Spanning Trails Joining Two Given Edges

Paul A. Catlin
Wayne State University

Hong-Jian Lai
University of Waterloo

ABSTRACT

Let G be a graph and let $e_1, e_2 \in E(G)$. If G has two edge-disjoint spanning trees, then either G has a spanning trail whose first edge is e_1 and last edge is e_2, or $\{e_1, e_2\}$ is an edge cut of G such that both components of $G - \{e_1, e_2\}$ contain at least one edge. This strengthens a result of S.-M. Zhan.

Keywords: Graph theory, spanning trail, spanning eulerian subgraph, hamiltonian line graph, edge-disjoint spanning trees

1. Notation

We shall use the notation of Bondy and Murty [1], except where noted otherwise. We forbid loops but allow multiple edges in graphs. An edge-cut X of a connected graph G is called *essential* if at least two components of G – X contain at least one edge. The symmetric difference of sets R and S is denoted $R \Delta S$.

Let $e_1, e_2 \in E(G)$. A trail in G whose first edge is e_1 and whose last edge is e_2 is called an (e_1, e_2)-*trail*. An (e_1, e_2)-trail T is called a *spanning* (e_1, e_2)-trail if $V(T) = V(G)$ and if every edge of G is incident with an internal vertex of T. For $v_1, v_2 \in V(G)$, a trail in G whose origin is v_1 and whose terminus is v_2 is called a (v_1, v_2)-*trail*, and it is a *spanning* (v_1, v_2)-*trail* if it contains every vertex of G.

The *line graph* of a graph G is the graph L(G) with E(G) as its vertex set, where e_1 and e_2 are adjacent vertices in L(G) whenever they are adjacent edges in G.

2. The Problem

S.-M. Zhan [11] proved the following result:

Theorem 1 *(Zhan [11]) If G is a 4–edge–connected graph, then for any edges $e_1, e_2 \in E(G)$ there is a spanning (e_1, e_2)–trail in G.*

A graph G is *Hamilton–connected* if for every pair of vertices v_1, v_2 of G, there is a Hamilton (v_1, v_2)–path in G. Combination of Theorems 1 and 2' gives this result:

Corollary 1A *If a graph G is 4–edge–connected, then L(G) is Hamilton–connected.*

In this paper we shall improve on Theorem 1 by using a weaker hypothesis. An exceptional case arises.

Zhan [11] emphasized Hamilton paths in L(G), as in Corollary 1A, and he did not state Theorem 1 in the form given above. However, Theorem 1 can be obtained as a case of Theorem 4 of [11].

Harary and Nash–Williams [5] demonstrated this relationship between trails in G and Hamilton cycles in L(G):

Theorem 2 *(Harary and Nash–Williams [5]) Let G be a graph of order at least 4. Then L(G) is hamiltonian if and only if G has a closed trail Γ such that each edge of E(G) has at least one end in $V(\Gamma)$.*

(In Theorem 2, $V(\Gamma)$ need not equal V(G), and so Γ may not be a spanning trail. Also, we regard a single vertex as a closed trail.) A slight change in the proof of Theorem 2 gives:

Theorem 2' *Let G be a graph and let $e_1, e_2 \in E(G)$. Then L(G) has a Hamilton (e_1, e_2)–path if and only if G has an (e_1, e_2)–trail whose internal vertices contain at least one end of each edge of G.*

For any $k \in N$, it is a consequence of a theorem of Tutte [10] and Nash–Williams [8] that a 2k–edge–connected graph has k edge–disjoint spanning tress (see, e.g., [7] or [4]). For the case $k = 2$ (the case of interest for this paper), Zhan (in the proof of his Lemma 6 [11]) proved the "$\Rightarrow$" part of the next result:

Theorem 3 *(Catlin [3]) Let $k \in N$, let G be a graph with $|E(G)| \geq k$, and let ε_k be the family of all k–element subsets of E(G). Then G is 2k–edge–connected if and only if for any $E \in \varepsilon_k$, the graph $G - E$ has k edge–disjoint spanning trees.*

We shall prove the following result which, by Theorem 3 with $k = 2$, is stronger than Theorem 1:

Theorem 4 *Let G be a graph and let $e_1, e_2 \in E(G)$. If G has two edge–disjoint spanning trees, then exactly one of the following holds:*

(a) G has a spanning (e_1, e_2)–trail;

(b) $\{e_1, e_2\}$ is an essential edge–cut of G.

Corollary 4A *Let G be a graph of order at least 3 containing two edge–disjoint spanning trees. Then $L(G)$ is Hamilton–connected if and only if $L(G)$ is 3-connected.*

The proof of Theorem 4 appears in subsequent section, and it requires Theorems 8 and 9 and an application of the following reduction method. Corollary 4A is proved by combining Theorems 2' and 4.

3. The Reduction Method

For any graph H, define

$$O(H) = \{\text{odd–degree vertices of } H\}.$$

Let G be a graph, and let S be an even subset of V(G). An *S–subgraph* Γ of G is a subgraph $\Gamma \subseteq G$ such that both

$$O(\Gamma) = S$$

and

$$G - E(\Gamma) \text{ is connected.}$$

We call G *collapsible* if G has an S–subgraph for every even set $S \subseteq V(G)$. The family of collapsible graphs is denoted $\mathcal{CL}$. If $G \in \mathcal{CL}$, then we can set $S = O(G)$ in the definition and see that $G - E(\Gamma)$ is a spanning eulerian subgraph of G, and hence that G has a spanning closed trail, by Euler's Theorem ([1], p. 51). Of course, $K_1 \in \mathcal{CL}$, and any nontrivial graph in $\mathcal{CL}$ must be 2–edge–connected.

For any graph G, define

$$F(G) = \max_{E \subseteq E(G)} 2[\omega(G - E) - 1] - |E|.$$

Thus, $F(G) = 0$ if and only if G has two edge–disjoint spanning trees (see [8], [9], or [10]). Let $F'(G)$ denote the minimum number of edges that must be added to E(G) in order to create a graph with two edge–disjoint spanning trees.

Proposition *For any graph G, $F(G) = F'(G)$.*

The proof of this proposition appears later.

Theorem 5 *(Catlin [2]) If a graph G satisfies $F'(G) \le 1$ (equivalently, $F(G) \le 1$), then exactly one of the following holds:*

(a) $G \in \mathcal{CL}$;

(b) G has a cut–edge.

Corollary 5A *(Jaeger [6]) If $F(G) = 0$ then G has a spanning closed trail.*

Let H be a connected subgraph of G and let $S \subseteq V(G)$. Let G/H denote the graph obtained from G by contracting H to a vertex called v_H in G/H. Contractions are defined so that $E(G/H) = E(G) - E(H)$. Define

$$S/H = \begin{cases} S - V(H) & \text{if } |S \cap V(H)| \text{ is even;} \\ S - V(H) \cup \{v_H\} & \text{if } |S \cap V(H)| \text{ is odd.} \end{cases}$$

We shall need the following result:

Theorem 6 *(Catlin [2]) Let G be a graph, let H be a subgraph of G, and let $S \subseteq V(G)$. If $H \in \mathcal{CL}$, then G has an S–subgraph if and only if G/H has an (S/H)-subgraph.*

Corollary 6A *[2] If H is a collapsible subgraph of G, then*

$$G \in \mathcal{CL} \Leftrightarrow G/H \in \mathcal{CL}.$$

Corollary 6B *[2] If H is a collapsible subgraph of G, then G has a spanning closed trail if and only if G/H has a spanning closed trail.*

For a graph G, let $H_1, H_2, ..., H_c$ be the maximal collapsible subgraphs of G. We proved in [2] that these H_i's are uniquely determined and pairwise vertex–disjoint. Each vertex of G is in some H_i $(1 \leq i \leq c)$, because $K_1 \in \mathcal{CL}$. Let G' denote the graph of order c obtained from G by contracting each H_i to a distinct vertex $(1 \leq i \leq c)$. We call G' the *reduction* of G. If G has no nontrivial subgraph in $\mathcal{CL}$, then we call G *reduced*. We also proved [2] that the reduction of G is reduced. Examples of reduced graphs include forests and $K_{2,t}$ $(t \geq 2)$. By Corollary 6B, G has a spanning closed trail if and only if the reduction G' has a spanning closed trail.

4. Associated Results

Lemma 7 *([2], Lemma 1) Let H be a graph and let $S \subseteq V(H)$ have evenly many vertices in each component of H. Then there is a subgraph $\Gamma \subseteq H$ such that $O(\Gamma) = S$.*

Proof Let $P_1, P_2, ..., P_m$ be $m = |S|/2$ paths in H that join the vertices of S in distinct pairs. Thus, each $x \in S$ is an end of exactly one of the m paths. Define Γ by the rule that $e \in E(\Gamma)$ if and only if e lies in an odd number of the paths P_i $(1 \leq i \leq m)$. ❑

We say that an edge $e \in E(G)$ is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denoted $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by that length 2 path is called *subdividing* e. For a graph G and edges $e_1, e_2 \in E(G)$, let $G(e_1)$ denote the graph obtained from G by subdividing e_1, and let $G(e_1, e_2)$ denote the graph obtained from G by subdividing both e_1 and e_2. Thus,

$$V(G(e_1, e_2)) - V(G) = \{v(e_1), v(e_2)\}.$$

Theorem 8 *Let G be a graph and let $e_1, e_2 \in E(G)$. If G has edge–disjoint spanning trees Γ_1 and Γ_2 such that $e_1, e_2 \notin E(\Gamma_1)$, then G has a spanning (e_1, e_2)–trail.*

Proof Suppose that G, e_1, e_2, Γ_1, and Γ_2 satisfy the hypothesis. Denote $H = G - E(\Gamma_1)$. Since $E(\Gamma_2) \subseteq E(H)$, H is connected, and so by Lemma 7, there is a subgraph Γ of $H(e_1, e_2)$ such that

$$O(\Gamma) = O(G) \cup \{v(e_1), v(e_2)\}.$$

It follows that

$$O(G(e_1, e_2) - E(\Gamma)) = \{v(e_1), v(e_2)\},$$

and hence $G(e_1,e_2) - E(\Gamma)$ has an Euler trail joining $v(e_1)$ and $v(e_2)$. This Euler trail induces a spanning (e_1, e_2)–trail in G. ❑

Note that Theorem 1 follows from Theorem 8 and the case $k = 2$ of Theorem 3. If G satisfies the hypothesis of Theorem 1, then by Theorem 3, edge–disjoint spanning trees Γ_1 and Γ_2 can be chosen so that $e_1, e_2 \notin E(\Gamma_1)$, and so the hypothesis of Theorem 8 is satisfied. This is essentially the method used by Zhan [11] to prove Theorem 1.

In Theorem 4, we consider a graph G having two edge–disjoint spanning trees, say Γ_1 and Γ_2. To apply Theorem 8, we would want to know whether Γ_1 and Γ_2 can be chosen so that $e_1, e_2 \notin E(\Gamma_1)$. This motivates the following definition and theorem.

Let G be a graph and let $e_1, e_2 \in E(G)$. An $\{e_1, e_2\}$*–forbidden subgraph* G_0 is any subgraph G_0 of G such that

(i) $\{e_1, e_2\}$ is an edge–cut of G_0; and

(ii) $F(G_0) = 0$.

An $\{e_1, e_2\}$–forbidden subgraph will also be called a *forbidden subgraph* if there is no confusion about the values of e_1 and e_2.

Theorem 9 *Let G be a graph, and let $e_1, e_2 \in E(G)$. If $F(G) = 0$ and if G has no $\{e_1, e_2\}$–forbidden subgraph, then G has 2 edge–disjoint spanning trees Γ_1 and Γ_2 such that $e_1, e_2 \notin E(\Gamma_1)$.*

Of course, if e_1 and e_2 are parallel edges in G, then $G_0 = G[\{e_1, e_2\}]$ is a forbidden subgraph. If $\{e_1, e_2\}$ is an edge–cut of G and if $F(G) = 0$, then $G_0 = G$ is a forbidden subgraph. In these two cases, it is obvious that the conclusion of Theorem 9 fails. There are other instances when forbidden subgraphs cause the conclusion of Theorem 9 to fail.

Theorem 10 *Let G be a graph and let $e_1, e_2 \in E(G)$. If $F(G) = 0$ and if G has no $\{e_1, e_2\}$-forbidden subgraph, then $G(e_1, e_2) \in \mathcal{CL}$ (i.e., the reduction of $G(e_1, e_2)$ is K_1).*

5. Proof of Theorem 9

Lemma 11 *Let G be a graph and let H be a connected subgraph of G. If $F(H) = 0$ and $F(G/H) = 0$, then H has edge–disjoint spanning trees, say U_1 and U_2, and G/H has edge–disjoint spanning trees, say T_1 and T_2. The pair (Γ_1, Γ_2) with $\Gamma_i = G[E(U_i) \cup E(T_j)]$ $(i, j \in \{1, 2\})$ is a pair of edge–disjoint spanning trees of G.*

Proof Suppose $F(H) = 0$ and $F(G/H) = 0$. By the theorem of Tutte [10] and Nash–Williams [8], H and G/H each have two edge–disjoint spanning trees. These trees can be combined as indicated to form the trees Γ_1 and Γ_2 that span G. ❑

Lemma 12 *If G is a counterexample to Theorem 9 with*

(1) $|V(G)| + |E(G)|$ minimized,

then for any proper nontrivial subgraph H of G, $F(H) \geq 1$.

Proof By way of contradiction, suppose that G is a counterexample to Theorem 9 that satisfies (1), and let H be a nontrivial proper subgraph of G with

(2) $$F(H) = 0.$$

It follows from (2) that H is connected.

Since G is a counterexample to Theorem 9,

(3) $$F(G) = 0.$$

By (3), G has two edge–disjoint spanning trees, and thus G/H does also.

Therefore,

(4) $$F(G/H) = 0.$$

Case 1 Suppose that $|V(H)| < |V(G)|$ and $\{e_1, e_2\} \cap E(H) = \varnothing$.

Suppose, by way of contradiction, that G_0 is a subgraph of G/H with $F(G_0) = 0$ and with $\{e_1, e_2\}$ as an edge–cut, i.e., that G_0 is a forbidden subgraph of G/H. Since G satisfies the hypothesis of Theorem 9, G_0 is not a subgraph of G, and so the vertex v_H of G/H corresponding to H must be in G_0. By (2) and $F(G_0) = 0$, both

H and G_0 have two edge–disjoint spanning trees. But then $F(G[E(G_0) \cup E(H)]) = 0$ and $\{e_1, e_2\}$ is an edge–cut of $G[E(G_0) \cup E(H)]$. Thus, $G[E(G_0) \cup E(H)]$ is a forbidden subgraph of G, contrary to the assumption that G is a counterexample.

Therefore, G/H has no forbidden subgraph, and since G is a smallest counterexample to Theorem 9, G/H has edge–disjoint spanning trees Γ'_1 and Γ'_2 with $e_1, e_2 \notin E(\Gamma'_1)$. Since $F(H) = 0$, Lemma 11 implies that Γ'_1 and Γ'_2 induce edge–disjoint spanning trees Γ_1 and Γ_2 of G with $e_1, e_2 \notin E(\Gamma_1)$. Thus, G satisfies the conclusion of Theorem 9, a contradiction.

Case 2 Suppose the $|V(H)| < |V(G)|$ and $|\{e_1, e_2\} \cap E(H)| = 1$.

Without loss of generality, suppose that

$$e_1 \in E(H) \text{ and } e_2 \notin E(H).$$

Let v_H be the vertex of G/H onto which H is contracted. By (2), there are edge–disjoint spanning trees U_1 and U_2 of H, with $e_1 \notin E(U_1)$; and by (4) there are edge–disjoint spanning trees T_1 and T_2 of G/H with $e_2 \notin E(T_1)$. By Lemma 11, G has the edge–disjoint spanning trees

$$\Gamma_i = G[E(T_i) \cup E(U_i)] \quad (1 \le i \le 2).$$

Thus, Γ_1 and Γ_2 satisfy the conclusion of Theorem 9, a contradiction.

Case 3 Suppose that H is a proper subgraph of G (possibly a spanning subgraph), such that $e_1, e_2 \in E(H)$.

No forbidden subgraph H_0 exists in H, for otherwise H_0 would be a forbidden subgraph of G, and G would not be a counterexample.

Since H has no forbidden subgraph, (2) and the minimality of G imply that the proper subgraph H has edge–disjoint spanning trees, say U_1 and U_2, with $e_1, e_2 \notin E(U_1)$. If H is a spanning subgraph of G, then $\Gamma_1 = U_1$ and $\Gamma_2 = U_2$ satisfy the conclusion of Theorem 9. If H is not a spanning subgraph of G, then by Lemma 11 and since $F(G/H) = 0$, $E(U_1)$ and $E(U_2)$ are contained in edge–disjoint spanning trees Γ_1 and Γ_2, say, of G, where $e_1, e_2 \notin E(\Gamma_1)$. Again G satisfies the conclusion of Theorem 9, contrary to our assumption.

Case 4 Suppose that H is a spanning proper subgraph of G and that $e_i \notin E(H)$ for some $i \in \{1, 2\}$.

Define $G' = G - e_i$. Since H is a spanning subgraph of G', G' has two edge–disjoint spanning trees, say Γ_1 and Γ_2, and since neither contains e_i, there is no loss of generality in assuming that $e_1, e_2 \in E(\Gamma_1)$. Thus, G satisfies the conclusion of Theorem 9, a contradiction. ❑

Proof of Theorem 9 By way of contradiction, let G and $\{e_1, e_2\}$ be a counterexample of Theorem 9 satisfying (1), i.e., a smallest counterexample. By Lemma 12, if G'' is a proper nontrivial subgraph of G, then

$$F(G'') \geq 1. \tag{5}$$

Since G is a counterexample to Theorem 9, G has no $\{e_1, e_2\}$–forbidden subgraph, and so

$$G - \{e_1, e_2\} \text{ is connected.}$$

Hence, G has a spanning tree T with

$$E(T) \subseteq E(G - \{e_1, e_2\}).$$

Let $F_1, F_2, ..., F_k$ be the $\omega(G - E(T)) = k$ components of $G - E(T)$.

If $G - E(T)$ is a forest, then G is exactly $k - 1$ edges short of having two edge–disjoint spanning trees, and so $0 = F(G) = k - 1$. Therefore, $k = 1$ and so $\Gamma_1 = T$ and $\Gamma_2 = F_1$ are spanning trees that satisfy the conclusion of Theorem 9.

Therefore, we suppose that $G - E(T)$ is not a forest, and hence that at least one component F_i has a cycle. If F_i has at least one cycle, then we call F_i a *cyclic component* $(1 \leq i \leq k)$. Define

$$\sigma(T) = \min_{1 \leq i \leq k} \{|E(F_i)| : F_i \text{ is a cyclic component of } G - E(T)\}.$$

Choose a spanning tree T of $G - \{e_1, e_2\}$ so that

$$\omega(G - E(T)) \text{ is minimized} \tag{6}$$

and, subject to (6), so that

$$\sigma(T) \text{ is minimized.} \tag{7}$$

Let H be a cyclic component of $G - E(T)$ with

$$|E(H)| = \sigma(T).$$

Let $T_1, T_2, ..., T_m$ denote the components of $T[V(H)]$, and denote

$$V_i = V(T_i) \quad (1 \leq i \leq m).$$

Set $H^* = G[V(H)]$. Since H is cyclic component, $|E(H)| \geq |V(H)|$, and so

$$\begin{aligned} |E(H^*)| &= |E(H)| + \sum_{i=1}^{m} |E(T_i)| \\ &\geq |V(H)| + (|V(H)| - m) \\ &= 2|V(H)| - m. \end{aligned} \tag{8}$$

Let E be a subset of $E(H^*)$ such that $F(H^*) = 2[\omega(H^* - E) - 1] - |E|$. If H' is any component of $H^* - E$, and E' a subset of $E(H')$ such that

$$F(H') = 2[\omega(H' - E') - 1] - |E'|,$$

then

$$\begin{aligned} F(H^*) &\geq 2[\omega(H^* - (E \cup E')) - 1] - |E \cup E'| \\ &= 2[\omega(H^* - E) + \omega(H' - E') - 2] - |E| - |E'| \\ &= 2[\omega(H^* - E) - 1] - |E| + 2[\omega(H' - E') - 1] - |E'| \end{aligned}$$

$= F(H^*) + F(H'),$

and hence $F(H') = 0$. Using (5), we conclude that every component of $H^* - E$ is trivial, implying that $E = E(H^*)$ and $F(H^*) = 2[|V(H^*)| - 1] - |E(H^*)|$. Again by (5),

(9) $$|E(H^*)| = 2|V(H^*)| - 2 - F(H^*) \le 2|V(H^*)| - 3.$$

Combination of (8) and (9) gives

(10) $$m \ge 3.$$

For $i, j \in \{1, 2, ..., m\}$, denote

$$Y_{i,j} = \{uv \in E(H) : u \in V_i, v \in V_j\},$$

and denote

$$Y = \bigcup_{i \ne j} Y_{i,j}.$$

Since H is connected, $Y \ne \varnothing$.

Case 1 Suppose $Y - \{e_1, e_2\} \ne \varnothing$. Therefore, $Y - \{e_1, e_2\}$ has an edge z_1z_2, say, and without loss of generality, suppose that $z_1 \in V_1$ and $z_2 \in V_2$. Let C be the unique cycle of $T + z_1z_2$. There are edges $u_1v_1, u_2v_2 \in E(T)$ with $u_i \notin V(H)$ and $v_i \in V_i$ $(1 \le i \le 2)$.

1A Suppose that z_1z_2 is a cut–edge of H.

Since H is cyclic, one of the components of $H - z_1z_2$ has a cycle. Without loss of generality, we assume that the component of $H - z_1z_2$ containing z_1 has a cycle. Then

$$T' = T + z_1z_2 - u_2v_2$$

is a spanning tree of $G - \{e_1, e_2\}$ such that

$$\sigma(T') \le \sigma(T) - 1 \text{ and } \omega(G - E(T)) = \omega(G - E(T')),$$

contrary to (6) and (7).

1B Suppose that z_1z_2 is not a cut–edge of H. Hence, $H - z_1z_2$ is connected. Define

$$T'' = T + z_1z_2 - u_2v_2$$

Then

$$\omega(G - E(T'')) = \omega(G - E(T)) - 1,$$

contrary to (6).

Case 2 Suppose that $Y \subseteq \{e_1, e_2\}$. Since H is connected, (10) forces $Y = \{e_1, e_2\}$, $m = 3$, and $H[V_1]$, $H[V_2]$, and $H[V_3]$ must all be connected. Therefore, each $G[V_i]$ is an edge–disjoint union of the spanning connected graphs T_i and $H[V_i]$ $(1 \le i \le 3)$, and hence $F(G[V_i]) = 0$. By (5) with $G'' = G[V_i]$, this forces $G[V_i] = K_1$ $(1 \le i \le 3)$.

Hence, H is acyclic, a contradiction. This completes Case 2 and the proof of Theorem 9. ❑

6. Proof of Theorem 10

Let G, e_1, and e_2 satisfy the hypothesis of Theorem 10. The hypothesis of Theorem 9 holds, and so G has two edge–disjoint spanning trees, say T and U, such that

$$e_1, e_2 \notin E(T).$$

If $e_1 \notin E(U)$ or if $e_2 \notin E(U)$, then $F(G(e_1, e_2)) \leq 1$ and hence by Theorem 5, $G(e_1, e_2) \in \mathcal{CL}$. Thus, assume that $e_1, e_2 \in E(U)$. To prove $G(e_1, e_2) \in \mathcal{CL}$, we must prove that $G(e_1, e_2)$ has an S–subgraph Γ, for any even set $S \subseteq V(G(e_1, e_2))$. Let S be an even subset of $V(G(e_1, e_2))$.

Case 1 Suppose that $v(e_1), v(e_2) \notin S$. By Lemma 7, there is a subgraph Γ in T with $O(\Gamma) = S$. Since $E(\Gamma) \subseteq E(G(e_1, e_2))$ and since $U(e_1, e_2)$ is a spanning tree in $G(e_1, e_2) - E(\Gamma)$, Γ is an S–subgraph of $G(e_1, e_2)$.

Case 2 Suppose $v(e_1), v(e_2) \in S$. By Lemma 7, $U(e_1, e_2)$ has a subgraph Γ with $O(\Gamma) = S$. Then T is a spanning tree of $G(e_1, e_2) - E(\Gamma) - \{v(e_1), v(e_2)\}$, and since $d(v(e_i)) = 1$ $(1 \leq i \leq 2)$ in $G(e_1, e_2) - E(\Gamma)$, the subgraph $G(e_1, e_2) - E(\Gamma)$ is connected and spans $G(e_1, e_2)$. Therefore, Γ is an S–subgraph of $G(e_1, e_2)$.

Case 3 Suppose that $v(e_1) \in S$ and $v(e_2) \notin S$. Let C be the unique cycle of $T + e_2$ in G. Then $C - e_2$ contains an edge, say e_3, that joins the two components of $U - e_2$ in G. Define the edge–disjoint spanning trees

$$T' = T + e_2 - e_3,\ U' = U - e_2 + e_3.$$

Thus, $e_1 \in E(U')$, $e_2 \in E(T')$, and $S \subseteq V(U'(e_1))$. By Lemma 7, $U'(e_1)$ has a subgraph Γ such that $O(\Gamma) = S$. Then $G(e_1) - E(\Gamma)$ is a spanning connected subgraph of $G(e_1)$ containing e_2, and so $G(e_1, e_2) - E(\Gamma)$ is a spanning connected subgraph of $G(e_1, e_2)$. Hence Γ is an S–subgraph of $G(e_1, e_2)$.

The case $v(e_1) \notin S$, $v(e_2) \in S$ is similar. This proves Theorem 10. ❑

7. Proof of Theorem 4

Lemma 13 *Let G be a graph, let H be a subgraph of G, let S be a subset of $V(G)$, and let R be an even subset of $V(H)$. If $H \in \mathcal{CL}$, then G has an S–subgraph if and only if G has an $(S\, \Delta\, R)$–subgraph.*

Proof Suppose G, H, R, and S satisfy the hypothesis. Since $H \in \mathcal{CL}$, Theorem 6 implies these two equivalences:

$$G \text{ has an } S\text{–subgraph} \Leftrightarrow G/H \text{ has an } (S/H)\text{–subgraph};$$

$$G \text{ has an } (S \,\Delta\, R)\text{–subgraph} \Leftrightarrow G/H \text{ has an } ((S \,\Delta\, R)/H)\text{–subgraph}.$$

By the definition of S/H and since R is an even subset of V(H),

$$S/H = (S \,\Delta\, R)/H,$$

and Lemma 13 follows. ❑

Proof of Theorem 4 Suppose that G and $\{e_1, e_2\}$ satisfy the hypothesis of Theorem 4. Thus, $F(G) = 0$.

Suppose that G has no $\{e_1, e_2\}$–forbidden subgraph. By Theorems 9 and 8, G has a spanning (e_1, e_2)–trail, and (a) of Theorem 4 holds.

Next, suppose that G has an $\{e_1, e_2\}$–forbidden subgraph, say G_0. If $\{e_1, e_2\}$ is an edge–cut of G, then either (b) of Theorem 4 holds, or one component of $G - \{e_1, e_2\}$ is a single vertex. In the latter case, Corollary 5A implies (a) of Theorem 4. Suppose henceforth that $\{e_1, e_2\}$ is not an edge–cut of G.

Let G_1 and G_2 be the two components of $G_0 - \{e_1, e_2\}$. Since G_0 is a forbidden subgraph, $F(G_0) = 0$, and so G_0 has two edge–disjoint spanning trees. It follows that each component G_1 and G_2 of $G_0 - \{e_1, e_2\}$ has two edge–disjoint spanning trees, and so

(11) $$F(G_1) = F(G_2) = 0.$$

By (11) and Theorem 5,

(12) $$G_1, G_2 \in \mathcal{CL}.$$

Since $\{e_1, e_2\}$ is not an edge–cut of G, it follows that $G - e_2$ is 2–edge–connected. Also, $F(G) = 0$ gives $F(G - e_2) \le 1$, and so $G - e_2 \in \mathcal{CL}$, by Theorem 5. By setting $S = O(G - e_2)$ in the definition of $\mathcal{CL}$, we see that $G - e_2$ has a spanning eulerian subgraph H, say. Define $e_1 = x_1x_2$ and $e_2 = y_1y_2$, where

$$x_1, y_1 \in V(G_1) \text{ and } x_2, y_2 \in V(G_2).$$

Case 1 Suppose that $e_1 \in E(H)$.

Then $H - e_1$ has an eulerian (x_1, x_2)–trail that spans $G - \{e_1, e_2\}$. Define

$$\Gamma = G - \{e_1, e_2\} - E(H),$$

and set $S = O(G - \{e_1, e_2\}) \,\Delta\, \{x_1, x_2\}$, i.e., $S = O(G - \{e_1, e_2\}) \,\Delta\, O(H - e_1)$. Then Γ is an S–subgraph of $G - \{e_1, e_2\}$, and Γ is the complement of $H - e_1$ in $G - \{e_1, e_2\}$. Set

$$R = \begin{cases} \{e_1, e_2\} & \text{if } x_2 \ne y_2 \\ \varnothing & \text{if } x_2 = y_2 \end{cases}$$

Then by Lemma 13, by (12), and since R is an even subset of $V(G_2)$, it follows that $G - \{e_1, e_2\}$ has an $(S \Delta R)$–subgraph Γ', say. Note that Γ' is the complement in $G - \{e_1, e_2\}$ of a spanning connected subgraph H', say, where

$$O(H') = O(H - e_1) \Delta R = \{x_1, y_2\}.$$

By Euler's Theorem ([1], p. 52), H' contains an eulerian (x_1, y_2)–trail that spans $V(G - \{e_1, e_2\})$. By adding e_1 and e_2 to this trail, we extend it to a spanning (e_1, e_2)–trail of G.

Case 2 Suppose that $e_1 \notin E(H)$.

We imitate Case 1. Define

$$\Gamma = G - \{e_1, e_2\} - E(H),$$

and

$$S = O(G - \{e_1, e_2\}),$$

so that Γ is an S–subgraph of $G - \{e_1, e_2\}$ and Γ is the complement in $G - \{e_1, e_2\}$ of H. Set

$$R = \begin{cases} \{e_1, e_2\} & \text{if } x_2 \neq y_2 \\ \varnothing & \text{if } x_2 = y_2 \end{cases}$$

By Lemma 13, by (12), and since $R = \{x_2, y_2\}$ is an even subset of $V(G_2)$ (set $R = \varnothing$ if $x_2 = y_2$), it follows that $G - \{e_1, e_2\}$ has an $(S \Delta R)$–subgraph Γ' that is the complement in $G - \{e_1, e_2\}$ of a spanning (x_2, y_2)–trail. By adding e_1 at x_2 and e_2 at y_2, we extend this trail to form a spanning (e_1, e_2)–trail in G. This completes the proof of Theorem 4. ❑

8. Proof of Proposition

In [3], we used the terminology

$$\mathcal{S}_{2,t} = \{G \mid F'(G) \leq t\},$$

where $t \geq 0$, and we proved that if a graph G has a subgraph H with 2 edge–disjoint spanning trees, then

(13) $$F'(G) = F'(G/H).$$

(In [3], this was expressed by saying that $\mathcal{S}_{2,0}$ is the "kernel" of $\mathcal{S}_{2,t}$, where $t = F'(G)$, and where "kernel" is defined in [3].) We shall now also show

(14) $$F(G) = F(G/H),$$

when H is a subgraph of G having two edge–disjoint spanning trees. Let $E'' \subseteq E(G/H)$ be such that

$$F(G/H) = 2[\omega((G/H) - E'') - 1] - |E''|.$$

Since $E'' \subseteq E(G/H) \subseteq E(G)$,

$$\begin{aligned} F(G/H) &= 2[\omega((G/H) - E'') - 1] - |E''| \\ &= 2[\omega(G - E'') - 1] - |E''| \leq F(G) \end{aligned}$$

Hence,

(15) $$F(G/H) \le F(G)$$

Suppose next that the subset $E' \subseteq E(G)$ is minimized, subject to

(16) $$F(G) = 2[\omega(G - E') - 1] - |E'|.$$

Let

$$E_1 = \{e \in E' \mid \text{both ends of } e \text{ are in } V(H)\} \text{ and } E_2 = E' - E_1.$$

Since H has two edge–disjoint spanning trees,

(17) $$F(H) = 0,$$

by a theorem of Tutte [10] and Nash–Williams [8]. By (16) and (17),

$$\begin{aligned} F(G) &= 2[\omega(G - E') - 1] - |E'| \\ &= 2[\omega(G - E_2) - 1] - |E_2| + 2[\omega(H - E_1) - 1] - |E_1| \\ &\le 2[\omega(G - E_2) - 1] - |E_2| + F(H) \le F(G) \end{aligned}$$

By the minimality of E', $E_1 = \varnothing$ and $E' = E_2$. Hence $E' \subseteq E(G/H)$ and so

(18) $$\begin{aligned} F(G) &= 2[\omega(G - E') - 1] - |E'| \\ &\le 2[\omega(G/H - E') - 1] - |E'| \le F(G/H). \end{aligned}$$

Combination of (15) and (18) yields (14).

The *arboricity* $a(G)$ of a graph G is the minimum number of edge–disjoint spanning trees whose union is G. Nash–William [9] proved

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil,$$

where the maximum is taken over all nontrivial subgraphs H of G. In [3] (Theorem 11), we used this to show that if no nontrivial subgraph H of G has two edge–disjoint spanning trees, then $a(G) \le 2$.

By way of contradiction, suppose that G is the smallest graph with $F(G) \ne F'(G)$. If G has a nontrivial subgraph H that contains two edge–disjoint spanning trees, then by (13) and (14),

$$F(G/H) = F(G) \ne F'(G) = F'(G/H),$$

contrary to the minimality of G. Thus, G has no nontrivial subgraph containing two edge–disjoint spanning trees, and it follows that $a(G) \le 2$, by prior remarks.

Since the arboricity of G is at most 2, the definition of $F'(G)$ yields

$$F'(G) = 2(|V(G)| - 1) - |E(G)|.$$

Let E be a subset of E(G) that attains the maximum in the definition of F(G):

$$F(G) = 2[\omega(G - E) - 1] - |E|,$$

and let $H_1, H_2, \ldots, H_c$ be the $c = \omega(G - E)$ components of $G - E$. To prove $F(G) = F'(G)$, it suffices to prove that each H_i $(1 \le i \le c)$ is a K_1, because this would imply $c = \omega(G - E) = |V(G)| - 1$. Since no nontrivial subgraph of G has two edge–disjoint spanning trees, it suffices to prove that H_i has two edge–disjoint spanning trees, for then H_i must be trivial. By way of contradiction, therefore, suppose that H_i

does not have two edge–disjoint spanning trees. By the theorem of Tutte [10] and Nash–Williams [8], $F(H_i) > 0$, and so there is a subset $X \subseteq E(H_i)$ such that

$$2[\omega(H_i - X) - 1] - |X| > 0.$$

Then

$$\begin{aligned} 2[\omega(G - (E \cup X)) - 1] - |E \cup X| &= 2[\omega(G - E) - 1 + \omega(H_i - X) - 1] - |E| - |X| \\ &= F(G) + 2[\omega(H_i - X) - 1] - |X| > F(G), \end{aligned}$$

a contradiction. As already remarked, the Proposition follows. ❑

9. Examples

The hypothesis of Theorem 4, that G has two edge–disjoint spanning trees, is equivalent to the statement $F(G) = 0$. The three connected graphs illustrated in Figure 1 have $F(G) = 1$. For the designated edges e_1 and e_2, each fails to satisfy the conclusion of Theorem 4, except when $\{e_1, e_2\}$ is an essential edge–cut in the first graph. Also, each has no Hamilton (e_1, e_2)–path in its line graph, except for the first one when the small circle represents a lone vertex.

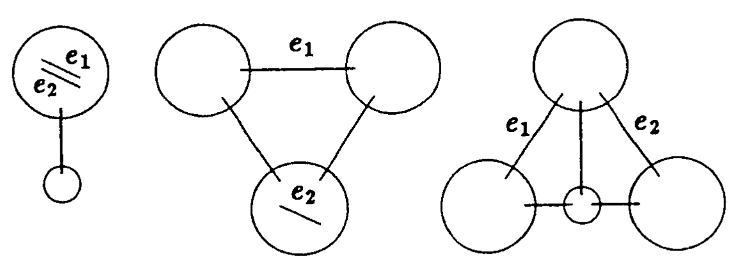

Figure 1: Three graphs with no spanning (e_1, e_2)–trail

In each graph of Figures 1 and 2, a circle denotes a subgraph having two edge–disjoint spanning trees, and if the circle is large, that subgraph is necessarily nontrivial.

In Figure 2, we give a typical example of a graph G with $F(G) = 0$ that has an $\{e_1, e_2\}$–forbidden subgraph G_0 such that $G_0 \neq G$ and $G_0 \neq G[\{e_1, e_2\}]$, where G has no edge–disjoint spanning trees Γ_1 and Γ_2 such that $e_1, e_2 \notin E(\Gamma_1)$. In this figure, $G_0 = G[V(G_1) \cup V(G_2)]$, and e_1 and e_2 are not parallel.

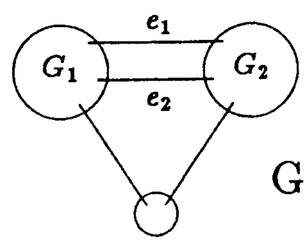

Figure 2

For the graph G of Figure 2, the reduction of $G(e_1, e_2)$ is $K_{2,3}$, which is not collapsible. Thus, the hypothesis in Theorem 10 that G has no $\{e_1, e_2\}$–forbidden subgraph is needed. We conjecture that if that hypothesis were omitted from Theorem 10, then the reduction of G (in the conclusion of Theorem 10) would be either K_1 or $K_{2,t}$ $(t \geq 2)$. This would follow from a conjecture of Catlin, that if a connected graph G satisfies $F(G) = 2$, then the reduction of G is either K_1, K_2, or $K_{2,t}$ $(t \geq 1)$.

Let $t \geq 3$ and let G_t denote the graph containing parallel edges e_1 and e_2 such that $G_t(e_1, e_2)$ is $K_{2,t}$. Then $F(G_t) = 0$ and G_t satisfies (a) of Theorem 4. However, every spanning (e_1, e_2)–trail in G is open (respectively, closed) if t is odd (respectively, even). Thus, even when e_1 and e_2 are adjacent and $F(G) = 0$ and (a) of Theorem 4 holds, we cannot guarantee that there is always a closed (resp., open) spanning (e_1, e_2)–trail in G.

Call the graph G *essentially 3–edge–connected* if for any $e, e' \in E(G)$, at most one component of $G - \{e, e'\}$ has an edge. A corollary of Theorem 4 is that if $F(G) = 0$ and if G is essentially 3–edge–connected, then G has a spanning (e_1, e_2)–trail, for any $e_1, e_2 \in E(G)$. However, this corollary does not appear to be sharp, because it may be possible to substitute $F(G) \leq 1$ for $F(G) = 0$ in the hypothesis. It would not be possible to substitute $F(G) \leq 2$, though, because $G = K_{2,t}$ $(t \geq 3)$ satisfies $F(G) = 2$, and G is essentially 3–edge–connected but does not have a spanning (e_1, e_2)–trail when t is odd and e_1 and e_2 are incident with a common divalent vertex of G.

Theorem 1 and Corollary 1A are best–possible in the sense that 3–edge–connectedness would not suffice. Let G be obtained by attaching ten disjoint copies of K_4 to Petersen graph, with just one K_4 attached at each vertex of the Petersen graph. Since the Petersen graph is not hamiltonian, there are edges of G (in a common K_4) such that G has no spanning (e_1, e_2)–trail.

Acknowledgment: The authors are grateful to the referee for many useful suggestions.

REFERENCES

[1] J.A. Bondy and U.S.R. Murty, "Graph Theory with Applications." American Elsevier, New York (1976).

[2] P.A. Catlin, A reduction method to find spanning eulerian subgraphs. J. Graph Theory 12 (1988) 29 – 44.

[3] P.A. Catlin, The reduction of graph families closed under contraction. Submitted.

[4] D. Gusfield, Connectivity and edge–disjoint spanning trees. Information Processing Letters 16 (1983) 87 – 89.

[5] F. Harary and C. St. J.A. Nash–Williams, On eulerian and hamiltonian graphs and line graphs. Canad. Math. Bull. 8 (1965) 701 – 709.

[6] F. Jaeger, A note on subeulerian graphs, J. Graph Theory 3 (1979) 91 – 93.

[7] S. Kundu, Bounds on the number of edge–disjoint spanning trees. J. Combinatorial Theory (B) 17 (1974) 199 – 203.

[8] C. St. J.A. Nash–Williams, Edge disjoint spanning trees of finite graphs. J. London Math. Soc. 36 (1961) 445 – 450.

[9] C. St. J.A. Nash–Williams, Decompositions of finite graphs into forests. J. London Math. Soc. 39 (1964) 12.

[10] W.T. Tutte, On the problem of decomposing a graph into n connected factors. J. London Math. Soc. 36 (1961) 221 – 230.

[11] S.–M. Zhan, Hamiltonian connectedness of line graphs. Ars Combinatoria 22 (1986) 89 – 95.