Graphs without nontrivial collapsible subgraphs

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Abstract

For any loopless graph G with the connected subgraph H, denote by G/H the graph obtained from G by contracting all edges of H and deleting any resulting loops. A graph is called supereulerian if it has a spanning connected subgraph whose vertices all have even degree. In a prior paper [3], we showed there is a family of graphs, which we call collapsible, such that if H is a collapsible subgraph of G, then G is supereulerian if and only if G/H is supereulerian. Any graph with two edge-disjoint spanning trees, for example, can be shown to be collapsible. Furthermore, for any graph G, repeated contractions of nontrivial collapsible subgraphs (until none remain) yields a unique graph, which we call the reduction of G. A graph G is supereulerian if and only if the reduction of G is supereulerian. Any graph that is the reduction of another is called reduced. A graph G_0 is reduced if and only if G_0 has no nontrivial collapsible subgraph. In this paper we discuss some conjectures about the class of reduced graphs, and we give a counterexample to a conjecture of H.-J. Lai.

1. INTRODUCTION

We use the notation of Bondy and Murty [2], except that graphs are considered to be loopless. A contraction of a graph G is any graph obtained from G by contracting a set (possibly empty) of edges and deleting all resulting loops. If H is a connected subgraph of G, then G/H denotes the graph obtained by contracting the edges of

$E(G[V(H)])$.

A graph is called supereulerian if it has a spanning closed trail, and K_1 is regarded as supereulerian. For any graph H, define

$$O(H) = \{\text{odd-degree vertices of } H\}.$$

By Euler's Theorem, G is supereulerian if and only if G has a connected spanning subgraph G_0 such that $O(G_0) = \emptyset$. Denote by $\mathcal{SL}$ the family of all supereulerian graphs.

In this paper, we give a counterexample to a conjecture of H.-J. Lai [8] on "reduced graphs" (defined below, and discussed in the abstract).

2. THE REDUCTION METHOD

A graph G is called collapsible if for every even set $X \subseteq V(G)$, there is a spanning connected subgraph G_X of G such that $O(G_X) = X$. Thus, K_1 is collapsible, and any collapsible graph is supereulerian. Denote by $\mathcal{CL}$ the family of all collapsible graphs. It was noted in [3] that $C_3 \in \mathcal{CL}$, and that if G has two edge-disjoint spanning trees, then $G \in \mathcal{CL}$. Since $\mathcal{CL} \subset \mathcal{SL}$, this improves Jaeger's Theorem [6], that any graph with at least two edge-disjoint spanning trees is supereulerian.

The following theorem is a corollary of Theorem 3 of [3].

Theorem 1 [3] Let H be a subgraph of G. If $H \in \mathcal{CL}$ then
(a) $G \in \mathcal{SL} \iff G/H \in \mathcal{SL}$; and
(b) $G \in \mathcal{CL} \iff G/H \in \mathcal{CL}$.

Catlin [4] conjectured that if $H \notin \mathcal{CL}$ then there is a supergraph G of H such that the equivalence (a) of Theorem 1 fails.

Theorem 2 [3, Theorem 4] Let H_1 and H_2 be subgraphs of G. If $H_1, H_2 \in \mathcal{CL}$ and if $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2 \in \mathcal{CL}$. □

Since $K_1 \in \mathcal{CL}$, any graph G has a collection $H_1, H_2, ..., H_c$ (say) of maximal collapsible subgraphs. It follows from Theorem 2 that the H_i's are disjoint and uniquely determined. Let G' denote the graph of order c obtained from G by contracting each H_i to a vertex

v_i $(1 \le i \le c)$. We call G' the reduction of G. (In [3], G' was denoted G_1.) Repeated applications of (a) of Theorem 1 give:

Theorem 3 [3, Theorem 8(vi)] For any graph G,

$$G \in \mathcal{SL} \iff G' \in \mathcal{SL}. \quad \Box$$

A graph is called reduced if it is the reduction of some graph.

Theorem 4 [3, Theorem 5] A graph is reduced if and only if it has no nontrivial collapsible subgraph. $\Box$

Theorem 5 [3, Theorem 7] If a graph G is at most one edge short of having two edge-disjoint spanning trees, then exactly one of these holds: $G \in \mathcal{CL}$; or G has a single cut edge.

Corollary The only reduced graphs of diameter at most 1 are K_1 and K_2.

3. REDUCED GRAPHS

In this section, we present a counterexample to a conjecture of Lai [8] on the diameter of reduced graphs. He characterized reduced graphs of diameter 2:

Theorem 6 [7,8] Let G be a graph of diameter 2. Then G is reduced if and only if one of these holds:
(a) G is a star of order at least 3;
(b) $G = K_{2,t}$ for some $t \in \mathbf{N}$;
(c) For some edge $e \in E(G)$ not parallel to any other edge of G, $G/G[e] = K_{2,t}$, for some $t \in \mathbf{N}$;
(d) G is the Petersen graph.

The next theorem is useful for constructing reduced graphs.

Lemma 1 A graph G is collapsible if and only if for any even set $S \subseteq V(G)$ there is a subgraph Γ of G with $O(\Gamma) = S$, such that $G - E(\Gamma)$ is connected.

Proof: In the definition of a collapsible graph, set $\Gamma = G - E(G_X)$ and $S = X \triangle O(G)$, where $\triangle$ denotes the symmetric difference. □

Theorem 7 Let G be a connected graph with nonadjacent edges e and e' such that $G - \{e, e'\}$ has two components, say G_1 and G_2, where e and e' both join G_1 and G_2 in G (see Figure 1). Then G is reduced if and only if both G_1 and G_2 are reduced.

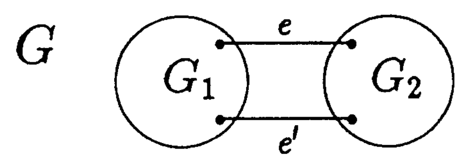

Figure 1

Proof: Let G, G_1, G_2, e and e' satisfy the hypothesis of Theorem 9. If G is reduced, then by Theorem 4, both G_1 and G_2 are reduced.

Conversely, suppose that both G_1 and G_2 are reduced. By way of contradiction, suppose that G has a nontrivial collapsible subgraph, say H. By Theorem 4, $H \not\subseteq G_1$ and $H \not\subseteq G_2$. Since a collapsible subgraph is 2-edge-connected, $e, e' \in E(H)$. Define

$$H_i = G_i \cap H, \quad i = 1, 2.$$

Since e and e' are not adjacent, both H_1 and H_2 are nontrivial. By Theorem 4 and since G_1 and G_2 are reduced, both H_1 and H_2 are reduced. Hence by Lemma 1, for each $i \in \{1, 2\}$ there is a nonempty even set $S_i \subseteq V(H_i)$ such that any subgraph $\Gamma_i \subseteq H_i$ with $O(\Gamma_i) = S_i$ has $H_i - E(\Gamma_i)$ disconnected.

Now, suppose that Γ is an arbitrary subgraph of H such that $O(\Gamma) = S_1 \cup S_2$. We claim that $H - E(\Gamma)$ is disconnected. Since Γ has evenly many odd-degree vertices in each set $V(H_i)$ ($|S_1|$ in $V(H_1)$, and $|S_2|$ in $V(H_2)$), Γ has evenly many edges in $[V(H_1), V(H_2)]$. Since $[V(G_1), V(G_2)] = \{e, e'\}$, either Γ contains both e and e' or Γ contains neither e nor e'. In the former case, $H - E(\Gamma)$ is disconnected, as claimed. Suppose that the latter case holds. Then e and e' are edges of $H - E(\Gamma)$, and for both $i \in \{1, 2\}$, $\Gamma[V(H_i)]$ is a subgraph, say Γ_i, of H_i with $O(\Gamma_i) = S_i$. As already remarked, both $H_1 - E(\Gamma_1)$ and $H_2 - E(\Gamma_2)$ are disconnected. If some component of $H_i - E(\Gamma_i)$

contains no end of $\{e, e'\}$ for some $i \in \{1, 2\}$, then $H - E(\Gamma)$ is disconnected, as claimed. If each component of $H_i - E(\Gamma_i)$ contains an end of $\{e, e'\}$ for both $i \in \{1, 2\}$, then e and e' lie in distinct components of $H - E(\Gamma)$. Thus, $H - E(\Gamma)$ is disconnected for any subgraph Γ of H with $O(\Gamma) = S_1 \cup S_2$. By Lemma 1, therefore, H is not collapsible, a contradiction. Thus, G has no nontrivial collapsible subgraph H, and so G is reduced. □

Note that the graph G of Figure 2 is reduced. This follows from the fact (Theorem 6) that $K_{2,2}$ and $K_{2,3}$ are reduced, and from two applications of Theorem 7.

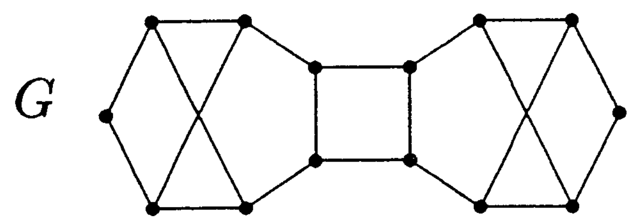

Figure 2

For any graph G, define $F(G)$ to be the minimum number of edges which, when added to G, form a graph with two edge-disjoint spanning trees. For example, $F(K_2) = 1$, $F(K_4) = 0$, and $F(K_{2,t}) = 2$, for any $t \in \mathbb{N}$. Hong-Jian Lai [8] conjectured that if G is a reduced graph, then $F(G)$ is at least as large as the diameter of G. The reduced graph G of Figure 2 is a counterexample, because $F(G) = 6$ and the diameter of G is 7. An infinite class of counterexamples can be constructed by attaching several copies of G end-to-end.

In the hypothesis of Theorem 7, it is necessary to assume that e and e' are not adjacent. For example, suppose that e and e' are adjacent edges of the 3-cycle G. Then the components of $G - \{e, e'\}$ are K_1 and K_2, which are both reduced. However, G is not reduced.

4. 3-EDGE-CONNECTED REDUCED GRAPHS

Z. H. Chen [5] has recently proved that the Petersen graph is the only simple 3-edge-connected graph of order at most 11 that is not collapsible. That result is best-possible, because there are several simple graphs G of order 12 containing a triangle H, such that G/H is the Petersen graph. Such graphs are not collapsible, by (b) of Theorem 1 and since the Petersen graph is not collapsible. (The Petersen graph cannot be collapsible, because it is not supereulerian). Lai [9] improved Chen's Theorem by showing that if $\delta(G) \geq 3$ and if G has order at most 11, then G is the Petersen graph or the reduction of G is either K_1 or K_2. The Blanuša snark [1] is a 3-regular, 3-edge-connected reduced graph of order 18.

5. REFERENCES

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