Graph whose edges are in small cycles

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Abstract


Paulraj (1987) conjectured the following:

(i) If every edge of a 2-connected graph \( G \) lies in a cycle of length at most 4 in \( G \), then \( G \) has a dominating closed trail.

(ii) If, in addition, \( \delta(G) \geq 3 \), then \( G \) has a closed spanning trail.

Collapsible graphs are defined and studied by Catlin (1988). Catlin showed that if \( H \) is a collapsible subgraph of \( G \), then \( G \) has a spanning closed trail if and only if \( G/H \), the graph obtained from \( G \) by contracting \( H \), has a spanning closed trail. Paulraj (1987) conjectured that a graph satisfying the hypothesis of (ii) is collapsible. In this paper, all three conjectures are proved.

Introduction

We shall use the notation of Bondy and Murty [1] except for contractions. A graph may have multiple edges but not loops. A spanning closed trail of \( G \) is called a spanning eulerian subgraph (SES) of \( G \). The collection of graphs that have an SES is denoted by \( SE \). Note that \( K_1 \in SE \). If a closed trail \( C \) of \( G \) satisfies \( E(G - V(C)) = \emptyset \), then \( C \) is called a dominating eulerian subgraph (DES) of \( G \).

Let \( G \) be a graph. An block of \( G \) that has exactly one cut-vertex of \( G \) is called an end block of \( G \). A block \( B \) is acyclic if \( B \cong K_2 \). For a subset \( X \subseteq E(G) \), the contraction \( G/X \) is the graph obtained from \( G \) by identifying the ends of each edge in \( X \) and then deleting the resulting loops. If \( H \) is a subgraph of \( G \), then we write \( G/H \) for \( G/E(H) \).

Let \( W \) be a subgraph of \( G \). If for some \( t \geq 2 \), \( W \cong K_{2,t} \), then \( W \) is called a \( W_{2,t} \)-subgraph of \( G \). Let \( C = v_1v_2v_3v_4v_5v_6v_1 \) be a 6-cycle. Define \( \Theta \) to be the graph with \( V(\Theta) = V(C) \) and \( E(\Theta) = E(C) \cup \{v_2v_5\} \).

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Let $H$ be a subgraph of $G$. Define $H^c = G[E(G) - E(H)]$ and $A_G(H) = V(H) \cap V(H^c)$. The vertices in $A_G(H)$ are called the vertices of attachment of $H$ in $G$. If $\kappa(H) \geq 2$ and $|A_G(H)| = 2$, then $H$ is called a 2-block of $G$.

A graph $G$ is an edge-disjoint union of $K_{2,n_i}$'s if there is a partition $E_1, E_2, \ldots, E_k$ of $E(G)$ such that $G[E_i] \cong K_{2,n_i}$'s ($n_i \geq 2$ and $1 \leq i \leq k$). Note that when $i \neq j$, we may have $n_i \neq n_j$.

We consider the following conditions.

\begin{align*}
\text{(1)} & \quad \text{Any edge of } G \text{ is an } m\text{-cycle of } G, \ m \leq 4; \\
\text{(2)} & \quad G \text{ is an edge-disjoint union of } K_{2,n_i}\text{'s.}
\end{align*}

Let $\mathcal{G} = \{G: G \text{ satisfies (1) with } \kappa(G) \geq 2\}$ and $\mathcal{G}_1 = \{G \in \mathcal{G}: \delta(G) \geq 3\}$. In [6–8], Paulrajaa raised the following two conjectures.

**Conjecture 1.** If $G \in \mathcal{G}_1$, then $G \in \mathcal{L}$.

**Conjecture 2.** If $G \in \mathcal{G}$, then $G$ has a DES.

A graph $G$ is collapsible if for every even subset $R \subseteq V(G)$, $G$ has a subgraph $\Gamma_R$ (called the $R$-subgraph of $G$) such that $G - E(\Gamma_R)$ is connected and $R$ is the set of odd-degree vertices of $\Gamma_R$. The collection of all collapsible graphs is denoted by $\mathcal{CL}$. Let $H_1, H_2, \ldots, H_c$ be all the maximal collapsible subgraphs of $G$. Denote by $(G)_1$ the graph of order $c$ obtained from $G$ by contracting $H_1, \ldots, H_c$ to $c$ distinct vertices. We call $(G)_1$ the reduction of $G$ and $H_1, H_2, \ldots, H_c$ the preimages of the vertices of $(G)_1$. A graph $G$ is reduced if $G = (G)_1$. In [2], Catlin showed that $(G)_1$ is well defined and unique. He also proved the following theorem.

**Theorem A** (Catlin [2]). Let $G$ be a graph.

- (i) If $G \in \mathcal{CL}$, then $G \in \mathcal{L}$.
- (ii) $G \in \mathcal{L}$ if and only if $(G)_1 \in \mathcal{L}$.
- (iii) If $H \in \mathcal{CL}$ is a subgraph of $G$, then $G \in \mathcal{CL}$ if and only if $G/H \in \mathcal{CL}$.
- (iv) $G$ is reduced if and only if $G$ has no nontrivial subgraphs in $\mathcal{CL}$. In particular, $G$ has no 3-cycles.
- (v) If $H_1, H_2 \in \mathcal{CL}$ are two subgraphs of $G$ and if $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2 \in \mathcal{CL}$.
- (vi) For any graph $G$, $(G)_1$ is reduced.

Catlin [3] made the following conjecture.

**Conjecture 3.** If $G \in \mathcal{G}_1$, then $G \in \mathcal{CL}$.

By (i) of Theorem A, we have the following proposition.
Proposition 1. Conjecture 3 implies Conjecture 1.

For each \( i = 1, 2, \ldots \), define

\[
D_i(G) = \{ v \in V(G) : \deg_G(v) = i \},
\]

and

\[
D^*_i(G) = \{ v \in V(G) : \deg_G(v) \geq i \}.
\]

Proposition 2. Conjecture 1 implies Conjecture 2.

Proof. Assume that Conjecture 1 holds. Let \( G \in \mathcal{G} \). Since an SES is always a DES, we may assume that \( G \not\in \mathcal{F} \mathcal{L} \) and \( G \in \mathcal{G} - \mathcal{G}_1 \). Thus \( \delta(G) = 2 \). Let

\[
D_2(G) = \{ x_1, x_2, \ldots, x_m \}.
\]

Since \( \kappa(G) \geq 2 \),

\[
\text{no vertex in } D_2(G) \text{ is the common end of multiple edges} ; \tag{3}
\]

and by (1) and \( \kappa(G) \geq 2 \),

\[
\text{no paths in } G \text{ contain three consecutive vertices in } D_2(G) . \tag{4}
\]

Choose \( y_1, y_2, \ldots, y_m \) such that \( e_i = x_i y_i \in E(G) \), \( 1 \leq i \leq m \), and such that if for some \( i, j \) with \( x_i x_j \in E(G) \) then \( y_i = x_j \) and \( x_i = y_j \). Note that the latter can happen only when \( x_i \) and \( x_j \) are the internal vertices of a path of \( G \) of length 3. By (3) and (4), the multiplicity of each \( e_i \) is one in \( G \). Let \( e'_i \) be an edge with the same ends as \( e_i \) but \( e'_i \not\in E(G) \), \( 1 \leq i \leq m \). Obtain a new graph \( G' \) by adding \( \{ e'_i, e'_2, \ldots, e'_m \} \) to \( G \). It follows that \( G' \in \mathcal{G} \) and so \( G' \in \mathcal{F} \mathcal{L} \), by Conjecture 1. Let \( C' \) be an SES of \( G' \). Since \( G \not\in \mathcal{F} \mathcal{L} \), we may assume that for some \( k \leq m \),

\[
\{ e'_1, e'_2, \ldots, e'_k \} \cup \{ e_1, e_2, \ldots, e_k \} \subseteq E(C') ; \tag{5}
\]

and that \( e'_j \not\in E(C') \), \( j > k \). Choose \( C' \) so that (5) is satisfied and that \( k \) is as small as possible. Let \( C = C' - \{ e_1, \ldots, e_k, e'_1, \ldots, e'_k \} \). Then every vertex of \( C \) has even degree in \( C \). If \( e'_s = x_s x_t \in E(C') \), for some \( 1 \leq s, t \leq m \), then since \( x_s \) and \( x_t \) are in \( D_2(G) \), \( x_s \) and \( x_t \) are in \( D_2(C') \) also. It follows that \( e_s \not\in E(C') \), contrary to the choice of \( C' \). So if \( x_s x_t \in E(C') \), then \( e_s \in E(C') \) and \( e'_s \not\in (C') \). This implies that \( C \) is connected, and that \( x_1, x_2, \ldots, x_k \) form an independent set in \( G \) and are the only vertices not in \( V(C) \). Thus \( C \) is a DES of \( G \). \( \Box \)

Reductions

Notation 1 (see Fig. 1). Let \( W \) be a \( K_{2,r} \)-subgraph of \( G \) and let \( H \) be a subgraph of \( G \) containing \( W \). Denote

\[
D_2(W) = \{ y_1, y_2, \ldots, y_t \}.
\]
Let $V_1$ be a subset of $D_2(W)$ such that $|V_1| = 2$, let $V_2 = V(W) - V_1$ and let
$\pi = \langle V_1, V_2 \rangle$ denote this partition of $V(W)$. Denote by $H/\pi$ the graph obtained
from $H$ by identifying all vertices of $V_1$ to form a single vertex $v_1$, by identifying
all vertices of $V_2$ to form a single vertex $v_2$, and by joining $v_1, v_2$ with a new edge
$e_\pi = v_1v_2$, so that
$$E(H) - E(W) = E(H/\pi) - \{e_\pi\}.$$  
And finally let $H' = (H/\pi) - e_\pi$.

**Theorem B** (Catlin [3]). Let $W$, $\pi$ and $G/\pi$ be defined in Notation 1. If
$G/\pi \in \mathcal{CL}$, then $G \in \mathcal{CL}$.

**Lemma 1.** If $G' = (G/\pi) - e_\pi \in \mathcal{CL}$, then $G \in \mathcal{CL}$.

**Proof.** It follows from Theorem B and (iii) of Theorem A. \qed

It is easy to see the following.

**Lemma 2** ([5]). If $G \in \mathcal{G}$, then $(G)_1$ has girth 4 and $(G)_1$ satisfies (1).

**Lemma 3** ([5]). Let $B$ be a block of $(G)_1$.

(i) If $G \in \mathcal{G}_1$, then $B \in \mathcal{G}_1$;

(ii) If $G \in \mathcal{G}$, then $B \in \mathcal{G}$.

**Lemma 4.** If $G$ is a counterexample to Conjecture 3 with $|V(G)|$ minimized then $G$
 is reduced.

**Proof.** Let $G$ satisfy the hypothesis of Lemma 4. If $G$ is not reduced, then $(G)_1$, the reduction
of $G$, has smaller order than $G$. By Lemmas 2 and 3, each block of $(G)_1$ is in \mathcal{G}_1 and so by the
minimality of $G$, each block of $(G)_1$ is in \mathcal{CL}. Thus by (v) and (iii) of Theorem A, $G \in \mathcal{CL}$, a contradiction. \qed
Notation 2. Let $C = x_1x_2x_3x_4x_1$ be a 4-cycle of $G$. Define:

$$G^* = (G - \{x_1x_2, x_3x_4\})/\{x_1x_4, x_2x_3\},$$

and denote by $v_1$ and $v_2$ the vertices of $G^*$ to which $x_1x_4$ and $x_2x_3$ are contracted, respectively. For convenience, we regard

$$V(G^*) = [V(G) - V(C)] \cup \{v_1, v_2\}, \quad E(G^*) \cup E(C) = E(G).$$

Lemma 5 [4–5]. If $G^* \in \mathcal{CL}$, then $G \in \mathcal{CL}$.

Lemma 6. Let $G$ be a counterexample to Conjecture 3 with $|V(G)|$ minimized. Then $G$ does not have a subgraph isomorphic to $\Theta$.

Proof. By contradiction, suppose that $G$ has $\Theta$ as a subgraph. Let $C = x_1x_2x_3x_4x_5x_6x_1$ denote the 6-cycle of $\Theta$ and let $C_1 = x_1x_2x_5x_6x_1$ and $C_2 = x_2x_3x_4x_5x_2$ be the two 4-cycles contained in $\Theta$. Define

$$\pi(1) = \langle \{x_1, x_3\}, \{x_2, x_6\} \rangle \quad \text{and} \quad \pi(2) = \langle \{x_2, x_4\}, \{x_3, x_5\} \rangle.$$

Define $G^1 = G/\pi(1)$ and $G^2 = G/\pi(2)$. Let $v_1^1$ and $v_2^1$ denote the vertices of $G^1$ to which $\{x_1, x_3\}$ and $\{x_2, x_6\}$ are mapped, respectively; and let $v_1^2$, $v_2^2$ be the vertices of $G^2$ to which $\{x_3, x_5\}$ and $\{x_2, x_4\}$ are mapped, respectively. Let $e_1 = v_1^1v_2^1$ and $e_2 = v_1^2v_2^2$. We shall regard $C_1 = x_1v_2^2v_1^1x_6x_1$ in $G^2$ and $C_2 = x_3v_2^2v_1^1x_4x_3$ in $G^1$ throughout the proof of this lemma.

Since $G \in \mathcal{G}_1$, we have $\delta(G^1) \geq 3$ and $\delta(G^2) \geq 3$, and both $G^1$ and $G^2$ satisfy (1).

Claim 1. $\kappa(G^1) = \kappa(G^2) = 1$, and all cut vertices of $G^1$ are in $\{v_1^1, v_2^1\}$ and all cut-vertices of $G^2$ are in $\{v_1^2, v_2^2\}$.

Proof. Clearly $G^1$ is connected. If $\kappa(G^1) \geq 2$, then $G^1 \in \mathcal{CL}$ follows by the minimality of $G$. Hence $G \in \mathcal{CL}$ by Theorem B, a contradiction. Thus $\kappa(G^1) = 1$. Since $\kappa(G) = 2$, the cut-vertices of $G^1$ must be in $\{v_1^1, v_2^1\}$. The proof for $G^2$ is similar. □

Claim 2 (see Fig. 2). For $i \in \{1, 2\}$, if $v_i^1$ is a cut-vertex of $G^i$ and if $L_i$ is an end-block of $G^i$ containing $v_i^1$ but not $v_i^2$, then $L_i \in \mathcal{CL}$. A similar result holds when we replace $v_i^1$ by $v_i^2$.

Fig. 2. Claim 2 of Lemma 6.
Proof. Without loss of generality, we assume \( i = 1 \) and \( v^i_1 \) is cut-vertex of \( G^i \) with \( L^i \) an end-block of \( G^i \) containing \( v^i_1 \) but not \( v^i_2 \). Let \( H = G[E(L^i)] \). Then by \( \kappa(G) \geq 2 \), we have \( A(C)(H) = \{ x_1, x_5 \} \). Since \( G \) satisfies (1), either \( H \) satisfies (1) or there is a vertex \( y \in V(H) \) with \( xy_1, xy_5 \in E(H) \).

If \( H \) satisfies (1), then \( L^i \in \mathcal{C}_1 \) and so \( L^i \in \mathcal{C}_2 \), by the minimality of \( G \).

Hence we assume that there is a vertex \( y \in V(H) \) such that \( xy_1, xy_5 \in E(H) \). By \( \kappa(G) \geq 2 \), \( y \) is not a cut-vertex of \( G \). Hence either \( x_1 \) or \( x_5 \) is adjacent to some vertex of \( V(H) - \{ y \} \). It follows that \( \delta(L^i) \geq 3 \). Since (1) holds for \( G \), (1) holds for \( L^i \) also. Thus \( L^i \in \mathcal{C}_2 \) by the minimality of \( G \). □

For \( i \in \{1, 2\} \), let \( B^i \) be the block of \( G' \) containing both \( v^i_1 \) and \( v^i_2 \).

Claim 3. Either \( v^i_1 \) or \( v^i_2 \) has degree less than 3 in \( B^i \).

Proof. By Claim 2, all blocks of \( G' \) other than \( B^i \) are in \( \mathcal{C}_2 \). Clearly \( B^i \) satisfies (1) and \( \kappa(B^i) \geq 2 \). If \( \delta(B^i) \geq 3 \), then \( B^i \in \mathcal{C}_2 \) follows from the minimality of \( G \) and so by (v) of Theorem A and by Theorem B, \( G \in \mathcal{C}_2 \), a contradiction. Hence \( \delta(B^i) < 3 \). Since \( \delta(G) \geq 3 \), every vertex in \( V(B^i) - \{ v^i_1, v^i_2 \} \) has degree at least 3 in \( B^i \). Hence either \( v^i_1 \) or \( v^i_2 \) has degree less than 3 in \( B^i \). □

Claim 4. For \( i, j \in \{1, 2\} \), if \( v^j_1 \) is not a cut-vertex of \( G^i \), then \( v^j_1 \) has degree at least 3 in \( B^i \).

Proof. Without loss of generality, we assume that \( i = j = 1 \). Since the degree of \( x_1 \) is at least 3 in \( G \), \( x_1 \) is incident with an edge \( e' \in E(G) - E(\Theta) \). Since \( v^i_1 \) is not a cut-vertex of \( G^i \), all the edges incident with \( v^i_1 \) in \( G^i \) are in \( B^1 \). Since we map \( x_1 \) and \( x_5 \) to \( v^1_1 \), and since \( e' \in E(G) - E(\Theta) \), \( v^i_1 \) has degree at least 3 in \( B^1 \). □

The proof of Lemma 6 will now be divided into the following cases.

Case 1: For some \( i \in \{1, 2\} \), \( v^i_1, v^i_2 \) are cut vertices of \( G^i \).

Without loss of generality, we assume \( i = 1 \) and that there are end blocks \( L^1_1 \) and \( L^1_2 \) in \( G^1 \), where \( B^1 \notin \{ L^1_1, L^1_2 \} \), with \( A(G)(L^1_j) = \{ v^j_1 \} \) (\( 1 \leq j \leq 2 \)). Let \( H_j = G[E(L^1_j)] \), (\( 1 \leq j \leq 2 \)). Then \( H_1 \) and \( H_2 \) are connected, and \( A(G)(H_1) = \{ x_1, x_5 \} \) and \( A(G)(H_2) = \{ x_2, x_6 \} \). Note that \( H_1 \) and \( H_2 \) also induce subgraphs in \( G^2 \) (which we also call \( H_1 \) and \( H_2 \)), with \( A(G)(H_1) = \{ x_1, v^2_1 \} \) and \( A(G)(H_2) = \{ x_6, v^2_2 \} \).

Therefore, the block \( B^2 \) of \( G^2 \) containing \( v^2_1 \) and \( v^2_2 \) also contains \( C_1 \), \( H_1 \) and \( H_2 \), and so \( v^2_1 \) and \( v^2_2 \) have degree at least 3 in \( B^2 \), contrary to Claim 3.

Case 2: The only cut-vertex of \( G' \) is \( v^1_1 \), (\( 1 \leq i \leq 2 \)).

Let \( L^i \) be an end block of \( G^i \) that does not contain \( v^i_1 \) and has \( v^i_1 \) as its only vertex of attachment in \( G^i \), and let \( H^i = G[E(L^i)] \). Then \( H^i \) is connected with \( A(G)(H^i) = \{ x_1, x_5 \} \). Thus \( x_1 \) is incident with an edge \( e'' \in E(H^i) \). Since \( H^i \) can be regarded as a subgraph of \( B^2 \), \( e'' \) is an edge incident with \( v^2_1 \) in \( B^2 \). Hence \( v^2_1 \) is incident with \( e'' \) and two edges in \( C_1 \) and so \( v^2_1 \) has degree at least 3 in \( B^2 \). Since
\(v_2^2\) is not a cut-vertex of \(G^2\), it follows by Claim 4 that \(v_2^2\) has degree at least 3 in \(B^2\), contrary to Claim 3.

**Case 3** (see Fig. 3): The only cut-vertex of \(G^1\) is \(v_1^1\) and the only cut-vertex of \(G^2\) is \(v_2^2\).

Let \(L^i\) be an end block of \(G^i\) that contains \(v_i^i\) but not \(v_{3-i}^i\) and let \(H^i = G[E(L^i)]\), \((1 \leq i \leq 2)\). Since \(L^i\) is an end block of \(G^i\), \((1 \leq i \leq 2)\), and since \(\kappa(G) \geq 2\),

\[A_G(H^1) = \{x_1, x_3\} \quad \text{and} \quad A_G(H^2) = \{x_2, x_4\}.\]

Now we show that \(G - E(\Theta)\) has a component \(H\) (say) with \(A_G(H) = \{x_3, x_6\} \).

Since \(\delta(G) \geq 3\), \(x_3\) is incident with an edge that is not in \(E(\Theta)\). By \(\kappa(G) \geq 2\), there is an \((x_3, x_j)\)-path \(P\) in \(G - E(\Theta)\) for some \(j \in \{1, 2, 4, 5, 6\}\). If \(j = 1\) or \(j = 5\), then in \(G^1\), \(v_1^1\) has degree at least 3 in \(B^1\). By Claim 4, \(v_2^1\) has degree at least 3 in \(B^1\), contrary to Claim 3. If \(j = 2\) or \(j = 4\), then in \(G^2\), \(v_2^2\) has degree 3 in \(B^2\), and a similar contradiction arises. Hence \(j = 6\) and so \(A_G(H) = \{x_3, x_6\}\).

**Claim 5.** There is no \(y' \in V(G) - \{x_2, x_3\}\) such that either \(y'x_6, y'x_e \in E(G)\) or \(y'x_1, y'x_3 \in E(G)\).

**Proof.** If \(y'x_4, y'x_6 \in E(G)\) and \(y' \neq x_5\), then since \(G\) has a connected subgraph \(H\) with \(A_G(H) = \{x_3, x_6\}\), in \(G^1\), both \(v_1^1\) and \(v_2^1\) have degree at \(\geq 3\) in \(B^1\), violating Claim 3. The proof when \(y'x_1, y'x_3 \in E(G)\) is similar. \(\Box\)

**Subcase 3.1:** There is a vertex \(y \in V(H^1)\) with \(yx_1, yx_3 \in E(G)\).

Then \(x_1yx_5x_2x_1\) is a 4-cycle in \(G\). Let \(\pi(3) = \langle \{y, x_2\}, \{x_1, x_3\} \rangle\) and let \(G^3 = G/\pi(3)\) Denote by \(v_1^3\) and \(v_2^3\) the vertices of \(G^3\) to which \(\{x_1, x_3\}\) and \(\{y, x_2\}\) are mapped, respectively. Since \(x_3v_2^3v_1^3x_4x_3\) is a 4-cycle in \(G^3\), and since \(\Theta_1 = G[\{x_1, x_2, x_3, x_4, x_5, y\}] \cong \Theta\), we can apply the previous proofs to \(G^3\) to make the following.
Claim 3'. Let $B^3$ be the block in $G^3$ that contains $v_1^3$ and $v_2^3$. Then either $v_1^3$ or $v_2^3$ has degree less than 3 in $B^3$.

**Proof.** Note that the path $P' = x_6v_1^3v_2^3x_3$ and the $(x_3, x_6)$-path $P$ in $G - E(\Theta)$ form a cycle in $G^3$ containing $v_1^3$ and $v_2^3$. The block $B^3$ in $G^3$ containing $v_1^3$ and $v_2^3$ contains $P$, $P'$, $H^2$ and the edge $v_3^3x_4$. Hence $v_1^3$ and $v_2^3$ have degree $\geq 3$ in $B^3$, contrary to Claim 3'. □

Subcase 3.2: There is no $y \in V(H^1)$ with $yx_1$, $yx_5 \in E(H^1)$.

Let $G^* = (G - \{x_1x_2, x_5x_6\})/\{x_1x_6, x_2x_5\}$, and let $v_1^*$, $v_2^*$ denote the vertices of $G^*$ to which $x_1x_6$ and $x_2x_5$ are contracted, respectively. The component $H$ of $G - E(\Theta)$ containing $x_6$ does not contain $x_2$. Hence, by the assumption of this subcase and by Claim 5, no 4-cycle of $G$ contains exactly 2 edges of $E(C_1)$. Since $G$ is reduced (and hence is simple), no 4-cycle of $G$ has exactly 3 edges in $E(C_1)$. By Claim 3, no 4-cycle of $G$ contains $x_5x_6$ and no other edges in $E(C_1)$; and no 4-cycle of $G$ contains $x_1x_2$ and no other edges of $E(C_1)$. Hence $G^*$ satisfies (1). Since $\kappa(G) \geq 2$, every cut-vertex of $G^*$ other than $v_1^*$ and $v_2^*$ must separate $v_1^*$ and $v_2^*$. Since $H^1$ is connected, there is a $(x_1, x_5)$-path $Q$ in $H^1$ that is disjoint from the $(x_3, x_6)$-path $P$. These paths $Q$ and $P$, together with $v_2^*x_3$, form a cycle in $G^*$ containing $v_1^*$ and $v_2^*$. Hence only $v_1^*$ or $v_2^*$ can be a cut-vertex of $G^*$.

If $v_1^*$ and $v_2^*$ are not cut-vertex of $G^*$, then $\kappa(G^*) \geq 2$. By the assumption of this subcase and by $A_{G}(H^1) = \{x_1, x_5\}$, (1) holds for $H^1$. Hence $x_1$ has degree $\geq 2$ in $H^1$ and so $v_1^*$ has degree $\geq 3$ in $G^*$. Similarly, $x_5$ has degree $\geq 2$ in $H^1$ and so $v_2^*$ has degree $\geq 3$ in $G^*$. It follows by the minimality of $G$ that $G^* \in \mathcal{CL}$ and so by Lemma 5, $G \in \mathcal{CL}$, a contradiction.

If $v_1^*$ is a cut-vertex of $G^*$, then $G^*$ has an end block $B_1^*$ containing $v_1^*$ but not $v_2^*$. Let $B_1 = G[E(B_1^*)]$. Since $\kappa(G) \geq 2$, $A_{G}(B_1) = \{x_1, x_6\}$. Hence $x_1$ is incident with an edge $e_1 \in E(B_1)$ and $x_6$ is incident with an edge $e_6 \in E(B_1)$. Since $e_1, e_6 \notin E(\Theta)$, so in $G^1$, the vertices $v_1^1$, $v_2^1$ have degree $\geq 3$ in $B^1$, contrary to Claim 3.

If $v_2^*$ is a cut-vertex of $G^*$, then $G^*$ has an end block $B_2^*$ that contains $v_2^*$ but not $v_1^*$. Let $B_2 = G[E(B_2^*)]$. Since $\kappa(G) \geq 2$, $A_{G}(B_2) = \{x_2, x_5\}$ and so in $G^1$, $v_1^1$ and $v_2^1$ have degree $\geq 3$ in $B^1$ contrary to Claim 3.

This proves Lemma 6. □

**Main result**

**Theorem 1.** If $G \in \mathcal{C}_1$, then $G \in \mathcal{CL}$.

By contradiction, let $G$ be a counterexample to Theorem 1 with $|V(G)|$ minimized. Immediately from Lemmas 4 and 6 and from definitions, we have the following observations.
Lemma 7 ([5]). Each of the following holds.

(i) $G$ is reduced and satisfies (2).
(ii) If $W$ is a maximal $K_{2,t}$-subgraph of $G$, $(t \geq 2)$, then
\[ D_2(W) \subseteq D^*_4(G). \tag{6} \]
(iii) If $L$ and $J$ are two subgraphs of $G$ with $|V(J) \cap V(L)| \leq 1$ and $J \cup L = G$, then both $J$ and $L$ satisfy (2).
(iv) If $W$ is a maximal $K_{2,t}$-subgraph of $G$ and if $L$ is a connected subgraph of $G - E(W)$ with $A_G(L) \subseteq V(W)$, then (2) hold for $L$.
(v) If $C = x_1x_2x_3x_4x_1$ is a maximal $K_{2,2}$-subgraph of $G$ and if $G - E(C)$ has an $(x_1, x_4)$-path $P_1$ and an $(x_2, x_3)$-path $P_2$, then $V(P_1) \cap V(P_2) \neq \emptyset$.

Lemma 8. $G$ cannot have a maximal $K_{2,2}$-subgraph.

Proof. By contradiction, let $C = x_1x_2x_3x_4x_1$ be a maximal $K_{2,2}$-subgraph of $G$. Define $G^*, v_1, v_2$ as in Notation 2. Then $G^*$ satisfies (2) and by (6), $\delta(G^*) \geq 3$. By Lemma 5 and by the minimality of $G$, $\kappa(G^*) \leq 1$. By (v) of Lemma 7, $G^*$ is connected. Hence $\kappa(G^*) = 1$.

Case 1: There is a cut-vertex $z \not\in \{v_1, v_2\}$ in $G^*$.

Then by $\kappa(G) \geq 2$, $z$ must separate $v_1$ and $v_2$ in $G^*$. By (v) of Lemma 7, we may assume that $z$ is in every $(x_1, x_4)$-path and in every $(x_2, x_3)$-path of $G - E(C)$. Hence there are connected subgraphs $H_1, H_2, H_3, H_4$ of $G - E(C)$ such that $A_G(H_i) = \{z, x_i\}$ and
\[ G - E(C) = H_1 \cup H_2 \cup H_3 \cup H_4. \]

Let $L^*$ be the graph obtained from $H_1 \cup H_4$ by identifying $x_1$ and $x_4$, $J^*$ be the graph obtained from $H_2 \cup H_3$ by identifying $x_2$ and $x_3$. Note that $G^* = J^* \cup L^*$.
By (iv) of Lemma 7, $G - E(C)$ satisfying (2), and so by (iii) of Lemma 7, both $H_1 \cup H_4$ and $H_2 \cup H_3$ satisfy (2). Thus $L^*$ and $J^*$ satisfy (2). By (2) and by the fact that $z$ is a cut-vertex of $H_1 \cup H_4$, the degree of $z$ in $L^*$ is at least 4 and so by (6), $\delta(L^*) \geq 3$. Similarly, $\delta(J^*) \geq 3$. It follows that $J^*, L^* \in \mathcal{G}_1$ and so by the minimality of $G$, $J^*, L^* \in \mathcal{E} \mathcal{L}$. By (v) of Theorem A, $G^* \in \mathcal{E} \mathcal{L}$ and so by Lemma 5, $G \in \mathcal{E} \mathcal{L}$, a contradiction.

Case 2: There is no cut-vertex in $G^*$ not in $\{v_1, v_2\}$.

Without loss of generality, let $v_1$ be a cut-vertex of $G^*$. Then $G^*$ has a nontrivial connected subgraph $B^*$ with $A_G(B^*) = \{v_1\}$ and $v_2 \not\in V(B^*)$. Let $B = G[E(B^*)]$. Then $A_G(B) = \{x_1, x_4\}$. By $\kappa(G) \geq 2$, $B$ is connected. By (v) of Lemma 7,
\[ \text{every } (x_2, x_3) \text{-path in } G - E(C) \text{ uses } x_1 \text{ or } x_4. \tag{7} \]
Since $G^*$ has no cut-vertex not in $\{v_1, v_2\}$, $G^* - E(B^*)$ has a cycle containing $v_1$ and $v_2$. Let $H^*$ be the block of $G^*$ that contains both $v_1$ and $v_2$ and let $H = G[E(H^*)]$. Then $A_G(H) \subseteq V(C)$ and so by (iv) of Lemma 7, $H$ satisfies (2). Without loss of generality we assume that $x_2 \in V(H)$. 
Claim. \( x_3 \in V(H) \).

Proof. Suppose not, we assume that \( x_3 \notin V(H) \). By \( \delta(G) \geq 3 \), \( x_3 \) is incident with an edge in \( G - E(C) \). Note that this edge is an edge on \( G^* \). Let \( K^* \) be a block of \( G^* \) containing \( v_2 \) but not \( v_1 \), and let \( K = G[E(K^*)] \). By \( \kappa(G) \geq 2 \), \( x_2, x_3 \in V(K) \). Hence there is an \((x_2, x_3)\)-path in \( K \), contrary to (7). This proves the Claim. \( \square \)

Since \( H^* \) contains \( v_1 \), either \( x_1 \) or \( x_4 \) is in \( V(H) \). If \( x_4 \notin V(H) \), then by (7), \( x_1 \) is a cut-vertex of \( H \) and so \( x_1 \in D^*_2(H) \); if \( x_4 \in V(H) \), then \( V(C) \subseteq D^*_2(H) \). Hence in either case, \( v_1 \) and \( v_2 \) have degree \( \geq 3 \) in \( H^* \) and so by the minimality of \( G, H^* \notin \mathcal{L} \). Note that \( (G[E(H) \cup E(C)])^* = H^* \). By Lemma 5, \( G \) is not reduced, contrary to Lemma 4. \( \square \)

Lemma 9 ([5]). If \( G \) has a \( K_{2,t} \)-subgraph \( W \) with \( t \geq 3 \), then \( t = 3 \) and \( D_3(W) \subseteq D_3(G) \).

Proof. The proof uses essentially the same technique as in the proof of Lemma 8 and is routine. \( \square \)

Notation 3 (see Fig. 4). Define \( D \) to be the union of two copies of \( K_{2,3} \)'s \( W_1 \) and \( W_2 \) such that

\[
V(W_1) \cap V(W_2) = D_2(W_1) \cap D_2(W_2) = \{y\} \subseteq D_4(G).
\]

\( G/\pi(1) \quad G'' + y \)

Fig. 4. Notation 3.
Denote

\[ D_1(W_i) = \{y_1, y_2, y\} \quad \text{and} \quad D_2(W_2) = \{y_3, y_4, y\}. \]

Suppose that \( H \) is a subgraph of \( G \) containing \( D \). Let

\[ V_{11} = \{y_1, y_2\}, \quad V_{12} = V(W_i) - V_{11} \quad \text{and} \quad \pi(1) = \langle V_{11}, V_{12} \rangle. \]

For convenience, we let \( u_1, y \) denote the vertices of \( G/\pi(1) \) to which \( V_{11} \) and \( V_{12} \) are mapped, respectively. Denote \( e_{\pi(1)} = u_1y \) and \( H' = (H/\pi(1)) - e_{\pi(1)} \). We regard \( W_2 \) as a subgraph of \( H' \). Let

\[ V_{21} = \{y_3, y_4\}, \quad V_{22} = V(W_2) - V_{21} \quad \text{and} \quad \pi(2) = \langle V_{21}, V_{22} \rangle. \]

Let \( u_2, y \) denote the vertices of \( H'/\pi(2) \) to which \( V_{21} \) and \( V_{22} \) are mapped, respectively and let \( e_{\pi(2)} = u_2y \). Define \( H'' \) to be the nontrivial component of \( H'/\pi(2) - e_{\pi(2)} \). Note that

\[ G'' = (G/\pi(1))/\pi(2) - \{y\}. \]

By (iii) of Theorem A, by Theorem B, and by \( \kappa(G) \geq 2 \), we have the following.

**Lemma 10 ([5]).** Let \( D, G'', u_1, u_2 \) be defined in Notation 3. Then:

(i) If \( G'' \in \mathcal{CL} \), then \( G \in \mathcal{CL} \).

(ii) Neither \( u_1 \) nor \( u_2 \) is a cut-vertex of \( G'' \).

Since \( G \in \mathcal{G}_1 \), \( G'' \) satisfies (1). By (6), \( \delta(G'') \geq 3 \). Since \( G \) is a minimum counterexample to Theorem 1, by (i) of Lemma 11, \( \kappa(G'') \leq 1 \). This, together with \( \kappa(G) > 2 \), implies

\[ \kappa(G'') = 1. \tag{8} \]

**Lemma 11 ([5]).** If \( H \) is a subgraph of \( G \) with \( |A(G)(H)| = 2 \), then either \( H \cong K_2 \) or \( H \) contains a 2-block of \( G \).

**Proof.** This follows from the fact that \( G \in \mathcal{G}_1 \). \( \Box \)

**Lemma 12 ([5]).** If \( H \) is a minimal 2-block of \( G \), then:

(i) \( H \) cannot have \( D \) as a subgraph.

(ii) \( H \) cannot have a \( K_{2,3} \)-subgraph \( W \) with

\[ D_2(W) \subseteq D_2^*(G). \tag{9} \]

**Proof.** The proof is routine with the help of Lemmas 10 and 11, and the minimality of \( H \). \( \Box \)

**Lemma 13.** \( G \) has a 2-block.
**Proof.** By (i) of Lemma 7, and by Lemmas 8 and 9, $G$ is an edge-disjoint union of $K_{2,3}$'s such that if $W$ is a $K_{2,3}$-subgraph of $G$, then $D_3(W) \subseteq D_3(G)$. Hence $G$ either has $D$ as a subgraph or has a subgraph $W \cong K_{2,3}$ satisfying (9).

**Case 1: $G$ has $D$ as a subgraph.**

We shall use the notations in Notation 3 and let $G''$, $u_1$, $u_2$ be defined as in Notation 3. By (8), by (ii) of Lemma 10, and by $\kappa(G) \geq 2$, $G''$ has a cut-vertex $z \notin \{u_1, u_2\}$ separating $u_1$ and $u_2$. Let $B''$ be an end block of $G''$ that contains $u_1$ but not $u_2$, and let $B = G[E(B'')]$. Then $A_G(B \cup W_1) = \{z, y\}$ and so by Lemma 11 with $H = B \cup W_1$, $G$ has a 2-block. □

**Case 2: $G$ has a subgraph $W \cong K_{2,3}$ satisfying (9).**

Define $G'$, $v_1$, $v_2$ as in Notation 1 with $i = 3$. By (9), $\delta(G') \geq 3$. It is easy to see that $\kappa(G') = 1$ and $G'$ has a cut-vertex $z \notin \{v_1, v_2\}$ that separates $v_1$ and $v_2$ in $G'$. Let $B'$ be an end block of $G'$ that contains $v_1$ but not $v_2$ and let $B = G[E(B')]$. Then $A_G(B) = \{z, y\}$ and so by Lemma 11, $G$ has a 2-block. □

**Proof of Theorem 1.** By Lemma 13, $G$ has a minimal 2-block $H$. By Lemma 7, 8 and 9, $H$ must be an edge-disjoint union of $K_{2,3}$'s. Since $H$ is a 2-block, and since $\delta(G) \geq 3$, $H \not\cong K_{2,3}$. Hence either $H$ has $D$ as a subgraph or $H$ has a subgraph $W \cong K_{2,3}$ that satisfies (9), contrary to Lemma 12. □

**References**