Cycle covers of planar graphs

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Abstract

Bondy conjectured in [1] that every 2-connected simple graph with $n$ vertices admits a cycle cover with at most $(2n - 1)/3$ cycles. In this paper we shall show that every 2-connected planar graph with $n \geq 6$ vertices admits a cycle cover with at most $(2n - 2)/3$ cycles. This bound is best possible.

1. Introduction

We follow the notation of Bondy and Murty [2], unless otherwise noted. In particular, $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of a graph $G$, respectively. An edge $e$ of a graph $G$ is called a multiple edge if $G - e$ has an edge $f$ that has the same ends as $e$ in $G$. We allow multiple edges but forbid loops. When $v, v' \in V(G)$, $vv'$ would denote any one edge in $G$ with ends $v$ and $v'$. For $X \subseteq E(G)$, the contraction $G/X$ is the graph obtained from $G$ by identifying the ends of each edge of $X$ and then deleting the resulting loops. We shall use $G/e$ for $G/\{e\}$ and when $H$ is a subgraph of $G$, we write $G/H$ for $G/E(H)$. For $v \in V(G)$, $N(v)$, the neighborhood of $v$ in $G$, denotes the set of vertices adjacent to $v$ in $G$. If $H$ is a subgraph of $G$ and $P = v_1v_2 \cdots v_k$ is a path of $G$, then we shall write $H + v_1v_2 \cdots v_k$ for the subgraph $G[E(H) \cup E(P)]$.

A cycle cover (CC) is a collection $\mathcal{C}$ of cycles in $G$ such that every edge in $G$ lies in at least one cycle in $\mathcal{C}$. It is clear that $G$ has a cycle cover if and only if $\kappa'(G) \geq 2$. For a 2-edge-connected graph, let

\[ cc(G) = \min \{|\mathcal{C}| : \mathcal{C} \text{ is a CC of } G\} \tag{1} \]

In [B], Bondy posed the following conjecture:

**Conjecture SCC:** (Bondy [1]) If $G$ is a 2-connected simple graph $G$ with $n$ vertices, then

\[ cc(G) \leq \frac{2n - 1}{3}. \tag{2} \]

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We shall work on a multigraph version of this conjecture. For a graph $G$, define a relation on $E(G)$ such that $e$ is related to $e'$ if and only if $e = e'$ or $e$ is parallel to $e'$ in $G$. It is easy to check that this is an equivalence relation. Let $[e]$ denote the equivalence class containing $e$, and $[G]$ the collection of all equivalence classes. Define

$$
\mu(G) = \sum_{[e] \in [G]} ([e] - 1).
$$

Thus $G$ is simple if and only if $\mu(G) = 0$, and so a multigraph version of Conjecture SCC can be stated as follows: If $G$ is a 2-edge-connected graph with order $n$, then

$$
cc(G) \leq \frac{2n - 1}{3} + \frac{\mu(G)}{2}.
$$

Call a multigraph $G$ a plane triangulation if $G$ can be embedded in the plane such that every face of the embedding has degree 2 or 3. In [3], we showed the following:

**Theorem 1.1** If $G$ is a planar triangulation with $n \geq 6$ vertices, then

$$
cc(G) \leq \frac{2n - 3}{3} + \frac{\mu(G)}{2}.
$$

In this note, we shall show that Conjecture SCC holds for planar graphs:

**Theorem 1.2** If $G$ is a planar graph with $n \geq 6$ vertices and with $\kappa(G) \geq 2$, then

$$
cc(G) \leq \frac{2n - 2}{3} + \frac{\mu(G)}{2}.
$$

This result is best possible, in the sense that there exists a collection of planar graphs in which the bound in (6) is attained (see [4]).

2. **Lemmas.** We shall argue with a minimum counterexample, and so we need to take care of graphs with small orders.

**Lemma 2.1** Let $H$ be a 2-connected simple planar graph with $4 \leq |V(H)| \leq 5$. If $\delta(H) = 3$, then $H \in \{K_4, J_1, J_2\}$, (see Figure 1 for definition of $J_1$ and $J_2$).

*Proof:* This is trivial if $|V(H)| = 4$ and so we assume that $|V(H)| = 5$. By $\delta(H) = 3$, we have $2|E(H)| \geq 15$ and so $|E(H)| \geq 8$. Since $H$ is a simple plane graph with 5 vertices, we have $|E(H)| \leq 9$, with equality if and only if $H$ is a triangulation. Hence either $G$ is a triangulation ($H \cong J_2$) or $H$ is a triangulation minus an edge with $\delta(H) = 3$ ($H \cong J_1$).

**Lemma 2.2** Let $H$ be a 2-edge-connected simple planar graph. Each of the following holds:

(i) If $|V(H)| = 4$ and if $e \in E(H)$ is given, then $H$ has two cycles that covers $H$ such that $e$ can be covered at twice.

(ii) If $|V(H)| = 5$ and if $\delta(H) = 2$, or if $H = J_1$, then $cc(H) \leq 2$.

(iii) Let $H \in \{J_1, J_2\}$. If $e \in E(H)$ is given, then $H$ has 3 cycles covering $H$ such that $e$ is covered at least twice.
(iv) Given \( e \in E(J_2) \), \( J_2 \) has 3 cycles covering \( J_2 \) so that \( e \) is covered 3 times.

(v) Given \( e_1, e_2 \in E(J_2) \), \( J_2 \) has 3 cycles covering \( J_2 \) so that \( e_1 \) and \( e_2 \) are covered at least twice.

**Proof:** We shall use the notation in Figure 1 in the proof. (i) of Lemma 2.2 is obvious. If \( \delta(H) = 2 \) in (ii), then one can contract an edge incident with a vertex of degree 2 to get (ii) from (i). Note that \( J_1 \) can be covered by \( z_1 z_2 z_3 z_4 z_1 \) and \( z_1 z_4 z_2 z_5 z_3 z_1 \). This shows that \( cc(J_1) = 2 \) and the case \( H = J_1 \) in (iii), since the edge \( e \) can be covered by the third cycle. The case \( H = J_2 \) in (iii) follows from (iv). By symmetry of \( J_2 \), we may assume that edge \( e \in \{ z_1 z_4, z_2 z_3, z_3 z_5 \} \). Table 1 below exhibits 3 desired cycles for (iv).

<table>
<thead>
<tr>
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Table 1

Let \( e_1, e_2 \in E(J_2) \) be given. There is a cycle \( C_1 \) containing both \( e_1 \) and \( e_2 \), together with a third edge \( e_3 \). If \( e_3 \) is either incident with a vertex of degree 3 and a vertex of degree 4 (say \( e_3 = z_4 z_5 \)), or \( e_3 \) is incident with two vertices of degree 4 (say \( e_3 = z_4 z_5 \)), then in any case, \( J_2 \) can be covered by two cycles and so (v) of Lemma 2.2 follows. \( \square \)

**Lemma 2.3** If \( H \in \{ L_8, L_9, L_{10}, L_{11} \} \) (see Figure 4 for definitions), then \( cc(H) \leq 4 \).

**Proof:** We shall use the notation in Figure 4. The cycles \( x_1 x_2 x_3 x_5 x_6 x_4 x_5 x_7 x_7 x_1, x_2 x_5 x_4 x_7 x_1, x_1 x_7 x_8 x_2 x_4 x_1 \) and \( x_2 x_4 x_3 x_5 x_2 \) cover \( L_8 \); \( x_1 x_2 x_3 x_6 x_4 x_8 x_7 x_1, x_1 x_7 x_2 x_5 x_3 x_4 x_1, x_2 x_4 x_5 x_2 \) and \( x_2 x_5 x_6 x_4 x_7 x_8 x_2 \) cover \( L'_8 \); \( x_1 x_2 x_3 x_6 x_4 x_8 x_7 x_1, x_1 x_7 x_2 x_5 x_4 x_6 x_5, x_2 x_4 x_3 x_5 x_2 \) and \( x_2 x_5 x_6 x_4 x_7 x_8 x_2 \) cover \( L''_8 \); and \( x_1 x_2 x_3 x_6 x_4 x_8 x_7 x_1, x_1 x_7 x_2 x_5 x_4 x_6 x_5, x_2 x_5 x_6 x_4 x_3 x_2 \) and \( x_2 x_5 x_6 x_4 x_7 x_8 x_2 \) cover \( L'''_8 \). \( \square \)

**Lemma 2.4** If \( H \in \{ J_3, J_4 \} \) (see Figure 3 for definitions), then \( cc(H) \leq 3 \).

**Proof:** We shall use the notation in Figure 3. The cycles \( z_1 z_2 z_3 z_1, z_1 z_4 z_5 u_1 z_2 z_3 z_1 \) and \( z_2 z_5 u_1 z_3 z_4 z_2 \) cover \( J_3 \); and the cycles \( z_1 z_4 z_2 z_5 u_1 z_3 z_4 z_1, z_1 z_2 u_1 z_3 z_4 z_1 \) and \( z_3 z_4 z_5 z_3 \) cover \( J_4 \). \( \square \)

3. **Proof of Theorem 1.2** From now on we assume that

\[ G \text{ is a counterexample to Theorem 1.2} \]

such that

\[ |V(G)| \text{ is minimized,} \]

and subject to (8),

\[ \mu(G) \text{ is minimized.} \]
(iv) Given $e \in E(J_2)$, $J_2$ has 3 cycles covering $J_2$ so that $e$ is covered 3 times.

(v) Given $e_1, e_2 \in E(J_2)$, $J_2$ has 3 cycles covering $J_2$ so that $e_1$ and $e_2$ are covered at least twice.

Proof: We shall use the notation in Figure 1 in the proof. (i) of Lemma 2.2 is obvious. If $\delta(H) = 2$ in (ii), then one can contract an edge incident with a vertex of degree 2 to get (ii) from (i). Note that $J_1$ can be covered by $z_1z_2z_3z_5z_4z_1$ and $z_1z_4z_2z_5z_3z_1$. This shows that $cc(J_1) = 2$ and the case $H = J_1$ in (iii), since the edge $e$ can be covered by the third cycle. The case $H = J_2$ in (iii) follows from (iv). By symmetry of $J_2$, we may assume that edge $e \in \{z_1z_4, z_2z_3, z_3z_5\}$. Table 1 below exhibits 3 desired cycles for (iv).

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<td>$z_2z_3z_2z_4$</td>
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Table 1

Let $e_1, e_2 \in E(J_2)$ be given. There is a cycle $C_1$ containing both $e_1$ and $e_2$, together with a third edge $e_3$. If $e_3$ is either incident with a vertex of degree 3 and a vertex of degree 4 (say $e_3 = z_4z_5$), or $e_3$ is incident with two vertices of degree 4 (say $e_3 = z_2z_4$), then in any case, $J_2 - e_3$ can be covered by two cycles and so (v) of Lemma 2.2 follows. □

Lemma 2.3 If $H \in \{L_6, L_6', L_6'', L_6'''\}$ (see Figure 4 for definitions), then $cc(H) \leq 4$.

Proof: We shall use the notation in Figure 4. The cycles $x_1x_2x_3x_6x_6x_4x_7x_1$, $x_2x_5x_4x_7x_2$, $x_1z_7x_8x_2x_4x_1$ and $x_2x_4x_3x_5x_6x_2$ cover $L_6$; $x_1x_2x_3x_6x_4x_7x_8x_1$, $x_1x_7x_2x_5x_3x_4x_1$, $x_2x_4x_7x_2$ and $x_2x_5x_6x_4x_7x_8x_2$ cover $L_6'$; $x_1x_2x_3x_6x_7x_8x_2$, $x_1x_7x_2x_5x_4x_1$, $x_2x_4x_3x_5x_2$ and $x_2x_5x_7x_4x_6x_5x_2$ cover $L_6''$; and $x_1z_6x_7x_2x_5x_6x_3x_4x_1$, $x_1x_2x_3x_5x_6x_4x_7x_1$, $x_2x_5x_4x_7x_2$ and $x_2x_5x_4x_2$ cover $L_6'''$. □

Lemma 2.4 If $H \in \{J_3, J_4\}$ (see Figure 3 for definitions), then $cc(H) \leq 3$.

Proof: We shall use the notation in Figure 3. The cycles $z_1z_2z_5u_1z_3z_1$, $z_1z_4z_5u_1z_2z_3z_1$ and $z_2z_5u_1z_3z_4z_2$ cover $J_3$; and the cycles $z_1z_4z_2z_5u_1z_3z_1$, $z_1z_2u_1z_3z_4z_1$ and $z_2z_4z_5z_3$ cover $J_4$. □

3. Proof of Theorem 1.2 From now on we assume that

\[ G \text{ is a counterexample to Theorem 1.2} \]  \hspace{1cm} (7)

such that

\[ |V(G)| \text{ is minimized,} \]  \hspace{1cm} (8)

and subject to (8),

\[ \mu(G) \text{ is minimized.} \]  \hspace{1cm} (9)
Lemma 3.1 \( \mu(G) \leq 1 \).

Proof: Suppose that \( \mu(G) \geq 2 \). Then either there is an edge \( e \) with \( |e| \geq 3 \) or there are two edges \( e_1, e_2 \) with \( |e_1| \geq 2, |e_2| \geq 2 \) and with \( e_1 \neq e_2 \). In either case we can pick two edges \( e, e' \) (say), in such a way that \( \mu(G - \{e, e'\}) = \mu(G) - 2 \). By \( \kappa(G) \geq 2 \), there is a cycle \( C \) containing \( e \) and \( e' \). By (9) and by the fact that any CC \( C \) of \( G - \{e, e'\} \) together with the cycle \( C \) containing \( e \) and \( e' \) will form a CC of \( G \), we have

\[
cc(G) \leq cc(G - \{e, e'\}) + 1 \leq \frac{2n - 2}{3} + \frac{(G) - 2}{2} + 1,
\]

contrary to (7). \( \square \)

Lemma 3.2 \( G \) does not have any vertex \( v \) of degree 3 that is incident with some multiple edges.

Proof: We argue by contradiction. Suppose that \( G \) has a vertex of degree 3 that is incident with some multiple edges. Let \( e_1, e_2, e_3 \) be the edges incident with \( v \). By \( \kappa(G) \geq 2 \) and by Lemma 3.1, we may assume that \( [e_1] = \{e_1, e_2\} \) and \( [e_3] = \{e_3\} \). Let the two vertices adjacent to \( v \) be \( u, u' \) such that \( u \) is incident with \( e_3 \) and \( u' \) with \( e_1 \) and \( e_2 \). Define

\[
G' = \begin{cases} 
G - v + uu' & \text{if } uu' \not\in E(G) \\
G - v & \text{if } uu' \in E(G)
\end{cases}
\]

(10)

It is then easy to see that

\[
cc(G) \leq cc(G') + 1.
\]

(11)

In fact, let \( C' \) be a CC of \( G' \) and let \( C' \in C \) be a cycle that contains the edge \( uu' \). Since every cycle in \( C' - \{C'\} \) can be extended to a cycle in \( G \) by possibly replacing \( uu' \) by \( e_3 \), we shall use \( C \) to denote the collection of cycles in \( G \) corresponding to the cycles in \( C' - \{C'\} \). Let \( C_1 = C' - uu' + \{e_1, e_3\} \). If \( uu' \not\in E(G) \), then let \( C_2 = G[\{e_1, e_2\}] \), and if \( uu' \in E(G) \), then let \( C_2 = G[\{uu', e_2, e_3\}] \). Thus \( C \cup \{C_1, C_2\} \) would form a CC for \( G \) in either case and so (11) holds.

Since \( \kappa(G) \geq 2 \) and by (10), \( \kappa(G') \geq 2 \) also. Note that \( \mu(G') = \mu(G) - 1 \). If \( |V(G')| \geq 6 \), then by the minimality of \( G \), we have

\[
cc(G) \leq cc(G') + 1 \leq \frac{2|V(G')| - 2}{3} + \frac{(G')}{2} + 1 \leq \frac{2|V(G)| - 2}{3} + \frac{(G)}{2},
\]

contrary to (7). Thus we assume that \( |V(G')| = 5 \) and so \( |V(G)| = 6 \).

Since the edges in \( [e_1] \) are deleted, by Lemma 3.1, \( G' \) is simple with \( \delta(G') \geq 2 \). If \( \delta(G') = 2 \), or if \( G' = K_4 \), then by (i) or (ii) of Lemma 2.2, \( cc(G') = 2 \) and so by (11), \( cc(G) \leq 3 \), contrary to (7).

Hence by Lemma 2.1 we assume that \( G' \in \{J_1, J_2\} \). If \( uu' \not\in E(G) \), then by (iii) of Lemma 2.2, \( G' \) can have a CC \( C \) with 3 cycles such that the edge \( uu' \) can be covered by 2 cycles, whence \( cc(G) = 3 \), contrary to (7). If \( G' \cong J_1 \) and \( uu' \in E(G) \), then since \( cc(J_1) = 2 \), it follows by (11) that \( cc(G) \leq 3 \) also, contrary to (7). Thus we must have \( G' \cong J_2 \) and \( uu' \in E(G) \). By the symmetry of \( J_2 \), either both \( u, u' \) are of degree 4 in \( J_2 \)
(say \( \{u, u'\} = \{z_2, z_3\} \)), or one of \( \{u, u'\} \) has degree 4 and the other has degree 3 in \( J_2 \) (say \( \{u, u'\} = \{z_1, z_2\} \)). Table 2 below uses the notation in Figure 1 and shows that \( cc(G) \leq 3 \) in any case.

\[
\begin{array}{ccc}
\text{u} & \text{u'} & \text{C}_1 & \text{C}_2 & \text{C}_3 \\
z_1 & z_2 & z_1z_2z_3z_5z_4z_1 & z_1z_2z_4z_3z_1 & z_1z_2z_5z_4z_1 \\
z_2 & z_1 & z_1z_2z_3z_5z_4z_1 & z_1z_2z_4z_3z_1 & z_1z_2z_5z_4z_1 \\
z_2 & z_3 & z_1z_2z_3z_5z_4z_1 & z_1z_3z_2z_4z_1 & z_4z_5z_2z_3z_4 \\
\end{array}
\]

Table 2

Thus \( G \) satisfies (6) in any case, contrary to (7). \( \square \)

Lemma 3.3 If \( w \in V(G) \) with \( N(w) = \{w_1, w_2, w_3\} \) and if \( w_1w_3 \notin E(G) \), then \( G[N(w)] \) is disconnected.

Proof: By contradiction, we may assume that \( w_1w_2, w_2w_3 \in E(G) \). By Lemma 3.1 and without loss of generality, we may assume that \( |w_1w_2| = |w_2w_3| = 1 \). Let \( G' = G/ww_3 \) and let \( \mathcal{C} \) be a CC of \( G' \). Denote \( e_1 = w_2w_3, e_2 = ww_2 \) and \( e_3 = w_1w_2 \), and let \( C_i \in \mathcal{C} \) be cycles containing \( e_i \), \( 1 \leq i \leq 3 \). Note that \( e_1 \) and \( e_2 \) become edges with the same ends in \( G' \), and so we may assume that \( C_1 \neq C_3 \).

If \( C_1 = C_2 \) is a 2-cycle, then let \( C'_3 = C_3 - e_3 + w_1ww_2 \) and \( F'' = w_1ww_3w_2, w_1, \) and extend any cycle \( L \in \mathcal{L} - \{C_1, C_3\} \) to a cycle \( L' \) in \( G \) by adding \( ww_3 \), if necessary. Thus \( \{L' | L \in \mathcal{L} - \{C_1\}\} \cup \{F''\} \) is a CC of \( G \). Thus we may assume that neither \( C_1 \) nor \( C_2 \) is a 2-cycle. In Table 3, \( C'_1 \) and \( C'_2 \) are defined according to the different situations of \( C_1 - e_1 \) and \( C_2 - e_2 \) in \( G \).

\[
\begin{array}{ccc}
C_1 - e_1 \text{ in } G & C_2 - e_2 \text{ in } G & \text{The new cycle } C'_1 \\
(w_2, w_3)-\text{path} & (w_2, w_3)-\text{path} & C_1 - e_1 + w_2ww_3 \\
(w_2, w_3)-\text{path} & (w_2, w)-\text{path} & C_1 - e_1 + w_2ww_3 \\
(w_2, w)-\text{path} & (w_2, w_3)-\text{path} & C_1 - e_1 + w_3w_2 \\
(w_2, w)-\text{path} & (w_2, w)-\text{path} & C_1 - e_1 + w_3w_2 \\
\end{array}
\]

The new cycle \( C'_2 \)

\[
\begin{array}{ccc}
C_2 & C_2 - e_2 + w_2ww_3 & C_2 - e_2 + w_2ww_3 \\
C_2 & C_2 - e_2 + w_2ww_3 & C_2 - e_2 + w_2ww_3 \\
C_2 & C_2 - e_2 + w_2ww_3 & C_2 \\
\end{array}
\]

Table 3

Extend any cycle \( C \in \mathcal{C} - \{C_1, C_2\} \) to a cycle \( C' \) in \( G \). Thus \( \{C' | C \in \mathcal{C}\} \) gives

\[ cc(G) \leq cc(G'). \] (12)

Since \( N(w) = \{w_1, w_2, w_3\} \) and by \( \kappa(G) \geq 2 \), \( \{w, w_3\} \) is not a vertex cut of \( G \) and so \( \kappa(G') \geq 2 \). By (8) and (12), if \( |V(G')| \geq 6 \), then \( G \) satisfies (6), contrary to (7). Since \( |V(G)| \geq 6 \), we must have \( |V(G')| = 5 \) and so \( G' \) is spanned by \( J_1 \) or \( J_2 \). By (iii) of Lemma 2.2 and by (12), \( G \) satisfies (6) also, contrary to (7). \( \square \)
If $S$ is a vertex cut of a connected graph $H$ and if the components of $H - S$ have vertex sets $V_1, V_2, \ldots, V_p$, then $H[V_i \cup S]$ ($1 \leq i \leq p$) is called an $S$-component of $H$.

Lemma 3.4 $G$ does not have a vertex cut $\{u_1, u_2\}$ with $u_1u_2 \in E(G)$ such that $G$ has a $\{u_1, u_2\}$-component of at most 3 vertices.

Proof: Suppose such an edge $u_1u_2$ exists. Let $L'$ be a $\{u_1, u_2\}$-components of $G$ with $V(L) = \{u_1, u_2, \ldots, u_k\}$ and with $k \leq 5$. Let $L = L'$ if $|u_1u_2| = 1$ and $L$ be the underlying simple graph of $L'$ if $|u_1u_2| = 2$. Throughout the proof, let $e \notin E(G)$ be an edge parallel to $u_1u_2$.

Suppose first that $k = 3$. By Lemma 3.2, $d(u_3) = 2$. Let $G_1 = (G - u_3) + e$. Then $cc(G) \leq cc(G_1)$ since the edge $e$ can be replaced by the path $u_1u_3u_2$ in any cycle containing $e$. By the minimality of $G$, we may assume that $|V(G_1)| = 5$. But then $|V(G)| = 6$, and so by (ii) or (iii) of Lemma 2.2, $cc(G) \leq cc(G_1) \leq 3$, contrary to (7). Thus $k \neq 3$.

Since we did not use the fact that $u_1u_2 \in E(G)$, we have in fact shown:

Corollary 3.5 $\delta(G) \geq 3$. □

We continue the proof of Lemma 3.4 and assume that $k = 4$. Let $G_2 = \begin{cases} G - \{u_3, u_4\} & \text{if $\mu(L) = 0$} \\ G - \{u_3, u_4\} + e & \text{if $\mu(L) = 1$.} \end{cases}$

Since $u_1u_2 \in E(G)$, $\kappa(G_2) \geq 2$. By $\delta(G) \geq 3$ and by Lemma 3.2, $L$ must be spanned by a $K_4$. We claim that $cc(G) \leq cc(G_2) + 1$. When $\mu(L) = 0$, let $C$ be a CC of $G_2$ and let $C \in \mathcal{C}$ be a cycle in $G_2$ containing $u_1u_2$. Since $C - u_1u_2 + u_1u_3u_4u_2$ and $u_1u_2u_3u_4u_1$ are two cycles in $G$, it follows that $cc(G) \leq cc(G_2) + 1$. The case when $\mu(L) = 1$ is similar. By (8), if $|V(G_2)| \geq 6$, then

$$cc(G) \leq cc(G_2) + 1 \leq \frac{2(n - 2) - 2}{3} + \frac{\mu(G)}{2} + 1 < \frac{2n - 2}{3} + \frac{\mu(G)}{2},$$

contrary to (7). Thus $|V(G_2)| \leq 5$. If $|V(G_2)| = 4$, then $|V(G)| = 6$ and by (i) of Lemma 2.2, $cc(G) \leq cc(G_2) + 1 \leq 3$, contrary to (7). If $|V(G_2)| = 5$, then $|V(G)| = 7$ and by (ii) or (iii) of Lemma 2.2, $cc(G) \leq cc(G_2) + 1 \leq 4$, contrary to (7) also. This excludes $k = 4$.

Assume that $k = 5$. Let $G_3 = G - \{u_3, u_4, u_5\}$. Let $C$ be a CC of $G_3$ and let $\{C_1, C_2, C_3\}$ be a CC of $L$ so that $u_1u_2$ is covered by $C_1$ and $C_2$. Let $C \in \mathcal{C}$ be a cycle containing $u_1u_2$. Then $(C - \{C\}) \cup \{C_2, C_3, G[E(C) \cup E(C_1) - \{e\}]\}$ is a CC of $G$ and so $cc(G) \leq cc(G_3) + cc(L) - 1$. By the minimality of $G$, we may assume $|V(G_3)| \leq 5$. But if $|V(G_3)| < 5$, then we are back to the cases of $3 \leq k \leq 4$. Therefore we assume that $|V(G_3)| = |V(L)| = 5$ and so $|V(G)| = 8$. If $cc(G_3) = 2$ or $cc(L) = 2$, then by (ii) or (iii) of Lemma 2.2 and by $cc(G) \leq cc(L) + cc(G_3) - 1$, we have $cc(G) \leq 4$, contrary to (7). If $\mu(G) = 1$, then by the same reasons, $cc(G) \leq 5$, contrary to (7) again. Thus we have $\mu(G) = 0$ and $G_3 \cong L \cong J_2$. By the symmetry of $J_2, G \in \{L_8, L_6, L_6', L_8''\}$, and so by Lemma 2.3, $G$ satisfies (6), contrary to (7). □
Fix a planar embedding of $G$. For a cycle $C$ in $G$, $IntC$, called the interior of $C$, denotes the vertices of $G$ lying inside $C$, excluding $V(C)$. The exterior of $C$ is defined similarly. A cycle $C$ of a plane graph $G$ is trivial if $IntC = \emptyset$. A vertex $v \in V(G)$ is cyclic if $G[N(v)]$ is spanned by an $m$-cycle $C_m$ with $m = \text{deg}(v)$. This cycle $C_m$ is called the rim cycle of $v$. It is clear that if every vertex of $G$ is cyclic, then $G$ has a triangulation planar embedding.

For any vertex $v$ in a simple plane graph $G$, the vertices in $N(v)$ can be viewed as an ordered string $<v_1, v_2, \ldots, v_m>$ such that $vv_i$ and $vv_{i+1}$ are incident with the same face of $G$, $i = 1, 2, \ldots, m, (mod\ m)$. We call this string an ordered neighborhood of $v$ in $G$. (The definition for multigraphs is similar.) Note that for each $v \in V(G)$, one can have two different ordered neighborhoods: the clockwise one and the anticlockwise one.

If $v \in V(G)$ is not a cyclic vertex, then for any plane embedding of $G$, there are two vertices $u, u' \in N(v)$ such that $vu$ and $vu'$ are incident with the same face but $uu' \not\in E(G)$. We called $u, u'$ a bad pair in $N(v)$.

**Lemma 3.6** Let $G$ be a connected planar graph satisfying (7), (8) and (9). Then $G$ has no cyclic vertices.

**Proof**: We shall show by contradiction that if $v \in V(G)$ is a cyclic vertex, then every vertex in $N(v)$ is also a cyclic vertex. Then by the connectedness of $G$, every vertex of $G$ is cyclic and so $G$ is a triangulation. A contradiction follows from Theorem 1.1.

Fix an embedding of $G$ such that the interior of the rim cycle of $v$ consists of $v$ only. Let $N(v) = <v_1, v_2, \ldots, v_m>$ be an ordered neighborhood of $v$ in $G$. By contradiction, we assume that $v_1$ is not cyclic. Let $N(v_1) = \{v'_1, v''_1, \ldots\}$ be the neighborhood of $v_1$ in $G$. Since $v_1$ is not cyclic, we may assume that $v'_1, v''_1 \in N(v_1)$ such that

$$v'_1 \text{ and } v''_1 \text{ are a bad pair in } N(v_1).$$

(13)

**Claim 1**: $\{v'_1, v''_1\} \neq \{v_2, v_m\}$.

In fact, if $\{v'_1, v''_1\} = \{v_2, v_m\}$, then since $v'_1v''_1 \not\in E(G)$ and since the interior of the rim cycle of $v$ consists of $v$ only, we have $|N(v_1)| = 3$, contrary to Lemma 3.3.

Therefore, we assume from now on that one can choose $v, v_1$ and $v'_1$ such that $v'_1 \not\in \{v_2, v_m\}$.

**Claim 2**: One can choose $v, v_1$ and $v'_1$ so that $v'_1 \not\in N(v)$.

Suppose that $v'_1v \in E(G)$. Since $v'_1 \not\in \{v_2, v_m\}$, we may assume that $v_2$ is in the interior of $C_v = vv_1v'_1v$ and $v''_1$ is in the exterior of $C_v$. For each cyclic vertex $v$ with a noncyclic neighbor $v_1$, we choose $v_1$ so that

$$|IntC_v| \text{ is minimized},$$

(14)

subject to the condition that $C_v$ separates $v_2$ and $v''_1$.

If $v_2$ is a cyclic vertex, then since $v''_1$ lies in $ExtC_v$, $v''_1 \not\in N(v_2)$. Thus one can replace $v, v_1, v'_1$ by $v_2, v_1, v''_1$, respectively, and so Claim 2 holds.
If \( v_2 \) is not a cyclic vertex, then there are two vertices \( v'_2, v''_2 \) that are a bad pair in \( N(v_2) \). If \( v'_2 \notin N(v) \), then \( v, v_1, v'_1 \) can be replaced by \( v, v_2, v'_2 \). Suppose that \( v'_2 \in N(v) \). Let \( C'_v = v_2v'_2v \). Then we may assume that \( IntC'_v = \emptyset \). For otherwise \( C'_v \) separates a vertex in \( N(v_2) \) and \( v''_2 \), violating (14).

If \( v''_2 = v_1 \), then \( d_G(v_2) = 3 \) and \( v_1, v'_2 \notin E(G) \), violating Lemma 3.3. Thus \( v''_2 \) must be a vertex in the interior of the cycle \( vv'_2v_2v_1v'_1v \) and so \( v''_2 \notin N(v) \). Hence we can replace \( v, v_1, v'_1 \) by \( v, v_2, v'_2 \) and so Claim 2 holds.

**Claim 3:** One can choose \( v, v_1 \) and \( v'_1 \) so that \( v'_1 \notin N(v) \) and \( v'_1v_m \notin E(G) \) or \( v'_1v_2 \notin E(G) \).

By Claim 2, we can have \( v, v_1 \) and \( v'_1 \) so that \( v'_1 \notin N(v) \). If \( v'_1v_2, v'_1v_m \in E(G) \), then by (13), \( v'_1 \notin \{ v_2, v_m \} \) and without loss of generality, we may assume that \( v'_1 \) is in the interior of \( vv_1v'_1v_2v \). Thus \( v'_1 \notin N(v) \) and \( v'_1v_m \notin E(G) \), and so one can replace \( v, v_1, v'_1 \) by \( v, v_1, v''_2 \) to establish Claim 3.

By Claim 3, we can choose a cyclic vertex \( v \), and a noncyclic vertex \( v_1 \in N(v) \) with ordered neighborhoods

\[
N(v) = \langle v_1, \ldots, v_m \rangle \quad \text{and} \quad N(v_1) = \langle u_1, \ldots, u_s \rangle,
\]

such that \( u_1 = v \), \( u_2 = v_2 \) and

\[
u_ju_{j+1} \in E(G), \quad (1 \leq j \leq i - 1), \quad u_i \notin N(v), u_iu_{i+1}, u_iv_m \notin E(G).
\]  \hspace{1cm} (15)

In other words, \( u_i, u_{i+1} \) are a bad pair. Let \( P \) denote the \((u_2, u_{i-1})\)-path \( u_2u_3 \cdots u_{i-1} \). (Note that \( P \) may consist of a single vertex \( u_2 = v_2 \) if \( u_{i-1} = v_2 \).

**Claim 4** \( \{v_1, u_i\} \) is not a vertex cut of \( G \).

**Proof of Claim 4:** Suppose that \( \{v_1, u_i\} \) is a vertex cut of \( G \). Let \( L_1 \) be the \( \{v_1, u_i\} \)-component of \( G \) containing \( v \). Note that \( v_2, v_m, u_{i-1} \) are also in \( V(L_1) \). Let \( L_2 \) be the union of other \( \{v_1, u_i\} \)-components of \( G \). For convenience, we assume that \( L_1 \) always contains only one edge in \([v_1u_i]\), even when \( |[v_1u_i]| = 2 \). Thus

\[
\mu(L_1) + \mu(L_2) = \mu(G),
\]  \hspace{1cm} (16)

Let \( L'_1 = (L_1 - u_iu_{i-1})/\{v_1u_i\} \) and denote by \( v' \) the vertex in \( L'_1 \) to which \( v_1u_{i-1} \) is contracted. Since \( v', v, u_{i-1}, u_{i-2} \in V(L'_1) \) and since \( u_i, v_1 \in E(L_2) \), both \( \kappa(L'_1) \geq 2 \) and \( \kappa(L_2) \geq 2 \). We claim that

\[
cc(G) \leq cc(L'_1) + cc(L_2).
\]  \hspace{1cm} (17)

Let \( C_1 \) and \( C_2 \) be two CC's of \( L'_1 \) and \( L_2 \), respectively. Let \( C_1 \in C_1 \) be a cycle containing \( v'u_{i-1} \) and let \( C_2 \in C_2 \) be a cycle containing \( v_1u_i \). Let \( C'_1 = C_1 - v'u_{i-1} + v_1u_i \) (if \( C_1 - v'u_{i-1} \) is a \((v_1, u_i)\)-path) or \( C'_1 = C_1 - v'u_{i-1} + u_{i-1}v_1u_i \) (if \( C_1 - v'u_{i-1} \) is a \((u_i, u_{i-1})\)-path), and let \( C'_2 = C_2 - v_1u_i + v_1u_{i-1}u_i \). Then \( (C_1 - \{C_1\}) \cup (C_2 - \{C_2\}) \cup \{C'_1, C'_2\} \) is a CC of \( G \), and so (17) holds.

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By Lemma 3.4, $|V(L_1)| \geq 6$ and $|V(L_2)| \geq 6$. If $|V(L'_1)| \geq 6$, or if $|V(L'_1)| = 5$ and $cc(L'_1) = 2$, then by (17) and by (8), $G$ satisfies (6), contrary to (7). Hence we may assume that $L'_1$ is spanned by $J_2$. If $\mu(L_1) = 1$, then by (16), $\mu(L_2)$ is simple, and so by (17) and by Lemma 2.2,

$$cc(G) \leq cc(L_2) + cc(L'_1) \leq \frac{2(n-4) - 2}{3} + 3 < \frac{2n-2}{3} + \frac{1}{2},$$

contrary to (7). Thus $\mu(L_1) = 0$. Then by (v) of Lemma 2.2, $L'_1$ can be covered by 3 cycles so that $v'u_{i-1}$ is covered 3 times. It follows that $L_1$ can have 3 cycles $C'_1, C''_1$ and $C''''_1$ (say) so that $v_1u_i$ is covered by $C'_1$ and $C''_1$. Let $C_2$ be a CC of $L_2$ with $C_2 \in C_2$ so that $v_1u_i \in E(C_2)$. Then $(C_2 - \{C_2\}) \cup \{C'_1, C''_1, C''''_1, E(C''''_1) \cup E(C_2) - \{v_1u_i\}\}$ is a CC of $G$ with at most $(2(n-4) - 2)/3 + \mu(G)/2 + 2 < (2n-2)/3 + \mu(G)/2$ cycles, contrary to (7). This proves Claim 4.

**Case 1** Both $|\{v_1\}| = |\{v_1u_i\}| = 1$.

Define $G_1 = (G - \{v_1v_2, u_{i-1}u_i, v_1v_m\})/\{v_1v_2, v_1u_i\}$ (see Figure 2). We shall show

$$cc(G) \leq cc(G_1) + 1. \quad (18)$$

Let $v'$ to denote the vertex in $G_1$ to which $v_1v$ and $v_1u_i$ are contracted. By Claim 4, $v'$ is not a cut vertex of $G$ and so $\kappa(G_1) \geq 2$. Let $C$ be a CC of $G_1$ and let $C_1, C_2, C_3 \in C$ such that $v'v_m \in C_1, v'v_2 \in C_2$ and $v'u_{i-1} \in C_3$, and such that $C_1 = C_2$ whenever $C_1$ contains 2 edges in $\{v'v_m, v'v_2, v'u_{i-1}\}$ that are assumed to be in $C_1$ and $C_2$, respectively. Denote

$$C'_1 = G[E(C_1) - \{v'v_m\}], C''_2 = G[E(C_2) - \{v'v_2\}], C''''_3 = G[E(C_3) - \{v'u_{i-1}\}]. \quad (19)$$

Note that any cycle $L$ in $G - \{C_1, C_2, C_3\}$ can be extended to a cycle $L'$ in $G$ by adding edges in $\{v_1v_2, u_{i-1}u_i, v_1v_m, v_1v, v_1u_i\}$, if necessary. In each of the subcases below, we shall exhibit a CC $\{L': L \in C\} \cup \{F\}$ of $G$ and so (18) follows.

**Case 1A** $|E(C'_j) \cap \{v'v_m, v'v_2, v'u_{i-1}\}| = 1, (1 \leq j \leq 3)$.

Then $C'_1$ is a $(v_m, w_1)$-path, $C''_2$ is a $(v_2, w_2)$-path and $C''''_3$ is a $(u_{i-1}, w_3)$-path, for some $w_j \in \{v_1, v_2, u_i\}, (1 \leq j \leq 3)$.

If $w_1 = v$ in $G$, then set $C'_1 = C''''_3 + v_mv_1v$ and extend $C''_2, C''''_3$ to cycles $C'_1$ and $C'''_3$ in $G$, respectively, so that $v_1v_2 \in E(C_2)$ and $v_1u_i \in E(C_3)$. Note that since $C''''_3$ is a $(u_{i-1}, w_3)$-path, either $v_1u_{i-1}$ or $u_{i-1}u_i$ is in $E(C''''_3)$. Let $F$ be a cycle in $G$ that contains $P$, $v_mv, v_2v$, and either $v_1u_{i-1}$ or $u_{i-1}u_i$, depending on whether $u_{i-1}u_i \in E(C''''_3)$ or $v_1u_{i-1} \in E(C''''_3)$, respectively.

If $w_1 = v_1$ or $w_1 = u_i$, then let $C'_1 = C''''_3 + v_mv_1v$ or $C'_1 = C''''_3 + v_mv_1u_i$, respectively. Extend $C''_2, C''''_3$ to cycles $C'_1$ and $C''''_3$ in $G$, respectively, so that $v_1v_2 \in E(C_2)$ and $v_1u_i \in E(C_3)$. Note again that either $v_1u_{i-1}$ or $u_{i-1}u_i$ is in $E(C''''_3)$. Let $F$ be a cycle in $G[\{v\} \cup N(v)]$ that contains $P$, $v_mv, v_2v$ and either $v_1u_{i-1}$ or $u_{i-1}u_i$, depending on whether $u_{i-1}u_i \in E(C''''_3)$ or $v_1u_{i-1} \in E(C''''_3)$, respectively. (See Table 5 at the end for details.)
Case 1B: $C_1 = C_2$. Then $C_1 \neq C_3$ and $C''_1$ is a $(v_m, u_2)$-path in $G$. Let $C''_1 = C''_1 + v_m u_1 v_2$ and extend $C''_1$ to a cycle $C'_3$ in $G$ so that $v_1 u_i \in E(C'_3)$. Note that either $v_1 u_{i-1}$ or $u_{i-1} u_i$ is in $E(C'_3)$. Let

$$F = \begin{cases} P + v_2 v_m v_1 u_{i-1} & \text{if } u_{i-1} v_i \in E(C'_3) \\ P + v_2 v_m v_1 u_i u_{i-1} & \text{if } u_{i-1} v_i \not\in E(C'_3). \end{cases}$$

Case 1C: $C_1 = C_3$. Then $C_1 \neq C_2$ and $C''_1$ is a $(v_m, u_{i-1})$-path in $G$. Let $C''_1 = C''_1 + v_m v_1 u_i u_{i-1}$ and extend $C''_1$ to a cycle $C'_2$ in $G$ so that $v_1 v_2 \in E(C'_2)$. Let $F = P + v_2 v_m v_1 u_{i-1}$.

Case 1D: $C_2 = C_3$. Then $C_2 \neq C_1$ and $C''_1$ is a $(v_2, u_{i-1})$-path in $G$. Let $C''_1 = C''_1 + v_2 v_1 u_i u_{i-1}$ and extend $C''_1$ to a cycle $C'_1$ in $G$ so that $v_1 v \in E(C'_1)$. Let $F = P + v_2 v_m v_1 u_{i-1}$.

By (18) and by (8), we may assume that $|V(G_1)| \leq 5$. Since $|V(G)| \geq 6$, we have $|V(G_1)| \geq 4$. By Corollary 3.5, $\delta(G_1) \geq 3$. If $cc(G_1) \leq 2$, then by (18), $cc(G) \leq 3$, contrary to (7). Hence we may assume that $G_1$ is spanned by $J_2$. By (iii) of Lemma 2.2, $cc(G_1) \leq 3$ and so by (18), $cc(G) \leq 4$. Since $n = |V(G_1)| + 2 = 7$, $G$ satisfies (6), contrary to (7).

Case 2 $|\{v_1 v_2\}| = 2$.

Let $G_2 = (G - \{v_1 v_m, v_1 v_2\})/\{v_1 v_2\}$. Let $v'$ denote the vertex in $G_2$ to which $v$ and $v_1$ are contracted, and let $C$ be a CC of $G_2$. We shall show that

$$cc(G) \leq cc(G_2) + 1. \quad (20)$$

Let $C_1, C_2 \in C$ be two cycles such that $v' v_m \in E(C_1)$ and $v' v_2 \in E(C_2)$. If $C_1 = C_2$, then let $C' = C_1 - \{v' v_m, v' v_2\} + v_m v_1 v_2$ and $F = vv_2 v_3 \cdots v_m v_1 v$. Note that both edges in $[v_1 v_2]$ are covered by $C'$ and $F$. Thus $(C - \{C_1\}) \cup \{C', F\}$ is a CC of $G$, and so (20) holds.

If $C_1 \neq C_2$, then $C''_i = C_i - \{v' v_n, v' v_2\}$ is a $(v_n, u_{i-1})$-path in $G$ for some $w_i \in \{v, v_1\}$, $(1 \leq i \leq 2)$. Table 4 defines $C'_i$ and $C''_i$ and $F$ so that $(C - \{C''_i, C'_{i+1}\}) \cup \{C'_i, C'_i\}$ is a CC of $G$.

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$C'_1$</th>
<th>$C''_1$</th>
<th>$C'_2$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$</td>
<td>$v$</td>
<td>$C'_1 + v_m v_1 v$</td>
<td>$C''_1 + v_2 v$</td>
<td>$v_1 v_2 \cdots v_m v$</td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>$v_1$</td>
<td>$C'_1 + v_m v_1 v$</td>
<td>$C''_1 + v_2 v_1$</td>
<td>$v_1 v_2 \cdots v_m v$</td>
<td></td>
</tr>
<tr>
<td>$v_1$</td>
<td>$v$</td>
<td>$C'_1 + v_m v_1 v$</td>
<td>$C''_1 + v_2 v_1 v$</td>
<td>$v_2 \cdots v_m v_1 v$</td>
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</tr>
<tr>
<td>$v_1$</td>
<td>$v_1$</td>
<td>$C'_1 + v_m v_1 v$</td>
<td>$C''_1 + v_2 v_1 v$</td>
<td>$v_2 \cdots v_m v_1 v$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4

Note that both edges in $[v_1 v_2]$ are covered by two of $C'_1, C'_2$ and $F$, and so (20) holds.
Since the rim cycle of \( v \) consists of \( v \) only, we have \( \kappa(G_2) \geq 2 \). By \( ||[v_1u_i]|| = 2 \) and by Lemma 3.1, \( G_2 \) is simple. If \( n \geq 7 \), then by (8) and (20),
\[
cc(G) \leq cc(G_2) + 1 \leq \frac{2(n-1) - 2}{3} + 1 < \frac{2n - 2}{3} + \frac{\mu(G)}{2},
\]
contrary to (7). Hence we assume that \( n = 6 \) and so \( |V(G')| = 5 \). By (15), \( G_2 \) is not a triangulation and so \( G_2 \neq J_2 \), which implies by Lemma 2.2 that \( cc(G_2) = 2 \). Thus by (20), \( cc(G) \leq 3 \), contrary to (7).

**Case 3** \( ||[v_1u_i]|| = 2 \).

Let \( G_3 = (G - u_iu_{i-1})/\{v_1u_i\} \) and let \( v' \) denote the vertex in \( G_3 \) to which \( v_1u_i \) is contracted. We shall show that
\[
cc(G) \leq cc(G_3) + 1 \tag{21}
\]

Let \( C \) be a CC of \( G_3 \) and let \( C \in C \) be a cycle with \( v'u_{i-1} \in E(C) \). Note that \( C'' = C - v'u_{i-1} \) is a \((u_{i-1}, w)\)-path in \( G \) for \( w \in \{v_1, u_i\} \). Let \( F = v_1u_{i-1}v_1 \). If \( w = v_1 \), then set \( C' = C'' + u_{i-1}u_{i+1}v_1 \), and if \( w = u_i \), then set \( C' = C'' + u_{i-1}u_{i+1}u_i \). Thus \( (C - \{C\}) \cup \{C', F\} \) is a CC of \( G \) and so (21) must hold.

By Claim 4, \( \kappa(G_3) \geq 2 \). By (21) and by (7), we may again assume that \( G_3 \cong J_2 \). By \( ||[v_1u_i]|| = 2 \) and by Lemma 3.2, \( d_{G}(u_i) \geq 4 \) and so \( d_{G_3}(v') \geq 4 \) also. It follows that \( v_2 = u_{i-1} \) in \( G \) and so \( v' \in \{z_3, z_2, z_4\} \) in \( J_2 \) (using the notation of Figure 1). By the symmetry of \( J_2 \), we may assume \( v' = z_2 \). It follows that either \( G \cong J_3 \) with \( ||[u_iz_2]|| = 2 \) (\( v = z_1, v_1 = z_3, v_2 = z_2, v_m = z_4 \), or \( G \cong J_4 \) with \( ||[u_iz_3]|| = 2 \) (\( v = z_4, v_1 = z_3, v_2 = z_3, v_m = z_1 \)). By Lemma 2.4, \( G \) satisfies (6), contrary to (7). \( \square \)

**Lemma 3.7** \( G \) does not have a nontrivial 4-cycle \( C \) with \( IntC = \{v\} \).

**Proof**: By contradiction, assume that \( C = v_1v_2v_3v_4v_1 \) is a 4-cycle in \( G \) with \( IntC = \{v\} \). Thus \( N(v) \subseteq V(C) \). By Lemma 3.6, \( N(v) \neq V(C) \). If \( |N(v)| = 2 \), then by Corollary 3.5, \( v \) must have degree 3, contrary to Lemma 3.2. If \( |N(v)| = 3 \), then by Lemma 3.6, \( v \) is not a cyclic vertex and so Lemma 3.3 must be violated. \( \square \)

**Lemma 3.8** If \( \Gamma_3 \) is a nontrivial 3-cycle of \( G \) such that
\[
|Int\Gamma_3| \text{ is minimized.} \tag{22}
\]

then each holds:

(i) \( |Int\Gamma_3| > 1 \);

(ii) for any \( v \in Int\Gamma_3 \) and for any consecutive bad pair \( vv_1, vv_2 \in N(v) \) \((vv_1 \text{ and } vv_2 \text{ are incident with the same face})\), neither \( vv_1 \) nor \( vv_2 \) lies in a 3-cycle of \( G \).

**Proof**: If \( |Int\Gamma_3| = 1 \), then either Lemma 3.6 or Lemma 3.3 would be violated, and so (i) of Lemma 3.8 follows.

We shall show (ii) by contradiction. Suppose that \( vv_1v_3v \) is a 3-cycle. Let \( e \notin E(G) \) be an edge parallel to \( vv_3 \). Since \( v \in Int\Gamma_3 \) and by (22), \( vv_1v_3v_1 \) must be a trivial 3-cycle. By
Lemma 3.1, \([|qq|] = 2\) in \(G\). Let

\[
G_4 = \begin{cases} 
G/qq & \text{if } |qq| = 1 \\
(G + e)/qq & \text{if } |qq| = 2 
\end{cases}
\]

We shall show \(cc(G) \leq cc(G_4)\). Let \(v'\) denote the vertex in \(G_4\) to which \(qq\) is contracted and let \(C'\) be a CC of \(G_4\). We consider two cases.

**Case 1:** \(|qq| = 2\) in \(G\). Then \(|v'v_3| = 3\) in \(G_4\) and we may assume that \([v'v_3] = \{e_1, e_2, e_3\}\). Let \(C' \in C'\) be a cycle containing \(e_i\), \((1 \leq i \leq 3)\). One can adjust the cycles in \(C' - \{C'_1, C'_2, C'_3\}\) to cycles in \(G\) and denote this family of cycles in \(G\) by \(C\).

Suppose that \(C_1 = C'_2 = G_4[\{e_1, e_2\}] \in C'\). Note that \(C'_2 - e_3\) is a \((v_3, w_1)\)-path, with \(w \in \{v, v_1\}\). If \(w = v\), then set \(C_3 = C'_3 = e_3 + v_3v_1v\), and if \(w = v_1\), then set \(C_3 = C'_3 - e_3 + v_3v_1v_1\). In any case, \(C \cup \{C_3\}\) is a CC of \(G\) and so \(cc(G) \leq cc(G_4)\).

Hence we may assume that the \(C_i\)'s are distinct. Note that \(C'_i - e_i\) is a \((v_3, w_1)\)-path with \(w_i \in \{v, v_1\}\), \((1 \leq i \leq 3)\). If \(w_i \neq w_j\) for some \(i\) and \(j\), say \(w_i = v\) and \(w_2 = v_1\), then set \(C_1 = C'_1 - e_1 + v_3v_1v, C_2 = C'_2 - e_2 + v_3v_1v\), and \(C_3\) be a cycle in \(G\) obtained by extending \(C'_2 - e_3\). If \(w_i = v, (1 \leq i \leq 3)\), then set \(C_1 = C'_1 - e_1 + v_3v, C_2 = C'_2 - e_2 + v_3v_1v\) and \(C_3 = C'_3 + e_3 + v_3v_1v\). If \(w_i = v_1, (1 \leq i \leq 3)\), then set \(C_1 = C'_1 - e_1 + v_3v_1, C_2 = C'_2 - e_2 + v_3v_1v\) and \(C_3 = C'_3 - e_3 + v_3v_1v\). In any case, \(C \cup \{C_1, C_2, C_3\}\) is a CC of \(G\) and so \(cc(G) \leq cc(G_4)\).

Since \(G\) is plane, \(v_1\) is incident with two faces, one being the 3-cycle \(v_1v_2v_3v\) and the other being \(v_1v_2 \cdots v_1\), which is a cycle of length at least 4. Therefore \(\mu(G_4) = \mu(G) + 1\), and so if \(n \geq 7\), then by (8),

\[
cc(G) \leq cc(G_4) \leq \frac{2(n - 1) - 2}{3} + \frac{\mu(G) + 1}{2},
\]

contrary to (7). Thus we may assume that \(n = 6\) and \(|V(G_4)| = 5\). Then by (ii) or (iii) of Lemma 2.2, \(cc(G) \leq cc(G_4) \leq 3\), contrary to (7).

**Case 2:** \(|qq| = 1\). The proof is similar to and simpler than that of Case 1, and so it is omitted.

Since the minimality of \(\Gamma_3\) is used in the proof of Lemma 3.8 only to guarantee that \(v_1v_2v\) is a trivial 3-cycle (and so \(\mu(G_4) = \mu(G) + 1\)), we have also proved:

**Corollary 3.9:** If \(G\) has no nontrivial 3-cycles, then for any \(v \in V(G)\) and for any consecutive bad pair \(v_1, v_2 \in N(v)\), neither \(v_1v\) nor \(v_2v\) lies in a 3-cycle of \(G\).

**Lemma 3.10** If \(\Gamma_3\) is a nontrivial 3-cycle of \(G\) such that (22) holds, then \(G[V(\Gamma_3) \cup Int \Gamma_3]\) does not contain a nontrivial 4-cycle.

**Proof:** By contradiction, we choose a nontrivial 4-cycle \(C\) with \(|Int C|\) is minimized.

\[
|Int C| \text{ is minimized.} \tag{23}
\]

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Let $C = u_1u_2u_3u_4u_1$ and let $v \in IntC$. Let $N(v) = < v_1, v_2, \ldots, v_m >$ be an ordered neighborhood of $v$. By Lemma 3.6, we may assume that $v_1v_2 \not\in E(G)$, and so at most one of $v_1$ and $v_2$ is in $V(\Gamma_3)$. Note that

$$|\{v_1, v_2\} \cap V(C)| \leq 1.$$  \hfill (24)

For if $v_1, v_2 \in V(C)$, then since $v_1v_2 \not\in E(G)$, we may assume that $v_1 = u_1$ and $v_2 = u_3$. By Lemma 3.7, $|IntC| > 1$ and so one of the 4-cycles $vu_1u_2u_3v$ and $vu_1u_4u_3v$ is nontrivial, contrary to (23). Similarly, by (23) and by Lemma 3.7, we have,

$$|N(v_1) \cap N(v_2) - \{v\}| \leq 1.$$  \hfill (25)

Define

$$G_5 = \begin{cases} 
G/\{v_1v, v_2v\} & \text{if } v_1v_2 \text{ does not lie in a 4-cycle of } G \\
(G - v_1v)/\{v_1v, v_2v\} & \text{if for some } w \in V(G), wv_1v_2w \text{ is a 4-cycle}
\end{cases}$$

and we shall show that

$$cc(G) \leq cc(G_5) + 1.$$  \hfill (26)

By Lemma 3.8, neither $vu_1$ nor $vu_2$ lies in a 3-cycle. This, together with (25), implies that no new multiple edges will be created in getting $G_5$, and so $\mu(G_5) \leq \mu(G)$.

**Case 1:** $v$ does not lie in a 4-cycle of $G$, and so $G_5 = G/\{v_1v, v_2v\}$.

Let $C'$ be a CC of $G_5$. Note that every cycle $L' \in C'$ can be extended to a cycle $L$ in $G$, by using edges in $\{vu_1, vu_2\}$, if necessary. By $\kappa(G) \geq 2$, there is a cycle $F$ in $G$ containing $vu_1vu_2$. Hence $\{L | L' \in C' \} \cup \{F\}$ is a CC of $G$, and so (26) holds.

**Case 2:** $F = vu_1vu_2v$ is a 4-cycle of $G$, and so $G_5 = (G - v_1v)/v_1v, v_2v$.

Note that $N(v_1) \cap N(v_2) = \{v, w\}$. Using the notation in Case 1, we conclude that $\{L | L' \in C' \} \cup \{F'\}$ is also a CC of $G$, and so (26) holds.

By (8) and (26), we may assume that $|V(G_5)| = 5$. By Lemma 2.2, $cc(G_5) \leq 3$, and so by (26), $cc(G) \leq 4$. Since $n = |V(G_5)| + 2 = 7$, $G'$ satisfies (6), contrary to (7). \hfill $\square$

Using Corollary 3.9 in place of (ii) of Lemma 3.8 in the proof of Lemma 3.10, we have, Corollary 3.11 If $G$ has no nontrivial 3-cycles, then $G$ has no nontrivial 4-cycles. \hfill $\square$

The argument in the proof for Lemma 3.10 also proves the following:

**Corollary 3.12** Let $C$ be a nontrivial cycle in $G$ such that $IntC$ has no nontrivial 4-cycles. Let $v \in IntC$ and define $G_5$ as in Lemma 3.10. Then $cc(G) \leq cc(G_5) + 1$.

**Proof of Theorem 1.2, continued** If $G$ has a nontrivial 3-cycle, then $G$ has a nontrivial 3-cycle $\Gamma_3$ such that (22) holds. Let $v \in Int\Gamma_3$. By Lemma 3.6, there are $v_1, v_2 \in N(v)$ forming a consecutive bad pair in $N(v)$. By Lemma 3.10, (25) must hold. Define $G_5$ as in
Lemma 3.10. By Lemma 3.10 and Corollary 3.12, \( cc(G) \leq cc(G_5) + 1 \), and so by (8) and (26), \( |V(G_5)| = 5 \), which implies that \( G \) is not a counterexample, contrary to (7).

If \( G \) does not have a nontrivial 3-cycle, then by Corollaries 3.11 and 3.12, one can define \( G_5 \) and argue as above to obtain a contradiction to (7).

Since contradiction arises in any case, the proof of Theorem 1.2 is now completed. \( \square \)

REFERENCES


[2] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications", American El-


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Figure 1: $J_1$ and $J_2$

Figure 2: $G$ and $G_1$ in Lemma 6

Figure 3: $J_3$ and $J_4$
Figure 4: Graphs $L_8, L'_8, L''_8$ and $L'''_8$