, and $n_2 = 3k_2 + 2$. Then $\lceil \frac{n_1 - 2}{3} \rceil = k_1$ and $\frac{n_2 - 2}{3} = k_2$, and hence $G_1$ is $k_1$-supereulerian, and $G_2$ is $(k_1 + 1)$-supereulerian. Thus, $G$ is $(k_1 + k_2)$-supereulerian. Since

$$k_1 + k_2 = \left\lceil \frac{3k_1 + 3k_2 + 2 - 1 - 2}{3} \right\rceil \leq \left\lceil \frac{3k_1 + j + 3k_2 + 2 - 1 - 2}{3} \right\rceil = \left\lceil \frac{n - 2}{3} \right\rceil,$$

$G$ is $\lceil \frac{n - 2}{3} \rceil$-supereulerian, a contradiction.

Case 3. For $i = 1, 2$, $n_i \equiv 0 \pmod{3}$, and $G_i \cong K_{2,3}(\frac{n_i - 2}{3}, \frac{n_i - 2}{3}, \frac{n_i - 2}{3})$.

Denote $n_i = 3k_i + 2$. Then $\frac{n_i - 2}{3} = k_i$, and hence $G_i$ is $(k_i + 1)$-supereulerian. Thus, $G$ is $(k_1 + k_2 + 1)$-supereulerian. Since

$$k_1 + k_2 + 1 = \left\lceil \frac{3k_1 + 2 + 3k_2 + 2 - 1 - 2}{3} \right\rceil = \left\lceil \frac{n - 2}{3} \right\rceil,$$

$G$ is $\lceil \frac{n - 2}{3} \rceil$-supereulerian, a contradiction.

This completes the proof of Claim 1. \hfill \Box

Claim 2. $\Delta(G) \leq 3$.

Proof of Claim 2. Suppose, to the contrary, that $\Delta(G) \geq 4$. Let $v$ be a vertex of $G$ with degree at least 4. By Claim 1, $G$ is 2-connected. Hence, by Theorem 22, $G$ contains two edges $vu_1$ and $vu_2$ incident to $v$ that can be split off such that the resulting graph, denoted by $G^*$ (i.e., $G^* = G - \{vu_1, vu_2\} + \{v_1v_2\}$), is connected and has no cut edges. Then $|V(G^*)| = |V(G)| = n$ and $|E(G^*)| = |E(G)| - 1 < |E(G)|$. By the minimality of $G$, we can obtain that $G^*$ is either $\lceil \frac{n - 2}{3} \rceil$-supereulerian or the graph $K_{2,3}(\frac{n - 2}{3}, \frac{n - 2}{3}, \frac{n - 2}{3})$.

First, suppose that $G^*$ is $\lceil \frac{n_i - 2}{3} \rceil$-supereulerian, i.e., $G^*$ has a spanning even subgraph $L^*$ with $n_e(L^*) \leq \lceil \frac{n_i - 2}{3} \rceil$. Then $v_1v_2 \in E(L^*)$; otherwise, $L^*$ is also a spanning even subgraph of $G$, and then $G$ is $\lceil \frac{n - 2}{3} \rceil$-supereulerian, a contradiction. Let $L_1^* \subset L^*$ be the component containing $v_1v_2$, $L_2^* \subset L^*$ the component containing $v$, and let

$$L = \begin{cases} (L^* - L_1^*) \cup (L_1^* - \{v_1v_2\}) \cup L_2^* \cup \{v_1v, v_2v\}, & \text{if } L_1^* \neq L_2^*; \\ (L^* - L_2^*) \cup (L_2^* - \{v_1v_2\}) \cup \{v_1v, v_2v\}, & \text{if } L_1^* = L_2^*. \end{cases}$$

Then $n_e(L) \leq n_e(L^*)$. Hence, $G$ has a spanning even subgraph $L$ with at most $\lceil \frac{n - 2}{3} \rceil$ components, i.e., $G$ is $\lceil \frac{n - 2}{3} \rceil$-supereulerian, a contradiction.

Next, suppose that $G^* \cong K_{2,3}(\frac{n_i - 2}{3}, \frac{n_i - 2}{3}, \frac{n_i - 2}{3})$. Then, by (a) of Lemma 23, $G$ is $\lceil \frac{n - 2}{3} \rceil$-supereulerian, a contradiction.

This completes the proof of Claim 2. \hfill \Box

Fig. 3. The subgraphs $G_1$ and $G_2$ of $G$. 
Claim 3.1. Both $P_{1}$ and $P_{2}$ have internal vertices in $G$. 

Proof of Claim 3.1. Suppose, to the contrary, that $P_{i}$ has no internal vertex. Denote $P_{i} = e = uv$ and $G_{1} = G - e$. Then, we claim that $G_{1}$ is 2-edge-connected. By way of contradiction, suppose that $G_{1}$ contains a cut edge $e'$. If $u$ and $v$ belong to the same component of $G_{1} - e'$, then $e'$ is also a cut edge of $G_{1}$, a contradiction; if $u$ and $v$ belong to two distinct components of $G_{1} - e'$, then $u$ is a cut vertex of $G_{1}$, contrary to Claim 1.

Hence, $G_{1}$ is 2-edge-connected. Note that $|V(G_{1})| = |V(G)| = n$ and $|E(G_{1})| = |E(G)| - 1 < |E(G)|$. By the minimality of $G$, either $G_{1}$ is $\lceil \frac{n-2}{3} \rceil$-superuniversal, and hence $G_{1}$ is $\lceil \frac{n-1}{3} \rceil$-superuniversal, a contradiction; or $G_{1} \cong K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3})$, and hence $G$ is $\lceil \frac{n-2}{3} \rceil$-superuniversal by (b) of Lemma 23, a contradiction. □

By Claim 3.1, for $i = 1, 2$, we may assume that $x_{i} \in V(P_{i})$ such that $ux_{i} \in E(G)$, i.e., $x_{i}$ is the neighbor of $u$ in $P_{i}$. To finish the proof of Claim 3, it suffices to consider the following two cases.

Case 1. $P_{3}$ has internal vertices. 

Let $x_{3} \in V(P_{3})$ such that $ux_{3} \in E(G)$, $G* = G/\{ux_{1}, ux_{2}, ux_{3}\}$, $P_{*} = P_{i}/\{ux_{i}\}$ the path in $G^{*}$ ($i = 1, 2, 3$), and $u^{*}$ the resulting vertex (of degree 3) obtained by contracting $\{ux_{1}, ux_{2}, ux_{3}\}$, depicted in Fig. 6. Then $n^{*} = |V(G^{*})| = n - 3$ and $|E(G^{*})| = |E(G)| - 3$. By the minimality of $G$, we can obtain that $G^{*}$ is either $\lceil \frac{n^{*}-2}{3} \rceil$-superuniversal or the graph $K_{2,3}(\frac{n^{*}-2}{3}, \frac{n^{*}-2}{3}, \frac{n^{*}-2}{3})$. The latter case does not hold; otherwise, $G \cong K_{2,3}(\frac{n^{*}-2}{3}, \frac{n^{*}-2}{3}, \frac{n^{*}-2}{3})$, a contradiction. So we need to consider the former case.

Let $L^{*}$ be a spanning even subgraph of $G^{*}$ with the least number of components. Then $n_{L^{*}} \leq \lceil \frac{n^{*}-2}{3} \rceil$. Let $L_{1}^{*}$ be the component of $L^{*}$ containing $u^{*}$. Then, we may assume that $L_{1}^{*}$ is nontrivial; otherwise, the vertices in $V(P_{1}^{*}) \cup V(P_{2}^{*})$ are all trivial in $L^{*}$, and we can replace these trivial components by $u^{*}P_{1}^{*}vP_{2}^{*}u^{*}$ to obtain a spanning even subgraph of $G^{*}$ with fewer components than $L^{*}$, contrary to the choice of $L^{*}$.
Since $L_1^*$ is nontrivial and $d_G(u^*) = 3$, we may assume that $P_i^*, P_j^* \subseteq L_1^*$, and that the internal vertices of $P_k^*$ are trivial components in $L^*$, where $(i, j, k) = (1, 2, 3)$. Then, let $L_1$ be the even subgraph of $G$ obtained from $L_1^*$ by replacing $P_i^*$ and $P_j^*$ by $P_i$ and $P_j$, respectively, and let $L = (L^* - L_1^*) \cup L_1 \cup \{x_1\}$. Then $L$ is a spanning even subgraph of $G$ with $n_2(L) = n(L^*) + 1 \leq \lceil \frac{n_2 - 2}{3} \rceil + 1 = \lceil \frac{n - 2}{3} \rceil$ since $n^* = n - 3$. Hence, $G$ is $\lceil \frac{n - 2}{3} \rceil$-supereularian, a contradiction.

**Case 2.** $P_3$ has no internal vertex.

Then, we can denote $P_3 = e_3 = uw$. Let $e_4, e_5$ be the two edges incident with $w$ excepting $e_3$, and let $P_4, P_5$ the maximal paths in $G$ corresponding to $e_4, e_5$, respectively. Let $G^* = G/\{x_1, x_2, e_3\}$, $P_i^* = P_i/\{x_1\}$ the path in $G^*$ corresponding to $P_i$ in $G$ $(i = 1, 2)$, and $u^*$ the resulting vertex (of degree 4) obtained by contracting $\{x_1, x_2, e_3\}$, depicted in Fig. 7. Then $n^* = n(G^*) = n - 3$ and $|E(G^*)| = |E(G)| - 3$. Since $d_G^*(u^*) = 4$ and $\Delta(K_{2,3}(k, k)) = 3$, and by the minimality of $G$, $G^*$ is $\lceil \frac{n - 2}{3} \rceil$-supereularian.

Let $L^*$ be a spanning even subgraph of $G^*$ with the least number of components. Then $n_2(L^*) \leq \lceil \frac{n - 2}{3} \rceil$. Let $L_1^*$ be the component of $L^*$ containing $u^*$. Then, by arguing similarly as Case 1, we may assume that $L_1^*$ is nontrivial. Hence, $d_{L^*}(u^*) = 2, 4$.

**Subcase 2.1.** $d_{L^*}(u^*) = 2$.

Then, exactly two of $\{P_1^*, P_2^*, P_4^*, P_5^*\}$ belong to $L_1^*$. By symmetry, we may assume that $P_1^*, P_2^* \subseteq L_1^*$, or $P_1^*, P_5^* \subseteq L_1^*$, or $P_2^*, P_5^* \subseteq L_1^*$.

**Subcase 2.1.1.** $P_1^*, P_2^* \subseteq L_1^*$.

In this case, the internal vertices of $P_1^*$ and $P_2^*$ are trivial components in $L^*$, and $L_1^* = u^*P_1^*vP_2^*u^*$. Let $L_1 = uP_1vP_2u$, and $L = (L^* - L_1^*) \cup L_1 \cup \{w\}$. Then $L$ is a spanning even subgraph of $G$ with $n_2(L) \leq \lceil \frac{n - 2}{3} \rceil$. Hence, $G$ is $\lceil \frac{n - 2}{3} \rceil$-supereularian, a contradiction.

**Subcase 2.1.2.** $P_1^*, P_4^* \subseteq L_1^*$.

In this case, the internal vertices of $P_2^*$ and $P_5^*$ are trivial components in $L^*$. Let $L_1$ be the graph obtained from $L_1^*$ by replacing $vP_1^*uP_4^*$ by $vP_1uwP_4$, and $L = (L^* - L_1^*) \cup L_1 \cup \{x_2\}$. Then $L$ is a spanning even subgraph of $G$ with $n_2(L) \leq \lceil \frac{n - 2}{3} \rceil$. Hence, $G$ is $\lceil \frac{n - 2}{3} \rceil$-supereularian, a contradiction.

**Subcase 2.1.3.** $P_4^*, P_5^* \subseteq L_1^*$.

In this case, the internal vertices of $P_1^*$, $P_2^*$ and $v$ are trivial components in $L^*$. Let $L_1^* = L_1^* \cup u^*P_1^*vP_2^*u^*$. Then, we can replace $L_1^*$ and the corresponding trivial components by $L_1^*$ in $L^*$, to reduce its number of components, contrary to the choice of $L^*$.

**Subcase 2.2.** $d_{L^*}(u^*) = 4$.

In this case, we can construct two even subgraphs $L_1'$ and $L_1''$ of $G$ from $L_1^*$: $L_1' = uP_1vP_2u$, and $L_1''$ is obtained from $L_1^*$ by deleting the vertices in $V(P_1^* \cup P_2^* \cup \{v\})$, and then replacing $P_4^*, P_5^*$ by $P_4, P_5$, respectively. Let $L = (L^* - L_1^*) \cup L_1' \cup L_1''$. Then $L$ is a spanning even subgraph of $G$ with $n_2(L) \leq \lceil \frac{n - 2}{3} \rceil$. Hence, $G$ is $\lceil \frac{n - 2}{3} \rceil$-supereularian, a contradiction.

This completes the proof of Claim 3. □

Now, we continue to prove Theorem 20. Note that $G^3$ is 2-edge-connected. By Theorem 24, there exists a constant $p$ and $3p$ perfect matchings $M_1, M_2, \ldots, M_{3p}$ such that each edge of $G^3$ is in $p$ of them. For $1 \leq i \leq 3p$, let $q(M_i) = \sum_{e \in M_i} q(e)$ be the weight of $M_i$. Without loss of generality, we can assume that $q(M_1) \leq q(M_2) \leq \cdots \leq q(M_{3p})$. By Theorem 24, $\sum_{i=1}^{3p} q(M_i) = p \sum_{e \in E(G^3)} q(e) = pd_2(G)$. Hence, $q(M_1) \leq \lceil d_2(G)/3 \rceil$. 

![Fig. 6. The demonstration of contraction when $P_3$ has internal vertices.](image)

![Fig. 7. The demonstration of contraction when $P_3$ is an edge.](image)
Since $M_1$ is a perfect matching, $G^3 - M_1$ is a 2-factor of $G^3$. By Claim 3, each component (i.e., cycle) of $G^3 - M_1$ contains at least three vertices. So $n_c(G^3 - M_1) \leq \lceil n(G^3)/3 \rceil = \lceil d_3(G)/3 \rceil$.

Now, we come back to consider the graph $G$. Let $L_1$ be the set of cycles (in $G$) which are the preimages of the cycles in $G^3 - M_1$, $L_2$ the set of vertices (in $G$) which are the internal vertices of the preimages of the edges in $M_1$, and let $L = L_1 \cup L_2$. Then $L$ is a spanning even subgraph of $G$ with

$$n_c(L) = n_c(L_1) + n_c(L_2) = n_c(G^3 - M_1) + q(M_1) \leq \left\lfloor \frac{d_3(G)}{3} \right\rfloor + \left\lfloor \frac{d_2(G)}{3} \right\rfloor.$$

Note that

$$\left\lfloor \frac{d_3(G)}{3} \right\rfloor + \left\lfloor \frac{d_2(G)}{3} \right\rfloor \leq \left\lfloor \frac{d_2(G) + d_3(G) - 2}{3} \right\rfloor = \left\lceil \frac{n - 2}{3} \right\rceil.$$

This implies that $G$ is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

This completes the proof of Theorem 20. \qed

Acknowledgments

The authors are extremely grateful to the referees for suggestions that led to correction and improvement of the paper. The former three authors are supported by the Natural Science Funds of China.

References