Some problems related to hamiltonian line graphs

Hong-Jian Lai and Yehong Shao

ABSTRACT. Part of this paper summarizes some of the recent developments in the study of hamiltonian line graphs and the related hamiltonian claw-free graphs. The last section of this paper solves some problems on the hamiltonian like indices from a paper by Clark and Wormald in 1983.

1. Definitions and Terminology

Graphs considered here are finite and loopless. Unless otherwise noted, we follow [2] for notations and terms. As in [2], κ(G), κ'(G) and δ(G) represent the connectivity, edge-connectivity, and the minimum degree of a graph G, respectively.

DEFINITION 1.1. A graph G is nontrivial if E(G) ≠ ∅.

DEFINITION 1.2. A vertex cut X of G is essential if G - X has at least two nontrivial components.

DEFINITION 1.3. For an integer k > 0, a graph G is essentially k-connected if G does not have an essential cut X with |X| < k.

DEFINITION 1.4. An edge cut Y of G is essential if G - Y has at least two nontrivial components.

DEFINITION 1.5. For an integer k > 0, a graph G is essentially k-edge-connected if G does not have an essential edge cut Y with |Y| < k.

DEFINITION 1.6. Let G be a graph and let X ⊆ E(G) be an edge subset. The contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops.

For convenience, we use G/e for G/{e} and G/∅ = G; and if H is a subgraph of G, we write G/H for G/E(H).

DEFINITION 1.7. For a graph G, O(G) denotes the set of all vertices of odd degree in G.

DEFINITION 1.8. A graph G is even if O(G) = ∅, is eulerian if G is both even and connected, and is super-eulerian if G contains a spanning eulerian subgraph.

1991 Mathematics Subject Classification. Primary 05C45; Secondary 05C38.
Key words and phrases. Line graph, claw-free graph, super-eulerian graphs, hamiltonian graphs, hamiltonian index, dominating Eulerian subgraph, essential connectivity.

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DEFINITION 1.9. A subgraph \( H \) of \( G \) is dominating if \( E(G - V(H)) = \emptyset \).

DEFINITION 1.10. A graph \( G \) is hamiltonian if \( G \) has a spanning cycle. A spanning cycle of \( G \) is a Hamilton cycle of \( G \).

DEFINITION 1.11. A graph \( G \) is hamiltonian connected if \( \forall u, v \in V(G) \), \( G \) has a spanning \((u, v)\)-path.

DEFINITION 1.12. The line graph of a graph \( G \), denoted by \( L(G) \), has \( E(G) \) as its vertex set, where two vertices in \( L(G) \) are adjacent if and only if the corresponding edges in \( G \) are adjacent.

The following theorem relates hamiltonian line graph \( L(G) \) and dominating eulerian subgraph in \( G \).

THEOREM 1.13. (Harary and Nash-Williams [10]) Let \( G \) be a connected graph with \( |E(G)| \geq 3 \). Then \( L(G) \) is hamiltonian if and only if \( G \) has a dominating eulerian subgraph.

DEFINITION 1.14. If \( P = v_0v_1\cdots v_k \) denotes a trail (or path, respectively), of \( G \), and if \( E(P) = \{e_1, e_2, \cdots, e_k\} \) is an edge set such that for \( i = 1, 2, \cdots, k \), \( e_i = v_{i-1}v_i \), then \( P \) is called a \((v_0, v_k)\)-trail (or path, respectively) of \( G \), and an \((e_1, e_k)\)-trail (or path, respectively) of \( G \). The vertices \( v_1, v_2, \cdots, v_{k-1} \) are called the internal vertices of \( P \). The edges \( e_1, e_k \) are called the end edges of \( P \).

DEFINITION 1.15. A trail \( P \) of \( G \) is dominating if every edge of \( G \) is incident with an internal vertex of \( P \).

With similar arguments (see page 74 in [10]), the following can be proved.

THEOREM 1.16. Let \( G \) be a connected graph with \( |E(G)| \geq 3 \). Then \( L(G) \) is hamiltonian connected if and only if for any pair of edges \( e_1, e_2 \in E(G) \), \( G \) has a dominating \((e_1, e_2)\)-trail.

DEFINITION 1.17. For a graph \( G \), an induced subgraph \( H \) isomorphic to \( K_{1,3} \) is called a claw of \( G \), and the only vertex of degree 3 of \( H \) is the center of the claw.

DEFINITION 1.18. A graph \( G \) is claw free if it does not contain a claw.

Beineke [1] and Robertson [11] independently proved that a graph \( G \) is a line graph if and only if \( G \) does not contain a list of 9 graphs as induced subgraphs. Šoltés [31], Lai and Šoltés [15] indicated for highly connected graphs, this forbidden list can be reduced to fewer graphs. But in any case, a line graph cannot have \( K_{1,3} \) as an induced subgraph.

The purpose of this article is to summarize some of the recent development on the study of hamiltonian line graphs, hamiltonian claw-free graphs, and to solve some of the problems from an earlier study by Clark and Wormald [7].

2. Sufficient Condition with Local Connectivity

DEFINITION 2.1. A vertex \( v \) is locally connected if \( N(v) \) is connected; and \( G \) is locally connected if every vertex of \( G \) is locally connected.

THEOREM 2.2. (D. J. Oberly and D. P. Sumner [26]) Every connected, locally connected claw-free graph is hamiltonian.
DEFINITION 2.3. A graph is vertex pancyclic if given any vertex \( v \in V(G) \), \( G \) has cycles \( C_i \) of length \( i \) containing \( v \), for each \( 3 \leq i \leq |V(G)| \).

THEOREM 2.4. (L. Clark [6], R. H. Shi [30], and C.-Q. Zhang [36]) Every connected, locally connected claw-free graph is vertex pancyclic.

DEFINITION 2.5. For a vertex \( v \in V(G) \), define
\[
N_2(v, G) = \{ e \in E(G) : e \text{ has at least one end in } N(v, G), \text{ but not incident with } v \}.
\]

DEFINITION 2.6. A vertex \( v \) is locally \( N_2 \)-connected in \( G \) if \( N_2(v, G) \) induces a connected subgraph in \( G \).

DEFINITION 2.7. A graph \( G \) is locally \( N_2 \)-connected if every vertex of \( G \) is locally \( N_2 \)-connected.

THEOREM 2.8. (Ryjáček [27]) Let \( G \) be a connected, \( N_2 \)-locally connected claw-free graph without vertices of degree 1, which does not contain an induced subgraph \( H \) isomorphic to either \( G_1 \) or \( G_2 \) (Figure below) such that \( N_1(x, G) \) of every vertex \( x \) of degree 4 in \( H \) is disconnected. Then \( G \) is hamiltonian.

![Figure 1](image)

The following was conjectured by Ryjáček [28].

THEOREM 2.9. (H.-J. Lai, Y. Shao and M. Zhan [16]) Every 3-connected, locally \( N_2 \)-connected claw-free graph is hamiltonian.

DEFINITION 2.10. For an integer \( k \geq 1 \), and a vertex \( v \in V(G) \),
\[
N^k(v, G) = \{ x \in V(G) : \text{dist}_G(v, x) \in \{1, 2, \cdots, k\} \}.
\]

DEFINITION 2.11. A vertex \( v \) is locally \( N^k \)-connected in \( G \) if \( N^k(v, G) \) induces a connected subgraph in \( G \).

DEFINITION 2.12. A graph \( G \) is locally \( N^k \)-connected if every vertex of \( G \) is locally \( N^k \)-connected.

CONJECTURE 2.13. (Li [22]) Every 3-connected, locally \( N^2 \)-connected claw-free graph is hamiltonian.

We now give a counterexample to Conjecture 2.13 as follows:

EXAMPLE 2.14. Let \( P_{10} \) denote the Petersen graph. For an integer \( k > 0 \) and let \( P_{10}(k) \) denote the graph obtained from \( P_{10} \) by adding \( k \) pendant edges at each vertex of \( P_{10} \). Let \( G(k) = L(P_{10}(k)) \) be the line graph of \( P_{10}(k) \). For an example when \( k = 3 \) see Figure 2.
By a well known result of Harary and Nash-Williams [10], $G(k)$ does not have a hamilton cycle. On the other hand, we shall check that each $G(k)$ is claw-free 3-connected (hence 3-edge-connected) and $N^2$-locally connected. Since line graphs are claw-free, $G(k)$ must also be claw-free. Since $P_{10}(k)$ does not have an essential edge cut with size less than 3, $\kappa(G(k)) \geq 3$. It remains to check that each $G(k)$ is $N^2$-locally connected.

We shall use the notation in Figure 2 to show that $G(k)$ is locally $N^2$-connected, by the symmetry of the Petersen graph, it suffices to show that both vertices $e_1$ and $e_2$ are locally $N^2$-connected.

Let $v_1, v_2$ and $v_3$ denote the vertices in $P_{10}(k)$ that are incident with both $e_1$ and $e_2$, both $e_2$ and $e_6$, and both $e_3$ and $e_7$, respectively. For each vertex $v \in V(P_{10}(k))$, let $K(v)$ denote the complete graph in $G(k)$ induced by the edge incident with $v$ in $P_{10}(k)$.

Since $e_1$ is a pendant edge in $P_{10}(k)$, it lies in a complete subgraph $K(v_1)$ of $G(k)$ containing $e_2, e_3, e_4$. Any vertices that are of distance 2 from $e_1$ in $G(k)$ must be a vertex adjacent to one of $e_2$, $e_3$ and $e_4$. Therefore, $e_1$ is a locally $N^2$-connected vertex in $G(k)$, see Figure 3.

It is not difficult to check that $e_2$ is also a locally $N^2$-connected vertex in $G(k)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{An illustration of the proof that $e_1$ is locally $N^2$-connected in $G(k)$}
\end{figure}

**Theorem 2.15.** (H.-J. Lai, M. Li, Y. Shao and L. Xiong [20])
(1) Every 2-edge-connected, locally $N^2$-connected claw-free graph has a spanning Eulerian subgraph with maximum degree at most 4.

(2) Every 3-edge-connected, locally $N^3$-connected claw-free graph has a spanning Eulerian subgraph with maximum degree at most 4.

**Definition 2.16.** A graph $G$ is trianually connected if for every pair of edges $e_1, e_2 \in E(G)$, $G$ has a sequence of 3-cycles $C_1, C_2, \ldots, C_l$ such that $e_1 \in C_1, e_2 \in C_l$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$, $(1 \leq i \leq l - 1)$.

Every connected, locally connected graph is trianually connected, but trianually connected graphs may not be locally connected.

**Theorem 2.17.** (H.-J. Lai, L. Miao, Y. Shao and L. Wan [29]) Every trianually connected claw-free graph is vertex pancyclic.

**Conjecture 2.18.** (M. Li, L. Xiong and H.-J. Lai, [23]) Every 3-connected 4-cycle connected claw-free graph $G$ with $|E(G)| \geq 3$ is hamiltonian.

We believe that such graphs may even be vertex pancyclic.

### 3. Degree Conditions

When $\kappa(H) = 2$, Kuipers and Veldman [13], and independently Favaron, Flaudrin, Li and Ryjáček [8], proved that if $H$ is a 2-connected claw-free graph with sufficiently large order $\nu$, and if $\delta(H) \geq \frac{\nu + c}{8}$ (where $c$ is a constant), then $H$ is hamiltonian except a member of ten well-defined families of graphs. When $\kappa(H) = 3$, the following have been proved and proposed.

**Theorem 3.1.** (Kuipers and Veldman[13]) If $H$ is a 3-connected claw-free simple graph with sufficiently large order $\nu$, and if $\delta(H) \geq \frac{\nu + 29}{8}$, then $H$ is hamiltonian.

**Theorem 3.2.** (Favaron and Fraisse [9]) If $H$ is a 3-connected claw-free simple graph with order $\nu$, and if $\delta(H) \geq \frac{\nu + 37}{10}$, then $H$ is hamiltonian.

**Conjecture 3.3.** (Kuipers and Veldman [13], see also [9]) Let $H$ be a 3-connected claw-free simple graph of order $\nu$ with $\delta(H) \geq \frac{\nu + 6}{10}$. If $\nu$ is sufficiently large, then $H$ is hamiltonian.

**Theorem 3.4.** (H.-J. Lai, Y. Shao and M. Zhan [18]) If $H$ is 3-connected claw-free simple graph with $\nu \geq 196$, and if $\delta(H) \geq \frac{\nu + 5}{10}$, then either $H$ is hamiltonian, or $\delta(H) = \frac{\nu + 5}{10}$ and $cl(H)$ is the line graph of $G$ obtained from the Petersen graph $P_{10}$ by adding $\frac{\nu - 15}{10}$ pendant edges at each vertex of $P_{10}$.

### 4. Matthews, Sumner and Thomassen Conjectures

**Conjecture 4.1.** (Thomassen [32]) Every 4-connected line graph is hamiltonian.

**Conjecture 4.2.** (Matthews and Sumner [24]) Every 4-connected claw-free graph is hamiltonian.

In 1986, Zhan proved:

**Theorem 4.3.** (Zhan [34]) If $G$ is a 4-edge-connected graph, then the line graph $L(G)$ is hamiltonian connected.

**Theorem 4.4.** (Ryjáček [28])
(1) Conjecture 4.1 and Conjecture 4.2 are equivalent.
(2) Every 7-connected claw-free graph is hamiltonian.

Theorem 4.5. (Chen, Lai, Lai, and Weng [14]) Every 4-connected line graph of a claw free graph is hamiltonian.

This has been improved by Kriesell.

Theorem 4.6. (Kriesell [12]) Every 4-connected line graph of a claw free graph is hamiltonian connected.

Definition 4.7. Let \( C_4 \) denote a 4-cycle in \( K_5 \). The graph \( K_5 - E(C_4) \) is called an hourglass.

Definition 4.8. A graph \( G \) is hourglass free if \( G \) does not have an induced subgraph isomorphic to \( K_5 - E(C_4) \).

Theorem 4.9. (Broersma, Kriesell and Ryjáček [3]) Every 4-connected hourglass free line graph is hamiltonian connected.

These results suggest that there may be a more general theorem behind them. In fact, the following is recently proved.

Theorem 4.10. (Lai, Shao, Yu and Zhan [19]) Let \( G \) be a connected graph with \( |V(G)| \geq 4 \). The core of this graph \( G \), denoted by \( G_0 \), is obtained by deleting all the vertices of degree 1 and contracting exactly one edge \( xy \) or \( yz \) for each path \( xyz \) in \( G \) with \( d_G(y) = 2 \). If every 3-edge-cut of the core \( G_0 \) has at least one edge lying in a cycle of length at most 3 in \( G_0 \), then the following statements are equivalent.

(1) \( L(G) \) is hamiltonian connected.
(2) \( \kappa(L(G)) \geq 3 \).

Corollary 4.11. Let \( G \) be a graph with \( |V(G)| \geq 4 \). Suppose that \( L(G) \) is hourglass free in which every 3-cut of \( L(G) \) is not an independent set. Then \( L(G) \) is hamiltonian-connected if and only if \( \kappa(L(G)) \geq 3 \).

Definition 4.12. A set \( B \subset V(G) \) is a dominating set if every vertex of \( G \) belongs to \( B \) or has a neighbor in \( B \).

Definition 4.13. The size of a minimum dominating set of \( G \) will be called dominating number of \( G \) and is denoted by \( \gamma(G) \). If \( \gamma(G) \leq k \), then \( G \) is \( k \)-dominated.

Definition 4.14. A graph \( G \) is almost claw free if the vertices that are centers of claws in \( G \) are independent and if the neighborhoods of the center of each claw in \( G \) is 2-dominated.

Note that every claw free graph is an almost claw free graph and there exist almost claw free graphs that are not claw-free.

Corollary 4.15. Every 4-connected line graph of an almost claw free graph is hamiltonian-connected.

We conclude this section with another result and a conjecture in this direction.

Theorem 4.16. (Lai, Shao, Wu and Zhou [21]) Every 3-connected, essentially 11-connected line graph is hamiltonian.

Conjecture 4.17. (H.-J. Lai and L. Šoltés, [31]) Every 7-connected claw-free graph is hamiltonian-connected.
5. Problems Related to the Hamiltonian Like Indices

**Definition 5.1.** For a nontrivial connected graph $G$, we define $L^0(G) = G$ and for any integer $k > 0$, $L^k(G) = L(L^{k-1}(G))$.

**Definition 5.2.** The *hamiltonian index* $h(G)$ of $G$ is the smallest positive integer $k$ such that $L^k(G)$ is hamiltonian.

The concept of hamiltonian index was first introduced by Chartrand and Wall [5], who showed that (Theorem A of [5]) if a connected graph $G$ is not a path, then $L^k(G)$ is defined for any positive integer $k$. For this reason, we shall assume, throughout this section, that the graph $G$ under discussion is simple, connected and not a path.

Clark and Wormald developed the idea and introduced the hamiltonian like indices.

**Definition 5.3.** A graph is *edge-hamiltonian* (eh) if each edge lies on a Hamilton cycle.

**Definition 5.4.** A graph is *pancyclic* (pc) if $G$ has a cycle of length $k$, for each $k$ with $3 \leq k \leq |V(G)|$.

**Definition 5.5.** A graph is *vertex pancyclic* (vp) if for every vertex $v \in V(G)$, $G$ has a cycle containing $v$ and of length $k$, for each $k$ with $3 \leq k \leq |V(G)|$.

**Definition 5.6.** A graph is *edge pancyclic* (ep) if for every edge $e \in E(G)$, $G$ has a cycle containing $e$ and of length $k$, for each $k$ with $3 \leq k \leq |V(G)|$.

**Definition 5.7.** A graph is *hamiltonian connected* (hc) if for every pair of vertices $u$ and $v$ of $G$, $G$ has a spanning $(u,v)$-path.

**Definition 5.8.** For a property $\mathcal{P}$ and a connected nonempty graph $G$ which is not a path, define the *$\mathcal{P}$-index* of $G$, denoted $\mathcal{P}(G)$, as

$$
\mathcal{P}(G) = \left\{ \begin{array}{ll}
\min\{k : L^k(G) \text{ has property } \mathcal{P} \} & \text{if at least one such integer } k \text{ exists} \\
\infty & \text{otherwise}
\end{array} \right.
$$

Clark and Wormald [7] showed that if a connected, nontrivial graph $G$ is not a path nor a cycle, then the indices $h(G), eh(G), pc(G), vp(G), ep(G), hc(G)$ exist as finite numbers.

The most studied index is $h(G)$, the hamiltonian index. By Theorem 1.13, if a graph $G$ is hamiltonian, then $L(G)$ is also hamiltonian. Therefore, a question arises: for any of the properties $\mathcal{P} \in \{eh, pc, vp, ep, hc\}$, if $G$ has property $\mathcal{P}$, does $L(G)$ also have property $\mathcal{P}$? We in this section will answer this question.

**Definition 5.9.** For a trail $P$ defined in Definition 1.14, let

$$
\partial(P) = \{ e \in E(G) : e \text{ is incident with an internal vertex of } P \}.
$$

If $\partial(P) = E(G)$, then $P$ is a dominating trail. We put the following facts in the form of a lemma, which follows directly from the definition of a line graph.

**Lemma 5.10.** Let $P = v_0v_1 \cdots v_k$ denote a (possibly closed) trail of $G$, and let $E(P) = \{ e_1, e_2, \cdots, e_k \}$ such that for $i = 1, 2, \cdots, k, e_i = v_{i-1}v_i$. Suppose that $X \subseteq \partial(P) - E(P)$ can be represented by $X = \{ x_1, \cdots, x_{n_1}, x_{n_1+1}, \cdots, x_{n_k-1} \}$, such
that for each $i = 0, 1, 2, \cdots, k - 2$, $x_{n_i + 1}, \cdots, x_{n_{i+1}}$ are incident with the vertex $v_{i+1}$, where we define $n_0 = 0$. Then the sequence $e_1, x_1, \cdots, x_{n_1}, e_2, x_{n_1 + 1}, \cdots, x_{n_k - 1}, e_k$ is a path $L$ in $L(G)$ with $V(L) = E(P) \cup X$.

**Proposition 5.11.** If $G$ is hamiltonian connected, then $L(G)$ is also hamiltonian connected.

**Proof.** By Theorem 1.16, it suffices to show that for any edges $e_1, e_2 \in E(G)$, $G$ has dominating $(e_1, e_2)$-trail.

We assume that $e_1 = v_1 v_2$ and $e_2 = v_2 v_3$. Since $G$ is hamiltonian connected, $G$ has a hamiltonian $(u_1, u_2)$-path $P$. If $e_1, e_2 \in E(P)$, then since $P$ is a $(u_1, u_2)$-path, $e_1$ and $e_2$ must be the end edges of the path and so $P$ is a dominating $(e_1, e_2)$-trail of $G$. If $e_1, e_2 \notin E(P)$, then $G[E(P) \cup \{e_1, e_2\}]$ is a dominating $(e_1, e_2)$-trail of $G$. If $|E(P) \cap \{e_1, e_2\}| = 1$, then we may assume that $e_1 \in E(P)$ and $e_2 \notin E(P)$. Since $P$ is a $(u_1, u_2)$-path, $e_1$ must be an end edge of $P$, and so $G[E(P) \cup \{e_2\}]$ is a dominating $(e_1, e_2)$-trail of $G$. \qed

**Proposition 5.12.** If $G$ is pancyclic, the $L(G)$ is also pancyclic.

**Proof.** For any integer $k$ with $3 \leq k \leq |E(G)|$, we need to show that $L(G)$ has a cycle of length $k$.

First assume that $3 \leq k \leq |V(G)|$. Since $G$ is pancyclic, $G$ has a cycle $C$ with $|E(C)| = k$. By Lemma 5.10, $L(C)$ is a cycle of length $k$ in $L(G)$. Thus we assume that $|V(G)| + 1 \leq k \leq |E(G)|$. Since $G$ is pancyclic, $G$ has a Hamilton cycle $C$, and so there exists an edge subset $X \subseteq E(G) - E(C)$ such that $|E(C) \cup X| = k$. It then follows by Lemma 5.10 that $L(G)[E(C) \cup X]$ contains a cycle of length $k$. Hence $L(G)$ must also be pancyclic. \qed

**Proposition 5.13.** If $G$ is edge-hamiltonian, the $L(G)$ is also edge-hamiltonian.

**Proof.** Let $f = e_1 e_2$ denote an arbitrary edge in $L(G)$, where $e_1, e_2$ are both adjacent to a vertex $u$ in $G$.

Suppose first that $u$ has degree 2 in $G$. Since $G$ is edge-hamiltonian, $G$ has a Hamilton cycle $C$ such that $e_1 \in E(C)$. It follows that $C$ is a dominating cycle of $G$. Since $u$ has degree 2 in $G$, $e_2 \in E(C)$. Therefore, $e_1$ and $e_2$ are adjacent in $C$ and $G$ has no other edges adjacent to $u$. It follows by Lemma 5.10 that $L(G)$ has a Hamilton cycle containing the edge $f = e_1 e_2$.

Hence we assume that $u$ has degree at least 3 in $G$. The there exists an edge $e_3 \in E(G) - \{e_1, e_2\}$ such that $e_3$ is also incident with $u$ in $G$. Since $G$ is edge-hamiltonian, $G$ has a Hamilton cycle $C$ such that $e_3 \in E(C)$. Since $C$ is a cycle containing $e_3$, and since $e_1, e_2, e_3$ are all incident with the same vertex $u$, we may assume that $e_2 \notin E(C)$. It then follows by Lemma 5.10 that $L(G)$ has a Hamilton cycle containing the edge $f = e_1 e_2$. \qed

**Proposition 5.14.** If $G$ is vertex pancyclic, the $L(G)$ is also vertex pancyclic.

**Proof.** For any integer $k$ with $3 \leq k \leq |E(G)|$, and for any $e \in V(L(G)) = E(G)$, we need to show that $L(G)$ has a cycle $L_k$ of length $k$ such that $e \in V(L_k)$. Let $e = uv$ such that $deg_G(u) \geq deg_G(v)$. We may assume that $G$ is not a 3-cycle, and since $G$ is vertex pancyclic, $deg_G(u) \geq 3$. Therefore, $e$ lies in a 3-cycle of $L(G)$. We assume that $k \geq 4$. 
Since $G$ is vertex pancyclic, $G$ has a cycle $C_k$ with $|E(C_k)| = k$, for $k = 3, 4, \ldots, |V(G)|$, such that $u \in V(C_k)$. If $e \in E(C_k)$, then by Lemma 5.10, $L_k = L(C_k)$ is a cycle of length $k$ containing $e$ in $L(G)$.

Therefore, we assume that $e \not\in E(C_k)$. As $k \geq 4$, $G$ has a cycle $C_{k-1}$ of length $k-1$ containing $u$. Since $\deg_G(u) \geq 3$, $|\partial(C_{k-1})| \geq k$ and $e \in \partial(C_{k-1})$. It follows by Lemma 5.10 that $L(G)$ has a cycle of length $k$ containing $e$.

When $|E(G)| \geq k \geq |V(G)| + 1$, since $G$ has a Hamilton cycle $C$ containing $u$, a similar argument shows that $L(G)$ has a cycle of length $k$ containing the edge $e$. Hence $L(G)$ must also be vertex pancyclic. \[\square\]

**PROPOSITION 5.15.** If $G$ is edge pancyclic, then $L(G)$ is also edge pancyclic.

**PROOF.** For any integer $k$ with $3 \leq k \leq |E(G)|$, and for any $e_1, e_2 \in V(L(G)) = E(G)$ with $f = e_1e_2 \in E(L(G))$, we need to show that $L(G)$ has a cycle $L_k$ of length $k$ such that $f \in E(L_k)$. Let $u$ denote the vertex incident to both $e_1$ and $e_2$.

We first assume $u$ has degree at least $d \geq 3$ in $G$, and so that $G$ has an edge $e_3 \in E(G) - \{e_1, e_2\}$. Thus $L(G)$ has a $k$-cycle containing the edge $f = e_1e_2$, for each $k = 3, 4, \ldots, d$.

Let $k \geq d + 1 \geq 4$ be an integer. Assume that $k \leq |V(G)|$. Since $G$ is edge pancyclic, $G$ has a cycle $C_{k-1}$ of length $k - 1$ such that $e_3 \in E(C_{k-1})$. If $e_1 \in E(C_{k-1})$, then $e_2 \not\in E(C_{k-1})$. Since $e_2 \in \partial(C_{k-1})$, it follows by Lemma 5.10 that $L(G)$ has a cycle of length $k$ containing $f = e_1e_2$. The case when $e_2 \in E(C_{k-1})$ can be similarly proved. Therefore, we assume that $e_1, e_2 \not\in E(C_{k-1})$. It follows that $d \geq 4$ and $G$ has an edge $e_4 \in E(G) - \{e_1, e_2, e_3\}$ incident with $u$ in $G$. Since $G$ is edge-pancyclic, $G$ has a cycle $C_{k-2}$ of length $k - 2$ containing $e_3$. Therefore, $\partial(C_{k-2}) \geq k$ and $e_1, e_2, e_3, e_4 \in \partial(C_{k-2})$. By Lemma 5.10, $L(G)$ has a cycle of length $k$ containing the edge $f = e_1e_2$. When $|E(G)| \geq k \geq |V(G)| + 1$, since $G$ has a Hamilton cycle $C$ containing $e_3$, a similar argument shows that $L(G)$ has a cycle of length $k$ containing the edge $f = e_1e_2$.

It remains to prove the case when $u$ has degree $2$ in $G$. In this case, any cycle of $G$ containing $e_1$ must also contain $e_2$. Using this fact and the same argument above, we can similarly prove that $L(G)$ has a cycle of length $k$ containing $f = e_1e_2$, for any $k$ with $3 \leq k \leq |E(G)|$. \[\square\]

There were a few open problems posted at the end of the paper of Clark and Wormald [7].

**DEFINITION 5.16.** For a property $P$ and integers $a, b$, with $1 \leq a \leq b$, define

$$P(a, b) = \left\{ \begin{array}{ll} \max\{P(G) : \kappa'(G) \geq a \text{ and } \delta(G) \geq b\} & \text{if the max exists} \\ \infty & \text{otherwise} \end{array} \right.$$ 

Clark and Wormald determined most of the values of $P(a, b)$ in [7]. At the end, they raised this question: What are the values of $h(a, b), ch(a, b), \text{ and } hc(a, b)$ when $4 \leq a \leq b$?

To answer this question, we shall apply the following theorem by Nash-Williams and Tutte.

**THEOREM 5.17.** (Nash-Williams [25] and Tutte [33]) Let $k > 0$ be an integer. A graph $G$ has $k$ edge-disjoint spanning trees if and only if for any partition \(\{V_1, V_2, \ldots, V_t\}\) of $V(G)$, the number of edges in $G$ joining distinct sets of these $V_i$'s is at least $k(t - 1)$. 


Using this powerful theorem of Nash-Williams and Tutte, Zhan proved the following.

**Theorem 5.18.** (Zhan [35]) If $G$ is 4-edge-connected, then for any edges $e_1, e_2 \in E(G)$, $G - \{e_1, e_2\}$ has two edge-disjoint spanning trees.

Catlin [4] proved the following.

**Theorem 5.19.** (Catlin [4]) If $G$ has two edge-disjoint spanning trees, then for any two vertices $u, v \in V(G)$, $G$ has a spanning $(u, v)$-trail.

It follows that if $\kappa'(G) \geq \kappa(G) \geq 4$, then by Theorem 5.18 and Theorem 5.19, for any pair of edges $e_1 = u_1v_1, e_2 = u_2v_2 \in E(G)$, the graph $G - \{e_1, e_2\}$ has a spanning $(u_1, u_2)$-trail $C$, which can be augmented by adding the edges $e_1$ and $e_2$ to result in a dominating $(e_1, e_2)$-trail of $G$. By Theorem 1.16, $L(G)$ is hamiltonian connected. This indicates that when $4 \leq a \leq b$, $hc(a, b) \leq 1$. By the definition of these indices, when $4 \leq a \leq b$, we have

$$h(a, b) \leq eh(a, b) \leq hc(a, b) \leq 1.$$

For any integer $m \geq 5$, since $K_{m,4}$ is 4-connected without a Hamilton cycle, $h(4, b) \geq 1$, for any $b \geq 4$. Therefore, when $4 \leq a \leq b$, we have

$$h(a, b) = eh(a, b) = hc(a, b) = 1.$$

This answers the question in [7].

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