A Reduction Method to Find Spanning Eulerian Subgraphs

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ABSTRACT

We ask, When does a graph G have a subgraph Γ such that the vertices of odd degree in Γ form a specified set $S \subseteq V(G)$, such that $G - E(\Gamma)$ is connected? If such a subgraph can be found for a suitable choice of S, then this can be applied to problems such as finding a spanning eulerian subgraph of G. We provide a general method, with applications.

We shall use the notation of Bondy and Murty [6].

The *arboricity* $a(G)$ of G is the minimum number of edge-disjoint forests whose union equals G. Nash-Williams [20] proved that, if G is nontrivial,

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil, \tag{1}$$

where the maximum in (1) is taken over all induced nontrivial subgraphs H of G.

For a graph G with a subgraph H, the *contraction* G/H is the graph obtained from G by replacing H by a vertex v_H, such that the number of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v in G to H. This differs from the notation of [5], in that multiple edges can arise in contractions, and *no edge of* E(G) − E(H) *is lost when* H *is contracted.*

For $S \subseteq V(G)$, we define an S-*subgraph* in G to be a subgraph Γ such that

(i) $G - E(\Gamma)$ is connected; and
(ii) $v \in S$ if and only if $d_\Gamma(v)$ is odd.

A graph G is *collapsible* if G has an S-subgraph for every even set $S \subseteq V(G)$. We regard K_1 as being collapsible. It is equivalent that a graph G is collapsible if and only if, for any even set $S' \subseteq V(G)$, G has a spanning connected subgraph G' having S' as its set of odd-degree vertices.

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In this paper, we study the question of whether a given graph G is collapsible. Since $G - E(\Gamma)$ is eulerian if Γ is an S-subgraph, where

$$S = \{v \in V(G) \mid d(v) \quad \text{is odd}\},$$

this can be applied to the study of whether G has a spanning eulerian subgraph. More important, we shall prove that, if G contains H as a subgraph and if H is collapsible, then G has a spanning eulerian subgraph if and only if G/H has a spanning eulerian subgraph. By a sequence of such contractions, starting from any graph G, we shall obtain a simple triangle-free graph G_1, with $a(G_1) \leq 2$, such that G has a spanning eulerian subgraph if and only if G_1 has a spanning eulerian subgraph.

Let H be a subgraph of G. If there is a graph Γ with

(i) $H \subseteq \Gamma \subseteq G$, and
(ii) each vertex of Γ has even degree in Γ,

then H is called *cyclable* in G.

A *bond* of G is a minimal edge set whose removal disconnects G.

Theorem 1 (Jaeger [16]). For any subgraph H of G, H is cyclable if and only if H contains no bond of G of odd cardinality.

Theorem 1 extends a result in [4], and the following lemma is related to some lemmas of [4]:

Lemma 1. If G has a spanning tree T, such that every component of $G - E(T)$ has evenly many vertices in S, then G has an S-subgraph.

Proof. Suppose that, for any component H of $G - E(T)$, $|V(H) \cap S|$ is even. Set $s = \frac{1}{2}|S|$ and let $P_1, P_2, \ldots, P_s$ be paths in $G - E(T)$ joining all vertices of S in pairs. Define Γ to be the subgraph of $G - E(T)$ induced by those edges occurring in an odd number of the paths $P_1, P_2, \ldots, P_s$. Then Γ is an S-subgraph of G. ∎

Lemma 2. For a graph G, define

$$S = \{v \in V(G) \mid d(v) \quad \text{is odd}\}.$$

Then G has an S-subgraph if and only if G has a spanning eulerian subgraph.

Proof. For the set S defined in Lemma 2, if Γ is an S-subgraph, then $G - E(\Gamma)$ is a spanning eulerian subgraph. Conversely, if H is a spanning eulerian subgraph of G, then $G - E(H)$ is an S-subgraph. ∎

Consider the following conditions:

(a) $\kappa'(G) \geq 4$;
(b) for any $E \subseteq E(G)$, $2[\omega(G - E) - 1] \leq |E|$;
(c) G has two edge-disjoint spanning trees;
(d) G is collapsible;
(e) G has a spanning connected subgraph with no vertex of odd degree;
(f) G has a spanning closed trail; and
(g) $L(G)$, the line graph of G, is hamiltonian.

Theorem 2. The following implications hold:

$$(a) \Rightarrow (b) \Longleftrightarrow (c) \Rightarrow (d) \Rightarrow (e) \Longleftrightarrow (f) \Rightarrow (g)\,.$$

Kundu [17] observed (a) $\Rightarrow$ (b). The equivalence of (b) and (c) is a theorem of Tutte [24] and Nash-Williams [19]. By Lemma 1, (c) $\Rightarrow$ (d). If G is collapsible, then G has an S-subgraph when S is the set of odd degree vertices of G, and hence by Lemma 2, G has a spanning eulerian subgraph. Therefore, (d) $\Rightarrow$ (e). (Theorem 1 gives (c) $\Rightarrow$ (e) directly, if H is one of two edge-disjoint spanning trees.) Euler's Theorem [14] (also [6], p. 51) is the equivalence of (e) and (f). The implication (f) $\Rightarrow$ (g) follows from a characterization of hamiltonian line graphs due to Harary and Nash-Williams [15] (see also Chartrand [11]).

Definition. Let $S \subseteq V(G)$. When a contraction of G reduces a subgraph H to a vertex v_H, we define the set $S/H \subseteq V(G/H)$ by

$$S/H = \begin{cases} S - V(H), & \text{if } |S \cap V(H)| \text{ is even;} \\ (S - V(H)) \cup \{v_H\}, & \text{if } |S \cap V(H)| \text{ is odd.} \end{cases}$$

Lemma 3. Let G be a graph, H be a subgraph of G, and $S \subseteq V(G)$. If G has an S-subgraph, then G/H has an (S/H)-subgraph.

Proof. Let Γ be a spanning S-subgraph of G. In Γ/H, the vertex v_H has degree

$$d_{\Gamma/H}(v_H) = \left(\sum_{u \in V(H)} d_\Gamma(u) \right) - 2|E(\Gamma[V(H)])|\,.$$

Since S is the set of odd-degree vertices of Γ, $d_{\Gamma/H}(v_H)$ is even if and only if $|S \cap V(H)|$ is even. Also, $d_{\Gamma/H}(v) = d_\Gamma(v)$ for all $v \in V(\Gamma/H) - \{v_H\}$. It follows that Γ/H is an (S/H)-subgraph of G/H. ∎

Theorem 3. Let G be a graph, H be a subgraph of G, and $S \subseteq V(G)$. If H is collapsible, then G has an S-subgraph if and only if G/H has an (S/H)-subgraph.

Proof. If G has an S-subgraph, then Lemma 3 applies.

Conversely, suppose that H is a collapsible subgraph of G; let S be an even subset of $V(G)$, and suppose that G/H has an (S/H)-subgraph Γ'. Let E' be the edges of Γ' incident in G/H with v_H. Since $E' \subseteq E(G)$, we can define S_1 to be the set of vertices of H incident in G with an odd number of edges of E'. Note that the symmetric difference

$$S_H = (S \cap V(H)) \Delta S_1$$

has even cardinality, since

$$|S \cap V(H)| \equiv d_{\Gamma'}(v_H) \equiv |S_1| \qquad (\text{mod } 2).$$

Since H is collapsible, there is an S_H-subgraph Γ_H of H. We claim that the subgraph

$$\Gamma = G[E(\Gamma') \cup E(\Gamma_H)]$$

is an S-subgraph of G. Since $G/H - E(\Gamma')$ is connected, and since $H - E(\Gamma_H)$ is connected, we know that $G - E(\Gamma)$ is connected. For any $v \in V(G) - V(H)$, we have $d_\Gamma(v) = d_{\Gamma'}(v)$, and so for such a vertex, $d_\Gamma(v)$ is odd if and only if $v \in S$. If $v \in V(H)$, then $d_\Gamma(v)$ is odd if and only if $v \in S$, because both of these equivalences hold:

$$d_{\Gamma_H}(v) \text{ is odd iff } v \in S_H\,;$$
$$d_{G[E(\Gamma')]}(v) \text{ is odd iff } v \in S_1\,.$$

This proves the claim and the theorem. ∎

Corollary. Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible.

Proof. Combine Theorem 3 and the definition of a collapsible graph. ∎

Theorem 4. Let H_1 and H_2 be subgraphs of H such that

$$H_1 \cup H_2 = H \tag{2}$$

and

$$V(H_1) \cap V(H_2) \neq \varnothing\,. \tag{3}$$

If H_1 and H_2 are collapsible, then so is H.

Proof. Let S be an even subset of $V(H)$. By Theorem 3, since H_1 is collapsible, it suffices to prove that

$$H/H_1 \text{ has an } (S/H_1)\text{-subgraph}. \tag{4}$$

By (3), we can pick $x \in V(H_1) \cap V(H_2)$. Define the even set

$$S_2 = \begin{cases} S - V(H_1), & \text{if } |S \cap V(H_1)| \text{ is even;} \\ (S - V(H_1)) \cup \{x\}, & \text{if } |S \cap V(H_1)| \text{ is odd.} \end{cases}$$

By (2) and the definition of S_2, $S_2 \subseteq V(H_2)$. Since H_2 is collapsible, it has an S_2-subgraph, and so Lemma 3 implies

$$H_2/(H_1 \cap H_2) \text{ has an } (S_2/(H_1 \cap H_2))\text{-subgraph}. \tag{5}$$

However, (2) implies

$$H/H_1 = H_2/(H_1 \cap H_2), \tag{6}$$

and the definition of S_2 implies

$$S/H_1 = S_2/H_1 = S_2/(H_1 \cap H_2). \tag{7}$$

Plugging (6) and (7) into (5), we get (4), and hence Theorem 4. ∎

Corollary 1. Let G be a graph. If G contains a spanning tree T such that each edge of T is in a collapsible subgraph of G, then G is collapsible.

Proof. Let H_1 be a maximum collapsible subgraph of G, and let T satsify the hypothesis. If $G \neq H_1$, then there is an edge $e \in E(T)$ with exactly one end in $V(H_1)$. By the hypothesis about T, e lies in a collapsible subgraph H_2. By Theorem 4, the subgraph $H = H_1 \cup H_2$ is collapsible, contrary to the maximality of H_1. Hence, $G = H_1$, and so G is collapsible. ∎

Suppose that each edge of some spanning tree of G lies in a K_3. Since K_3 is collapsible, Corollary 1 implies that G is collapsible. By (d) $\Rightarrow$ (e) $\Rightarrow$ (g) of Theorem 2, this implies several previous results. Balakrishnan and Paulraja [1] showed that a graph G has a spanning eulerian subgraph if each edge of G lies in a triangle. Oberly and Sumner [22] had shown that the line graph of such a graph is hamiltonian, and they noted that theorems of Chartrand and Wall [12] and Nebesky [21] follow.

Corollary 1 also implies a recent result of P. Paulraja [23], which asserts that the graph G has a spanning closed trail if each edge of G lies in a simple subgraph G' of G, such that for some edge $e \in E(G')$, a spanning subgraph of $G' - e$ is a member of the set $\{C_5, K_{2,t}\}$ for some $t \geq 1$. In particular, such a subgraph G' contains a 3-cycle H. Since H is collapsible and since G'/H is collapsible, the Corollary of Theorem 3 asserts that G' is collapsible, and hence Corollary 1 applies.

Corollary 2. Let E'' be a minimal edge set such that every component of $G - E''$ is collapsible. Let E' be the edges of G that lie in no collapsible subgraph of G. Then $E'' = E'$.

Proof. Clearly, if $e \in E(G) - E''$ then $e \notin E'$. Hence, $E' \subseteq E''$.

By way of contradiction, suppose that there is an edge $xy \in E'' - E'$. Denote by H_x and H_y the collapsible components of $G - E''$ that contain x and y, respectively. Since $xy \notin E'$, xy lies in a collapsible subgraph H_{xy}. By Theorem 4, $H_x \cup H_{xy}$ is collapsible, and $(H_x \cup H_{xy}) \cup H_y$ is collapsible, and so each component of $G - (E'' - E(H_{xy}))$ is collapsible. This contradicts the minimality of E''. ∎

Definitions. Let $E'' \subseteq E(G)$ be a minimal edge set such that each component of $G - E''$ is collapsible. Let $H_1, H_2, \ldots, H_c$ denote the components of $G - E''$. Denote by G_1 the graph of order c obtained from G by contracting the subgraphs $H_1, H_2, \ldots, H_c$ to distinct vertices. We refer to G_1 as the *reduction of* G, and any graph G_1 that is the reduction of some graph G is called *reduced*. We let

$$\theta : G \to G_1$$

denote the associated contraction-mapping determining G_1. Each H_i $(1 \leq i \leq c)$ is a preimage under θ of a vertex v_i of G_1.

Note that

$$E(G_1) = E'',$$

and since E'' is unique, by Corollary 2, graph G_1 is uniquely determined, and θ is uniquely determined. In the rest of this paper, G_1 and θ will have the meaning of this definition.

Lemma 4. Let H be a collapsible subgraph of G. Let E'' be a minimal subset of $E(G)$ such that every component of $G - E''$ is collapsible, let E^{**} be a minimal subset of $E(G/H)$ such that every component of $(G/H) - E^{**}$ is collapsible, and let

$$E' = \{e \in E(G) \mid e \text{ lies in no collapsible subgraph of } G\}$$

and

$$E^* = \{e \in E(G/H) \mid e \text{ lies in no collapsible subgraph of } G/H\}.$$

Then

$$E'' = E' = E^* = E^{**}. \tag{8}$$

Proof. The first and last equalities of (8) are instances of Corollary 2 of Theorem 4. It remains to prove $E' = E^*$.

Let $e \in E'$ and suppose $e \notin E^*$, by way of contradiction. By $e \notin E^*$, G/H has a collapsible subgraph, say G'/H, containing e, where G' is a subgraph of G. Since H is also collapsible, G' is a collapsible subgraph of G, by the Corollary of Theorem 3. Since $e \in E(G')$, this contradicts $e \in E'$. Therefore,

$$E' \subseteq E^*. \tag{9}$$

Let $e \in E(G) - E'$. By the definition of E' and by Theorem 4, G has a unique maximal collapsible subgraph, say H_0, containing e. If H_0 and H are disjoint, then H_0 is a collapsible subgraph of G/H containing e, and so by the definition of E^*,

$$e \notin E^*. \tag{10}$$

On the other hand, if $V(H_0) \cap V(H) \neq \emptyset$, then by Theorem 4, $H_0 \cup H$ is collapsible, and so $H \subseteq H_0$, by the maximality of H_0. Either $e \in E(H)$ or $e \notin E(H)$. If $e \in E(H)$, then $e \notin E(G/H)$, and so (10) holds, since $E^* \subseteq E(G/H)$. If instead $e \notin E(H)$, then $e \in E(H_0/H)$, and since H_0 and H are collapsible, the Corollary of Theorem 3 asserts that H_0/H is collapsible, whence (10) holds. Since (10) holds for all $e \in E(G) - E'$, we have $E^* \subseteq E'$ and thus by (9), $E' = E^*$. ∎

Theorem 5. The graph G_1 is reduced if and only if every collapsible subgraph of G_1 is trivial.

Proof. If every collapsible subgraph of G_1 is trivial, then G_1 is the reduction of itself, and hence G_1 is reduced.

Suppose that G_1 is reduced. Then G_1 is the reduction of some graph G, and we denote by $H_1, H_2, \ldots, H_\omega$ the components of $G - E''$, where E'' is the set defined in Lemma 4. Let E_1 denote the edges of G_1 that lie in no collapsible subgraph of G_1. Since G_1 is obtained from G by contracting the subgraphs $H_1, H_2, \ldots, H_\omega$ to distinct vertices, repeated applications of (8) of Lemma 4 ensure that $E' = E_1$, where E' is defined as in Lemma 4. But $E' = E'' = E(G_1)$ also, whence $E_1 = E(G_1)$, and so no collapsible subgraph of G_1 has an edge. ∎

Corollary. Every subgraph of a reduced graph is reduced.

Proof. Immediate. ∎

Theorem 6. If H is a collapsible subgraph of G, then the reduction of G equals the reduction of G/H.

Proof. Let G and H satisfy the conditions of Theorem 6, and let $E'' \subseteq E(G)$ and $E^{**} \subseteq E(G/H)$ satisfy the conditions of Lemma 4. By that lemma, $E'' = E^{**}$. Since H is in a single component of $G - E'' = G - E^{**}$, the reduction of G (obtained from G by contracting each component of $G - E'' = G - E^{**}$ to a distinct vertex) equals the reduction of G/H [obtained from G/H by contracting each component of $(G/H) - E^{**}$ to a distinct vertex]. ∎

Next, we improve (c) $\Rightarrow$ (d) of Theorem 2. For the proof, we define, for any graph H and vertex $v \in V(H)$,

$$M(v,H) = \{w \in V(H) \mid H \text{ has a } (v,w)\text{-path}\}.$$

Define $D(G)$ to be the minimum number of edges whose addition to G is necessary to create a spanning supergraph containing two edge-disjoint spanning trees.

Theorem 7. Suppose that G is one edge short of having two edge-disjoint spanning trees. Then these are equivalent:

(i) G is collapsible
(ii) $\kappa'(G) \geq 2$.

Proof. If G is one edge short of having two edge-disjoint spanning trees, then G is connected. Now

$$\kappa'(G) = 1 \Rightarrow G \text{ is not collapsible},$$

because if G has a cut-edge uv, then G has no S-subgraph when $S = \{u, v\}$.

Conversely, suppose $\kappa'(G) \geq 2$, where G is a minimal counterexample to Theorem 7. By way of contradiction, suppose that G has a nontrivial collapsible subgraph, say H. Since G is a counterexample, the reduction G_1 of G is not K_1 (i.e., G is not collapsible). Of course, $D(G) = 1$ implies $D(G/H) \leq 1$. If $D(G/H) = 0$, then G/H is collapsible, by (c) $\Rightarrow$ (d) of Theorem 2, and so G is collapsible, by the Corollary of Theorem 3. This contradiction gives $D(G/H) = 1$. Also, $\kappa'(G) \geq 2$ implies $\kappa'(G/H) \geq 2$, and Theorem 6 implies that the reduction of G/H is G_1, which is not K_1. Thus, G/H violates the minimality of G, and so G has no nontrivial collapsible subgraph H. By Theorem 5, G is reduced.

By the hypothesis, there are two edge-disjoint spanning forests T and U, where

$$\omega(T) = 1, \omega(U) = 2.$$

Let U_1 and U_2 be the two components of U, and assume T and U are chosen to minimize $|V(U_2)|$. We shall prove $U_2 = K_1$.

By way of contradiction, we first suppose that $T[V(U_2)]$ is disconnected. Then there is an edge $e \in E(U_2)$ whose ends lie in distinct components of $T[V(U_2)]$. The unique cycle in $T + e$ contains an edge $e' \in [V(U_1), V(U_2)]$, and so the pair

$$T' = T + e - e', \qquad U' = U - e + e'$$

are edge-disjoint spanning forests in G, with $\omega(T') = 1$ and $\omega(U') = 2$, where one of the components of U' is also a component of $U_2 - e$. Since this contradicts the minimality of $|V(U_2)|$, the graph $T[V(U_2)]$ is connected.

The connected subgraph $T[V(U_2)]$ and the subgraph U_2 are two edge-disjoint trees that span $G[V(U_2)]$. By (c) $\Rightarrow$ (d) of Theorem 2, $G[V(U_2)]$ is collapsible, and since G is reduced, U_2 is trivial. Therefore, we can set $V(U_2) = \{u\}$.

Let $V_1, V_2, \ldots, V_r$ denote the vertex sets of the components of $T - u$, where $r = d_T(u)$. All edges of G incident with u are in $E(T)$, otherwise G would have two edge-disjoint spanning trees. Hence $r = d_T(u) = d_G(u) \geq \kappa'(G) \geq 2$. Since T is a tree, we have $|V_i \cap N(u)| = 1$ for $1 \leq i \leq r$, and we can denote

$$V_i \cap N(u) = \{v_i\}.$$

Define the subtrees U_3 and U_4, where U_3 is the smallest subtree of U_1 such that $N(u) \subseteq V(U_3)$, and where U_4 is the smallest subtree of U_3 that contains every edge $vw \in E(U_3)$ satisfying

$$v \in V_i, \qquad w \in V_j \qquad i \neq j.$$

Choose $xy \in E(U_4)$, such that x is an endvertex of U_4, and without loss of generality, suppose $x \in V_1$. Define U_x and U_y to be the two components of $U_1 - xy$, where $x \in V(U_x)$ and $y \in V(U_y)$. Since $x \in V_1$, $V(U_x \cap U_3) \subseteq V_1$. By the minimality of U_3, each endvertex of U_3 is in $N(u)$, and U_x must contain an endvertex of U_3. Therefore,

$$v_1 \in M(x, U_3 - xy) \subseteq M(x, U_1 - xy) = V(U_x)$$

and

$$\{v_2, \ldots, v_r\} \subseteq M(y, U_3 - xy) \subseteq M(y, U_1 - xy) = V(U_y).$$

From the minimality of U_4, we conclude that $y \in V_k$ for some $k \geq 2$. Hence, the unique cycle in $T + xy$ contains the edges uv_1 and uv_k, and these two edges join u to the distinct components U_x and U_y of $U_1 - xy$. Let S be an even subset of $V(G)$. If $u \notin S$, then $|U_1 \cap S|$ is even, and G has an S-subgraph, by Lemma 1. If $u \in S$, then exactly one of $|V(U_x) \cap S|$, $|V(U_y) \cap S|$ is odd, and so there is a unique $i \in \{1, k\}$ such that the edge-disjoint tree and forest

$$T' = T + xy - uv_i, \qquad U' = U - xy + uv_i$$

satisfy the hypothesis of Lemma 1, that each component of U' has evenly many vertices in S. Theorem 7 thus follows from Lemma 1. ∎

Theorem 8. Let G be a graph, and G_1 be the reduction of G. Then each of the following holds:

(i) G_1 is simple.
(ii) G_1 has no K_3.
(iii) $a(G_1) \leq 2$.
(iv) For any subgraph H of G_1, either $H \in \{K_1, K_2\}$, or

$$|E(H)| \leq 2|V(H)| - 4 .$$

(v) Let $S \subseteq V(G)$ be an even set, and define

$$S_1 = \{x \in V(G_1) : |\theta^{-1}(x) \cap S| \text{ is odd}\} .$$

Then G has an S-subgraph iff G_1 has an S_1-subgraph.
(iv) G has a spanning eulerian subgraph iff G_1 has a spanning eulerian subgraph.
(vii) $L(G)$ is hamiltonian iff G_1 has a closed trail containing at least one vertex of each edge of G_1 and containing each vertex $x \in V(G_1)$ for which $|\theta^{-1}(x)| > 1$.

Proof. Let G_1 be the reduction of G. By Theorem 5, no nontrivial subgraph of G_1 is collapsible. Therefore, since a 2-cycle is collapsible, (i) follows; since a 3-cycle is collapsible, (ii) follows.

Suppose (iii) is false. Let G be a reduced graph with $a(G) \geq 3$. For every subgraph H of G, the Corollary of Theorem 5 asserts that H is reduced. The minimality of G thus implies that if H is a proper subgraph of G, then $a(H) \leq 2$, and so by (1),

$$|E(H)| \leq 2(|V(H)| - 1) . \tag{11}$$

By (11) and since $a(G) \geq 3$, the only subgraph of G that attains the maximum in (1) is G itself, and so

$$|E(G)| > 2(|V(G)| - 1) . \tag{12}$$

The minimality of G implies that G is connected, and so the inequality of (b) of Theorem 2 holds when $E = \varnothing$. Let E be a nonempty subset of $E(G)$, and denote the components of $G - E$ by $H_1, H_2, \ldots, H_\omega$, where $\omega = \omega(G - E)$.

By (12) and (11),

$$\begin{aligned}|E| &= |E(G)| - \sum_{i=1}^{\omega} |E(H_i)| \\ &> 2(|V(G)| - 1) - \sum_{i=1}^{\omega} 2(|V(H_i)| - 1) \\ &= -2 + 2\omega = 2(\omega(G - E) - 1)\,, \end{aligned} \tag{13}$$

and so (b) of Theorem 2 holds. By (b) $\Rightarrow$(d), of Theorem 2, G is collapsible. Since G is thus collapsible and reduced, Theorem 5 implies that $G = K_1$, and so $a(G) \geq 3$ is contradicted. Thus, (iii) has no counterexample.

Next, we prove (iv). Suppose that H is a subgraph of G_1 satisfying

$$|E(H)| \geq 2|V(H)| - 3\,. \tag{14}$$

Since $a(H) \leq a(G_1) \leq 2$, (14) implies $D(H) \leq 1$. In particular, H is connected. By Theorems 2 and 7, either H is collapsible or $\kappa'(H) = 1$. If H is collapsible, then Theorem 5 implies that $H = K_1$. If $\kappa'(H) = 1$, then H has a cut-edge e, and (14) and (iii) imply that each component of $H - e$ has two edge-disjoint spanning trees, and by Theorem 2, they are both collapsible. By Theorem 5, each component of $H - e$ is therefore a vertex, and hence $H = K_2$. This proves (iv).

Part (v) follows from applications of Theorem 3, since G_1 is obtained from G by a sequence of contractions of disjoint collapsible subgraphs. If, in (v)

$$S = \{x \in V(G) \,|\, d_G(x) \text{ is odd}\},$$

then

$$S_1 = \{x \in V(G_1) \,|\, d_{G_1}(x) \text{ is odd}\}\,.$$

Hence, (vi) follows from (v) and application of Lemma 2 to both G and G_1.

The proof of (vii) uses the Harary and Nash-Williams Theorem [15], which says that $L(G)$ is hamiltonian if and only if G contains an independent set $U \subseteq V(G)$ such that $G - U$ has a spanning eulerian subgraph. By (vi), these conditions are each equivalent to the condition that G has an independent set U such that the reduction of $G - U$ has a spanning eulerian subgraph, and this is equivalent to the latter condition of (vii). ∎

The inequality in part (iv) is best possible, because, if $G_1 = H = K_{2,t}$, then $|E(H)| = 2|V(H)| - 4$, and G_1 is reduced, as we now show. If S is a nonadjacent pair of degree r vertices of $K_{2,r}$, where $r \leq t$, then $K_{2,r}$ has no S-subgraph, and hence is not collapsible. Any nontrivial subgraph G_2 of G_1 is not collapsible, because either $\kappa'(G_2) \leq 1$, or $G_2 = K_{2,r} (r \leq t)$. By Theorem 5, G_1 is reduced.

Denote by V_4 the set of vertices of G_1 with degree less than 4.

Lemma 5. If G_1 is a nontrivial 2-edge-connected reduced graph, then $|V_4| \geq 4$. If $|V_4| = 4$, then G_1 is eulerian and G_1 has 4 vertices of degree 2.

Proof. Write $V(G_1) = \{v_1, v_2, \ldots, v_c\}$, where G_1 is a 2-edge-connected nontrivial reduced graph. By (iv) of Theorem 8,

$$|E(G_1)| \leq 2c - 4. \tag{15}$$

Hence,

$$\sum_1^c d(v_i) = 2|E(G_1)| \leq 4c - 8. \tag{16}$$

Since G is 2-edge-connected, $\delta(G) \geq 2$, and hence (16) implies $|V_4| \geq 4$. Furthermore, if $|V_4| = 4$, then (16) implies that $d(v_i) = 2$ if $v_i \in V_4$, and $d(v_i) = 4$ if $v_i \notin V_4$. Therefore, G_1 is eulerian. ▮

Theorem 9. Let G be a 2-edge-connected simple graph on n vertices. Let $b \in \{4, 5\}$. If

$$\delta(G) \geq \frac{n}{b} - 1 \tag{17}$$

and if $n > 4b$, then exactly one of the following holds:

(i) Equality holds in (17), and G is contractible to $K_{2,b-2}(b \in \{4, 5\})$, such that the preimage of each vertex of $K_{2,b-2}$ is a collapsible subgraph of G on exactly n/b vertices;
(ii) $b = 5$ and G satisfies (e) of Theorem 2; and
(iii) $b = 4$ and G satisfies (d) of Theorem 2.

Proof. Let G satisfy the stated conditions. Since $n > 4b$, for some $k \geq 4$,

$$bk \leq n < b(k + 1), \tag{18}$$

where $n = bk$ implies $k \geq 5$. By (17) and (18),

$$\delta(G) \geq \frac{n}{b} - 1 \geq k - 1, \tag{19}$$

with equality in (19) only if $n = bk$ and $k \geq 5$.

Let G_1 be the reduction of G, and let c denote the order of G_1. Index the vertices $v_i \in V(G_1)$ such that

$$d(v_1) \leq d(v_2) \leq \cdots \leq d(v_c), \tag{20}$$

and define the induced subgraph H_i to be the preimage of v_i $(1 \leq i \leq c)$ in the contraction $\theta : G \to G_1$. If G_1 is trivial, then (d) and (e) of Theorem 2 hold.

Suppose G_1 is not trivial. By (20) and Lemma 6, either

$$\{v_1, v_2, \ldots, v_5\} \subseteq V_4 \tag{21}$$

or

$$\delta(v_4) = 2, \quad \text{and} \quad G_1 \text{ is eulerian.} \tag{22}$$

Suppose $b = 5$. If (21) fails, then by (22), G_1 is eulerian. Hence, by (vi) of Theorem 8, (ii) of Theorem 9 holds, and so we may assume

$$\text{If } b = 5, \qquad \text{then (21) holds.} \tag{23}$$

Case 1. Suppose

$$\delta(G) \geq k\,. \tag{24}$$

If $v_i \in V_4$, then (24) and $k \geq 4 > d(v_i)$ imply that some $x_i \in V(H_i)$ has $N(x_i) \subseteq V(H_i)$. By (24)

$$|V(H_i)| \geq d(x_i) + 1 \geq \delta(G) + 1 \geq k + 1, \qquad (v_i \in V_4)\,. \tag{25}$$

If $b = 5$, then (21) holds, by (23). Hence, for $b \in \{4, 5\}$, (23) and Lemma 5 imply $\{v_1, v_2, \ldots, v_b\} \subseteq V_4$, and so $c \geq b$. Also, (25) holds for $i = 1, 2, \ldots, b$. Thus,

$$n = \sum_{i=1}^{c} |V(H_i)| \geq \sum_{i=1}^{b} |V(H_i)| \geq b(k + 1)\,,$$

contrary to (18).

Case 2. Suppose $\delta(G) < k$. By (19),

$$\delta(G) = k - 1\,. \tag{26}$$

Since (19) holds with equality, we have

$$n = bk, \qquad k \geq 5\,. \tag{27}$$

By (26) and $k \geq 5$, if $v_i \in V_4$, then $d(v_i) < \delta(G)$, implying that some $x_i \in V(H_i)$ has $N(x_i) \subseteq V(H_i)$, and so

$$|V(H_i)| \geq d(x_i) + 1 \geq \delta(G) + 1 = k, \qquad (v_i \in V_4)\,. \tag{28}$$

For $b \in \{4, 5\}$, (28) and the fact that $\{v_1, v_2, \ldots, v_b\} \subseteq V_4$ imply

$$n \geq \sum_{i=1}^{b} |V(H_i)| \geq bk\,. \tag{29}$$

Now, (27) forces equality in (29), and so (29) and (28) imply

$$|V(H_1)| = |V(H_2)| = \cdots = |V(H_b)| = k \tag{30}$$

and

$$|V(G_1)| = b\,. \tag{31}$$

If $b = 5$ and G_1 has a spanning eulerian subgraph, then (vi) of Theorem 8 implies that G has one too, and so (ii) holds. If $b = 5$ and if G_1 has no spanning eulerian subgraph, then $\kappa'(G_1) \geq \kappa'(G) \geq 2$ and (31) imply $G_1 = K_{2,3}$, and so (i) of Theorem 9 holds, by (30).

If $b = 4$, then by (31) and Lemma 6, $G_1 = K_{2,2}$. Then (i) of Theorem 9 holds, by (30). ∎

The bound $n > 4b$ in Theorem 9 is best possible, for $b \in \{4, 5\}$. Let G_t be the graph constructed by taking the union of K_t and the Petersen graph, and by identifying a pair of vertices, one from each component. Thus, G_t has order $t + 9$ and when $t \geq 4$,

$$\delta(G_t) = 3\,.$$

If $n = 4b$ and $t = n - 9$, then G_t fails (i), (ii), and (iii) and

$$\delta(G_t) = 3 = \frac{n}{b} - 1\,.$$

Therefore, examples on $4b$ vertices ($b \in \{4, 5\}$) show that $n > 4b$ in Theorem 9 cannot be improved.

Bauer [2] has conjectured that, if $\delta(G) > n/5 - 1$ and $n \geq 20$, then $L(G)$ is hamiltonian. By (e) $\Rightarrow$ (g) of Theorem 2, Theorem 9 proves this conjecture when $n > 20$, and Theorem 9 characterizes the extremal graphs. A slight change in the proof also proves the case $n = 20$ of Bauer's conjecture, where (17) is strict.

We have recently obtained this related result [10], to appear shortly:

Theorem 10. Let G be a connected simple graph of order n and let $p \geq 2$. If $n \geq 4p^2$ and if

$$d(u) + d(v) > \frac{2n}{p} - 2 \tag{32}$$

whenever $uv \notin E(G)$, then exactly one of the following holds:

(a) G has a spanning eulerian subgraph;
(b) The reduction of G is a graph G_1 of order less than p, where G_1 has no spanning eulerian subgraph;
(c) $p = 2$, and $G - x = K_{n-1}$ for some $x \in V(G)$ with $d(x) = 1$. ∎

Lesniak-Foster and Williamson proved the case $p = 2$ of Theorem 10 [18]. Benhocine, Clark, Köhler, and Veldman [3] recently conjectured the case $p = 5$, which is related to Theorem 8, and they proved a result virtually the same as the case $p = 3$ of Theorem 10. The only possible values of G_1 in (b) of Theorem 10 is when $p = 5$ are trees. The inequality (32) is best possible.

Theorem 10 gives a sufficient condition for G to have a spanning eulerian subgraph in terms of a bound on $d(u) + d(v)$, when $uv \notin E(G)$. By (f) $\Rightarrow$ (g) of Theorem 2, this also gives a condition for $L(G)$ to be hamiltonian. Sufficient conditions involving a bound on $d(u) + d(v)$ when $uv \in E(G)$ have been considered by Brualdi and Shanny [7], Clark [13], Veldman [25], Benhocine, Clark, Köhler, and Veldman [3], and Catlin ([8], [9]).

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