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Vertex arboricity and maximum degree

Paul A. Catlin[a,*], Hong-Jian Lai[b,1]

[a] *Department of Mathematics, Wayne State University, Detroit, MI 48202, USA*
[b] *West Virginia University, Morgantown, WV 26506, USA*

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Abstract

The vertex arboricity of graph G is the minimum number of colors that can be used to color the vertices of G so that each color class induces an acyclic subgraph of G. We prove results such as this: if a connected graph G is neither a cycle nor a clique, then there is a coloring of $V(G)$ with at most $\lceil \Delta(G)/2 \rceil$ colors, such that each color class induces a forest and one of those induced forests is a maximum induced forest in G. This improves prior results of Brooks (1941), Kronk and Mitchem (1974/75), and Lovász (1966), and it is analogus to a result of Catlin (1976, 1979) on the chromatic number that improves Brooks' theorem.

Keywords: Arboricity; Vertex arboricity; Chromatic number

1. Introduction

We follow the notation of Bondy and Murty [1], unless otherwise stated. As in [1], $\Delta(G)$ denotes the maximum degree of G and $\chi(G)$ denotes the chromatic number of G. The vertex arboricity of G, denoted by $a(G)$, is the minimum number of colors that can be used to color the vertices of G so that each "color class" induces an acyclic subgraph of G. (We use the term color class to refer either to a vertex set that induces one of the forests in that partition of G into induced forests, or to that induced forest itself.)

An easy bound on $\chi(G)$ in terms of $\Delta(G)$ is $\chi(G)\leqslant\Delta(G)+1$. The analogous upper bound $a(G)\leqslant\lceil(\Delta(G)+1)/2\rceil$ was obtained by Chartrand et al. [5], but it is also a special case of an older result of Lovász [8], in which the stronger condition $\Delta(H)\leqslant 1$ is also shown to hold for each color class H of G for some coloring satisfying that upper bound on $a(G)$.

The bound on $\chi(G)$ was sharpened by Brooks.

* Corresponding author.
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Theorem A (Brooks [2]) *Let G be a simple connected graph. If G is neither an odd cycle nor a clique, then*

$$\chi(G) \leqslant \Delta(G).$$

Catlin generalized Brooks' result in the following form (and Mitchem [11] gave a short proof).

Theorem B (Catlin [3,4]). *Let G be a simple graph. If G is neither an odd cycle nor a clique, then G has a proper coloring in at most $\Delta(G)$ colors such that one color class can be chosen as a maximum independent set of G.*

The analogue to Theorem A for vertex arboricity is due to Kronk and Mitchem.

Theorem C (Kronk and Mitchem [7]). *Let G be a simple connected graph. If G is neither a cycle nor a clique of odd order, then*

$$a(G) \leqslant \lceil \Delta(G)/2 \rceil.$$

Our objective in this paper is to do for Theorem C what Theorem B does for Theorem A: we shall show that the coloring satisfying Theorem C exists such that one color class is a *maximum induced forest*, i.e., an induced forest of the maximum possible order.

Matula [10] generalized Theorems A and C to allow colorings in which each "color class" is defined to have no $(k+1)$-edge-connected subgraph, for some fixed $k \geqslant 0$. Thus, $k=0$ is the special case of chromatic number, and $k=1$ is vertex arboricity. For $k \geqslant 2$, there is an unspecified collection of exceptional graphs (larger than the family of cycles and cliques), but all exceptional graphs are shown to be k-regular in Matula's generalization of Theorems A and C.

Let H be an arbitrary color class of the coloring of G in at most $\lceil \Delta(G)/2 \rceil$ colors, indicated in Theorem C. Harary et al. [6] showed that there is such a coloring satisfying $\Delta(H) \leqslant 2$, and Matsumoto [9] showed that this coloring could be achieved such that each color class induces paths and K_1's as its components.

Our two main results are the following theorems.

Theorem 1. *Let k be a natural number and let G be a connected simple graph with $\Delta(G) \leqslant 2k$ that is not a complete graph (if $k \geqslant 1$) nor a cycle (if $k=1$). Then $a(G) \leqslant k$ and there is a k-coloring of G such that each color class induces a forest, and such that one color class is a maximum induced forest in G.*

Theorem 2. *Let G be a connected simple graph, and let k be a positive integer. If $\Delta(G)=2k+1$ then G has a $(k+1)$-coloring, where each color class is a forest. Furthermore, if G is not a complete graph then for each property below, this coloring can be* *chosen to satisfy that property*:

(a) *one color class is edgeless and one color class may be assumed to be a maximum induced forest, or*

(b) *one color class may be assumed to be a maximum independent set.*

2. Some lemmas

Lemma 1. *Let G be a graph. If $S \subseteq V(G)$ is a maximal vertex subset so that $G[S]$ is a forest, then for each vertex $v \in V(G)-S$, $G[S \cup \{v\}]$ has a cycle containing v, and if*

$$d_{G-S}(v)+2 \geqslant \Delta(G),$$

then this cycle is unique.

Proof. It follows from the maximality of S that $G[S \cup \{v\}]$ has a cycle. Since $G[S]$ is a forest and if $d_{G-S}(v)+2 \geqslant \Delta(G)$, then this cycle is unique. □

Corollary 1. *If $S \subseteq V(G)$ is maximal such that $G[S]$ is a forest, then either $V(G)=S$ or*

$$\Delta(G-S) \leqslant \Delta(G)-2. \tag{1}$$

Moreover, if $\Delta(G) \leqslant 4$ and if $G-S$ contains a cycle C, then

$$\text{every vertex } v \in V(C) \text{ has degree 4 in } G. \tag{2}$$

Proof. It follows from Lemma 1. □

Lemma 2. *Let $m \geqslant 3$ be an integer and let G be a connected simple graph with $\Delta(G)=m$. Let $S \subseteq V(G)$ be a maximum subset such that $G[S]$ is acyclic and such that*

$$G-S \text{ has a few } K_{m-1}\text{'s as possible.} \tag{3}$$

If $G-S$ has a K_{m-1}, the each of these holds:

(a) *For any vertex v of this K_{m-1}, there are two neighbors of v in S, say s and s'; they are joined by a unique (s, s')-path in $G[S]$; and that path is a component of $G[S]$;*

(b) *$G \cong K_{m+1}$.*

Proof. First we prove conclusion (a). Let G and S satisfy the hypothesis of Lemma 2. By Corollary 1,

$$\Delta(G-S) \leqslant m-2. \tag{4}$$

If $G-S$ has no K_{m-1}, then there is nothing to prove. Let $H \cong K_{m-1}$ be a subgraph of $G-S$, and note that by (4), H is a component of $G-S$. Pick any vertex $v \in V(H)$. By the maximality of S,

$$d_{G-S}(v)+2=\Delta(G)=m.$$

By Lemma 1, there are distinct vertices s and s' in S such that vs and vs' are edges in a unique cycle C of $G[S\cup\{v\}]$.

We claim that C is a component of $G[S\cup\{v\}]$. If not, then C has a vertex z with at least 3 neighbors in $S\cup\{v\}$, and hence at most $m-3$ neighbors in $G-(S\cup\{v\})$. Then (3) is violated when S is replaced by $S\cup\{v\}-\{z\}$.

Thus, $C-v$ is an (s,s')-path in $G[S]$, and it is a component of $G[S]$. This proves (a) of Lemma 2, and only (b) remains.

Set $S_0=S$, let H_0 be a K_{m-1} is $G-S_0$, and pick vertices $v_0, v_1\in V(H_0)$. By (a) of Lemma 2, v_0 has neighbors s_0 and s_0' in $S=S_0$ such that $G[S]$ has a component that is an (s_0,s_0')-path. Call this path $P(0)$. Also by (a) of Lemma 2, v_1 has neighbors s_1 and s_1' in S such that $G[S]$ has a component $P(1)$ (say) that is an (s_1,s_1')-path.

Suppose $P(0)=P(1)$. Then $\{s_0,s_0'\}=\{s_1,s_1'\}$, and (3) implies both that the component of $G-(S\cup\{v_0\}-\{s_0\})$ containing v_1s_0 is a K_{m-1} and that the component of $G-(S\cup\{v_0\}-\{s_0'\})$ containing v_1s_0' is a K_{m-1}. If $s_0s_0'\notin E(G)$, then (3) is violated when S is replaced by $S\cup\{v_0,v_1\}-\{s_0,s_1\}$. If $s_0s_0'\in E(G)$, then $G\cong K_{m+1}$ follows, and (b) holds.

Suppose $P(0)\neq P(1)$. By (a) of Lemma 2,

$$\{s_0,s_0'\}\cap\{s_1,s_1'\}=\emptyset. \tag{5}$$

Set $S_1=S_0\cup\{v_1\}-\{s_1\}$. By (3), s_1 is in a K_{m-1} of $G-S_1$, and we denote that K_{m-1} containing s_1 by H_1. By (5), H_0 and H_1 are disjoint, and since $m\geqslant 3$, H_1 has order at least 2. Pick any $v_2\in V(H_1-s_1)$, and note that v_2 has $m-2$ neighbors in $G-S_1$, and two neighbors in S_1.

Inductively, for $i\geqslant 2$, let $\{s_1,s_i'\}$ denote the neighborhood of v_i in S_{i-1}, and define the subset

$$S_i=S_{i-1}\cup\{v_i\}-\{s_i\}.$$

By (a) of Lemma 2, one component of $G[S_{i-1}]$ is an (s_i,s_i')-path that we shall call $P(i)$. Also,

$$|S_i|=|S| \quad\text{and}\quad G[S_i] \text{ is acyclic,} \tag{6}$$

and S_i has the property that

$$G-S_i \text{ has as many } K_{m-1}\text{'s as does } G-S. \tag{7}$$

Hence, s_i must be in a K_{m-1} in $G-S_i$, and we denote this K_{m-1} by H_i. Since $m\geqslant 3$, we can arbitrarily pick $v_{i+1}\in V(H_i-s_i)$. Denote two neighbors of v_{i+1} in S_i by s_{i+1} and s_{i+1}'.

Since G is finite, there is a least natural number k such that H_k overlaps some H_j for some j such that $0\leqslant j<k$. Without loss of generality, suppose that $j=0$. Also, suppose that k is minimum in the sense that no other choice of s_i's and v_i's in the sequence $(v_0,v_1,s_1,v_2,s_2,\dots,v_k,s_k)$ would yield a shorter such sequence. By (5), $k\geqslant 2$. Since $j=0$, the minimality of k implies that H_k is a component of $G[V(H_0)\cup\{s_k\}-\{v_1\}]$, and hence that in G, $v_0v_1s_1v_2s_2\cdots v_ks_kv_0$ is a $(2k+1)$-cycle. Of course, $s_k\in\{s_0,s_0'\}$, and

without loss of generality, we suppose

$$s_k = s_0. \tag{8}$$

It also follows from the minimality of k that the $P(i)$'s $(0 \leqslant i \leqslant k)$ are disjoint paths, and that they are all components in $G[S_0]$ as well as in the respective $G[S_{i-1}]$'s $(1 \leqslant i \leqslant k)$.

We consider the relation of $P(k)$ and $P(0)$. Recall that $P(k)$ is the (s_k, s'_k)-path (and also a component) in $G[S_{k-1}]$ and also in $G[S_0]$. Also, $P(0)$ is an (s_0, s'_0)-path in $G[S_0]$ that is also a component of $G[S_0]$, too. This and (8) imply $P(0) = P(k)$, and

$$s'_0 = s'_k. \tag{9}$$

By (3) and by (8), s_0 and v_k are in a K_{m-1} in $G - (S \cup \{v_0\} - \{s_0\})$, and we denote this K_{m-1} by J. By the definitions of v_0 and s_0 and since $s_0 \in V(J)$, $G[V(J) \cup \{v_0\}]$ is a K_m. Similarly, by (3) and by (9), s'_0 and v_k are in a K_{m-1} in $G - (S \cup \{v_0\} - \{s'_0\})$, and we call this one J', and similarly (by the definitions of v_0 and s'_0 and since $s'_0 \in V(J')$), $G[V(J') \cup \{v_0\}]$ is also a K_m.

First suppose that $s_0 s'_0 \notin E(G)$. Then $G[V(J) \cup V(J')]$ is a K_{m+1}, and by $\Delta(G) = m$ and by the connectedness of G, we have (b) of Lemma 2.

Finally, suppose that $s_0 s'_0 \notin E(G)$. Then $G[S \cup \{v_0\}]$ has a unique cycle $C(0)$ (say) containing the path $s_0 v_0 s'_0$, and since $s_0 s'_0 \notin E(G)$, $C(0)$ has length at least 4. Hence,

$$s_0 \text{ has degree 2 in } G[S \cup \{v_0\} - \{s'_0\}]. \tag{10}$$

Consider the vertex $v_k \in V(J')$, where J' is that aforementioned K_{m-1} in $G - (S \cup \{v_0\} - \{s'_0\})$. By (a) of Lemma 2, the two neighbors of v_k in $S \cup \{v_0\} - \{s'_0\}$ each have degree 1 in $G[S \cup \{v_0\} - \{s'_0\}]$. But this violates (10), since one of those two neighbors is s_0. Lemma 2 follows. □

Lemma 3. *Let G be a connected simple graph with $\Delta(G) \leqslant 4$. Suppose that $S \subseteq V(G)$ is a maximum subset such that $G[S]$ is acyclic and such that*

$$\text{the number of cycles in } G - S \text{ is minimized.} \tag{11}$$

If $G - S$ has a cycle, then $G \cong K_5$.

Proof. Let G and S satisfy the hypothesis of the lemma. By Corollary 1, $\Delta(G - S) \leqslant 2$. Set $S_0 = S$. Let C_0 be a cycle in $G - S_0$ and pick $v_1 \in V(C_0)$. By Lemma 1, there is a vertex $s_1 \in S_0$ adjacent to v_1 in G such that $s_1 v_1$ lies in a cycle of $G[S_0 \cup \{v_1\}]$. Set $S_1 = S_0 \cup \{v_1\} - \{s_1\}$. By $\Delta(G) \leqslant 4$ and the maximality of S, s_1 is adjacent to at most 2 vertices in $V(G) - S_1$. If s_1 is adjacent to only one vertex in $V(G) - S_1$, then S_1 violates (11). Hence

$$s_1 \text{ is in a cycle in } G - (S_0 - \{s_1\}). \tag{12}$$

Inductively, for $i \geqslant 0$, we have an S_i such that

$$G[S_i] \text{ is acyclic such that } G - S_i \text{ contains as few cycles as possible.} \tag{13}$$

By (11) and by the hypothesis that $G-S$ has a cycle, $G-S_i$ also has a cycle C_i. Let $v_{i+1} \in V(C_i)$, We argue as before to conclude that there is a vetex $s_{i+1} \in S_i$ such that

$$s_{i+1}v_{i+1} \text{ is in a cycle in } G[S_i \cup \{v_{i+1}\}]. \tag{14}$$

Set $S_{i+1} = S_i \cup \{v_{i+1}\} - \{s_{i+1}\}$. Since G is a finite graph, it will eventually occur that some cycle C_k overlaps C_j for $j<k$. Without loss of generality, we assume that $j=0$, for otherwise we can reassign subscripts and start with C_j. By Lemma 1, s_k and v_1 must have the same neighbors in $C_0 - \{v_1\}$.

Let w and x be the two neighbors of v_1 in C_0. Then $w, x \in N(s_k)$. By (2), w and x have degree 4 in G and so each of them is adjacent to exactly one vertex of $S_0 - \{s_k\}$. Therefore, if $wx \notin E(G)$, then $G[S_0 \cup \{w, x\} - \{s_i\}]$ is a forest, contrary to the maximality of $|S_0|$. Thus $wx \in E(G)$ and so by (2), C_0 is a 3-cycle. Thus Lemma 3 follows from Lemma 2 with $m=4$. □

Corollary 2. *If G is a connected simple graph with $\Delta(G) \leqslant 4$, then either $G \cong K_5$ or G can be partitioned into two vertex disjoint forests, one of which is a maximum induced forest in G.*

Proof. Let S be a maximum subset of $V(G)$ such that $G[S]$ is a acyclic and such that (3) is satisfied. If $G-S$ has a cycle, then by Lemma 3, $G \cong K_5$. If $G-S$ is acyclic or if $S=V(G)$, then $a(G) \leqslant 2$. □

3. Proofs of main results

Proof of Theorem 1. We shall prove Theorem 1 by induction on k. The case when $k=1$ is trivial. Corollary 2 takes care of the case when $k=2$. Hence we assume that $k \geqslant 3$ and Theorem 1 holds for smaller values of k. We shall show then that either $G \cong K_{2k+1}$ or G has the desired k-coloring.

Let S be a maximum subset of $V(G)$ such that

$$G[S] \text{ is acyclic} \tag{15}$$

and

$$G-S \text{ contains a few } K_{2k-1}\text{'s as possible.} \tag{16}$$

By (1), each component of $G-S$ has maximum degree at most $2k-2$. If $G-S$ contains no K_{2k-1}, then since $k \geqslant 3$, it follows by induction that $a(G-S) \leqslant k-1$. Thus $a(G) \leqslant k$ and we are done. Hence we assume that $G-S$ contains some subgraphs isomorphic to K_{2k-1}. By (15), (16) and by (b) of Lemma 2 with $m=2k$, we have $G \cong K_{2k+1}$, as desired. □

We give examples to show that the phrase "there is a k-coloring of G" in Theorem 1 cannot be replaced by the phrase "there is an a ((G)-coloring of G". Let p be a natural number, and define

$$c=\binom{2p}{2}.$$

Let G_{2p} (the desired example) be the graph obtained from the complete graph K_{2p} as follows: for each edge $e\in K_{2p}$, add 2 paths of length 2 satisfying both of these conditions:

(i) The two paths of length 2 have the same ends as e; and

(ii) The internal vertices of these length 2 paths have degree 2 in G_{2p}.

Thus, G_{2p} has order $2p+2c$ and size $5c$. Any maximum induced forest F in G_{2p} contains all $2c$ vertices of degree 2 in G_{2p} and one vertex of degree $\Delta(G_{2p})$, so

$$|V(F)|=2c+1.$$

Note that $G_{2p}-V(F)$ is a K_{2p-1}, so any partition of G_{2p} into vertex induced forests, one of which is F, has $p+1$ classes. However, G_{2p} can be partitioned into $a(G_{2p})=p$ induced forests, where each forest contains exactly two vertices of the K_{2p} and at most $2(c-1)$ vertices of degree 2 in G, so the largest forest in this partition has order at most $2c$, which is less than the order of F. Thus, G_{2p} is the desired example.

Lemma 4. *Let G be a simple graph and let R be a maximal independent subset of $V(G)$. Then for any vertex v in $G-R$, $G[R\cup\{v\}]$ contains an edge.*

Proof. It follows from the maximality of R. □

Lemma 5. *Let $m\geqslant 2$ be an integer, let G be a connected simple graph with $\Delta(G)=m$, and let $R\subseteq V(G)$ be a maximum independent subset such that*

$$\textit{the number of } K_m\textit{'s in } G-R \textit{ is minimized.} \tag{17}$$

If $G-R$ has a K_m, then either $G\cong K_{m+1}$ or $m=2$ and G is an odd cycle.

Proof. Let G and R satisfy the hypothesis of the lemma. The case when $m=2$ is trivial and so we assume that $m\geqslant 3$. Set $R_0=R$ and let $H_0\cong K_m$ be a subgraph in $G-R$. By Lemma 4, we have

$$\Delta(G-R)\leqslant m-1. \tag{18}$$

For any vertex $v\in V(H_0)$, by Lemma 4, there is a vertex $u_1\in R_0$ such that $vu_1\in E(G)$. Thus if $|R_0|=1$, then we must have $G\cong K_{m+1}$. Hence we assume that

$$|R|\geqslant 2. \tag{19}$$

Similarly, we may assume that, by (19), there is a vertex $w\in V(H_0)$ such that

$$wu_1\notin E(G). \tag{20}$$

Fix some $v_1 \in V(H_0)$ such that $v_1 u_1 \in E(G)$. By (17), $G-(R_0 \cup \{v_1\} - \{u_1\})$ has a subgraph $H_1 \cong K_m$ with $u_1 \in V(H_1)$. Set $R_1 = R_0 \cup \{v_1\} - \{u_1\}$. Inductively, for $i \geqslant 1$, there is a vertex $u_i \in R_{i-1}$ and $R_i \subseteq V(G) - \{u_i\}$ with

$$|R_i| = |R| \quad \text{and} \quad E(G[R_i]) = \emptyset \tag{21}$$

such that

$$G - R_i \text{ has a few } K_m\text{'s as possible} \tag{22}$$

and such that

$$G - R_i \text{ has a subgraph } H_i \cong K_m \text{ containing } u_i. \tag{23}$$

Pick $v_{i+1} \in V(H_i) - \{u_i\}$. With a similar argument, there is a vertex $u_{i+1} \in R_i$ such that $v_{i+1} u_{i+1} \in E(G)$ and such that

$$G - (R_i \cup \{v_{i+1}\} - \{u_{i+1}\}) \text{ has a subgraph } H_{i+1} \cong K_m \text{ containing } u_{i+1}. \tag{24}$$

Set $R_{i+1} = R_i \cup \{v_{i+1}\} - \{u_{i+1}\}$.

Since G is finite, it will eventually occur that some H_i overlaps H_j for $0 \leqslant j < i$. Without loss of generality, one can assume that $j=0$ and that i is minimal. Note that the only way that H_i overlaps H_0 is that v_1 and u_i have the same neighbors in $V(H_0) - \{v_1\}$. Thus $wu_i \in E(G)$ and

$$G[V(H_0) \cup \{u_i\} - \{v_1\}] \cong K_m. \tag{25}$$

If $u_1 = u_i$ or if $v_1 u_i \in E(G)$, then by (25), $G[V(H_0) \cup \{u_1\}] \cong K_{m+1}$ and so Lemma 5 is proved. We thus assume that

$$u_1 \neq u_i \quad \text{and} \quad v_1 u_i \notin E(G). \tag{26}$$

By (25), by $\Delta(G) = m$, and by the maximality or R_0, any vertex in $V(H_0) - \{v_1\}$ has degree m in G. By $wu_i \in E(G)$ and by the fact that all the other neighbors of w are in $V(H_0)$, we conclude that $E(G[R_0 \cup \{w\} - \{u_i\}]) = \emptyset$. Since u_i is adjacent to all vertices of $V(H_0)$ except possibly v_1, where $|V(H_0)| = m \geqslant 3$, and by $u_i v_1 \notin E(G)$ (from (26)), it follows that $G - (R_0 \cup \{w\} - \{u_i\})$ has one K_m less than $G - R$, contrary to (17). Thus (26) must be false and so $G \cong K_{m+1}$. This proves Lemma 5. □

By Lemma 5, the following corollary is immediate.

Corollary 3. *Let $m \geqslant 2$ be an integer and let G be a simple graph with $\Delta(G) = m$. If G is neither an odd cycle nor a clique, then there is a partition of $V(G)$ into m independent sets;*

$$V(G) = V_1 \cup V_2 \cup \cdots \cup V_m,$$

such that for $1 \leqslant i \leqslant m-1$,

$$V_i \text{ is a maximum independent set of } G_i,$$

where $G=G_1$ *and* $G_{i+1}=G_i-V_i$.

It is clear that Corollary C generalizes both Theorem A and Theorem B.

Lemma 6. *Let G be a connected simple graph with* $\Delta(G)=3$ *and let* $R\subseteq V(G)$ *be a maximum independent subset such that*

$$G-R \text{ has a few cycles as possible.} \tag{27}$$

If $G-R$ *has a cycle,* $G\cong K_4$.

Proof. Let G and R satisfy the hypothesis of Lemma 6. Set $R_0=R$. Let C_0 be a cycle in $G-R$. Pick a vertex $v_1\in V(C_0)$. By Lemma 4, there is a vertex $u_1\in R_0$ such that $v_1u_1\in E(G)$. Set $R_1=R_0\cup\{v_1\}-\{u_1\}$. By the maximality of R and by (27), $G-R_1$ has a cycle containing u_1. Inductively, we can find $u_i\in R_{i-1}$ and $R_i\subseteq V(G)-\{u_i\}$ with $|R_i|=|R|$ and $E(G[R_i])=\emptyset$, such that (27) holds with R replaced by R_i. Then by (27) and by the existence of C_0, there is a cycle C_i in $G-R_i$ containing u_i. Choose $v_{i+1}\in V(C_i)-\{u_i\}$. Then by Lemma 4, one can find $u_{i+1}\in R_i$ with $v_{i+1}u_{i+1}\in E(G)$. Then define $R_{i+1}=R_i\cup\{v_{i+1}\}-\{u_{i+1}\}$. Again, we may assume that C_0 overlaps C_i for some $i>0$. Thus u_i and v_1 have exactly the same neighbors. Suppose that w and x are in $V(C_0)$ such that $wv_1, xv_1, wu_i, xu_i\in E(G)$.

We claim that C_0 is not a 3-cycle. If C_0 were the 3-cycle xv_1wx in $G-R$ and if $u_1=u_i$, then $G[V(C_0)\cup\{u_1\}]$ is a K_4 (and Lemma 6 holds), because u_1 is adjacent to v_1 and u_i is adjacent to x and w. If C_0 were the 3-cycle xv_1wx and if $u_1\neq u_i$, then (27) is violated: $G[(G-R)\cup\{u_i\}-\{x\}]$ has fewer cycles than $G-R$, because its maximum degree is at most 2 and the component containing u_i has v_1 as an endvertex.

Thus $|V(C_0)|\geqslant 4$ and so w and x are nonadjacent in G, by $\Delta(G)=3$. Hence, $E(G[R_0\cup\{w,x\}-\{u_i\}])=\emptyset$, contrary to the maximality of $|R|$. □

Proof of Theorem 2. We shall show (b) of Theorem 2 first. Choose a maximum independent subset $R\subseteq V(G)$ satisfying (17). By Lemma 4 $\Delta(G-R)\leqslant 2k$. Apply Theorem 1 to each component of $G-R$: then either $a(G-R)\leqslant k$ and we are done, or $G-R$ has a K_{2k+1}. When $G-R$ has a K_{2k+1}, by Lemma 5 with $m=2k$, we have $G\cong K_{2k+2}$, a contradiction. This proves (b) of Theorem 2.

To show (a), we assume that $k\geqslant 2$ first. Choose a maximum subset $S\subseteq V(G)$ such that $G[S]$ is acyclic and such that (3) holds with $m=2k+1$. By (1), each component of $G-S$ has maximum degree at most $2k-1$. Then choose a maximum independent set R in $G-S$ satisfying (17) with each component of $G-S$ replacing G in (17) and with $m=2k-1$. By Lemma 4, $\Delta(G-(S\cup R))\leqslant 2k-2$. If $k=2$ and $G-(S\cup R)$ has no cycles, or if $k\geqslant 3$ and $G-(S\cup R)$ has no subgraphs isomorphic to K_{2k-1}, then by Theorem 1, we have $a(G-(S\cup R))\leqslant k-1$, and so $a(G)\leqslant k+1$ and (a) of Theorem 2 holds.

If $k\geqslant 3$ and $G-(S\cup R)$ has a K_{2k-1}, then by applying Lemma 5 to the components of $G-S$, we conclude that $G-S$ has a K_{2k} and so by Lemma 2, G has a K_{2k+2}. Since G is connected and since $\Delta(G)=2k+1$, we must have $G\cong K_{2k+2}$. If $k=2$ and

$G-(S \cup R)$ has a cycle, then along with Lemma 2 we use Lemma 6 in place of Lemma 5 to conclude that $G \cong K_6$.

Now consider the case when $k=1$ and $\Delta(G)=3$. Again we choose $S \subseteq V(G)$ as above. By (1), each component of $G-S$ is either a K_1 or a K_2. If all are K_1's, then (a) of Theorem 2 holds. Thus $G-S$ has a K_2. By Lemma 2 with $m=3$, $G \cong K_4$. This completes the proof of Theorem 2. □

It is easy to see that $a(K_{2k})=k$ with each color class inducing an edge, and this is the only way to color $V(K_{2k})$ with k colors. In this sense, Theorem 2 is best possible.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier, New York, 1976).
[2] R. Brooks, On colouring the nodes of a network, Proc. Cambridge Phil. Soc. 37 (1941) 194–197.
[3] P.A. Catlin, Embedding graphs and coloring graphs under extremal degree conditions, Ph.D. Dissertation, Ohio State University, 1976.
[4] P.A. Catlin, Brooks' graph-coloring theorem and the independence number, J. Combin. Theory Ser. B 27 (1979) 42–48.
[5] G. Chartrand, H.V. Kronk and C.E. Wall, The point arboricity of a graph, Isreal J. Math. 6 (1968) 169–175.
[6] F. Harary, R. Maddox and W. Staton, On the point linear arboricity of a graph, Mathematiche (Catania) 44 (1989) 281–286.
[7] H.V. Kronk and J. Mitchem, Critical point arboritic graphs. J. London Math. Soc. 9 (1974/75) 459–466.
[8] L. Lovász, On decomposition of graphs, Studia Sci. Math. Hungar. 1 (1966) 237–238.
[9] M. Matsumoto, Bounds for the vertex linear arboricity, J. Graph Theory 14 (1990) 117–126.
[10] D.W. Matula, An extension of Brooks' theorem, CNA 69, preprint, 1973.
[11] J. Mitchem, A short proof of Catlin's extension of Brooks' theorem, Discrete Math. 21 (1978) 213–214.