Nearly-Eulerian Spanning Subgraphs

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Abstract

Let $k \geq 0$ be an integer, and let G be a connected graph. If G is at most $2k+1$ edges short of having two edge-disjoint spanning trees, then either G has a connected spanning subgraph with at most $2k$ vertices of odd degree or G can be contracted to a tree of order $2k+2$, whose vertices all have odd degree.

We shall use the notation of Bondy and Murty [1]. except that n shall denote the order of the graph G. The minimum number of trees whose union contains G is called the arboricity of G, and is denoted $a(G)$.

Let $F(G)$ denote the minimum number of edges that must be added to G. in order to obtain a supergraph that has two edge-disjoint spanning trees. By results of Tutte [6] and Nash-Williams ([4]. [5]).

$$(1) \qquad F(G) = \max_{E \subseteq E(G)} 2(\omega(G-E)-1) - \mid E \mid,$$

where the maximum in (1) is taken over all subsets $E \subseteq E(G)$, and where $\omega(G-E)$ denotes the number of components of $G-E$. We observe that if G has arboricity $a(G) \leq 2$, then it follows that

$$(2) \qquad F(G) = 2n - 2 - \mid E(G) \mid .$$

Of course. $F(G) \geq 0$, with equality if and only if G has two edge-disjoint spanning trees.

Let H be a connected subgraph of G. The contraction G/H is the graph obtained from G by replacing H by a vertex v_H, where any $v \in V(G) - V(H)$

is joined in G/H to v_H with as many edges as join v and $V(H)$ in G. If G' can be obtained from G by a sequence of contractions, then we say that G can be contracted to G'. If G can be contracted to G' then any cut-edge of G' is a cut-edge of G.

Throughout this paper, define, for the graph G, the set

$$R = \{v \in V(G) \mid d(v) \text{ is odd}\}.$$

Let H be a subgraph of G. The subgraph H will be called R-odd if $|V(H) \cap R|$ is odd; and H will be called R-even if $|V(H) \cap R|$ is even. When this terminology is used, H will be a forest in G. Define $\omega_{odd}(H)$ to be the number of R-odd components of H, where H is a subgraph of G.

Theorem Let k be a nonnegative integer, and let G be a connected graph. If

(3) $$F(G) \leq 2k + 1,$$

then exactly one of the following holds:

(a) G has a spanning connected subgraph with at most $2k$ vertices of odd degree;

(b) G can be contracted to a tree of order $2k + 2$, whose vertices all have odd degree.

Before beginning the proof, we will prove two lemmas, but first we need a definition. For any edge $xy \in E(G)$, define the subdivision $G * xy$ of G to be the graph obtained from $G - xy$ by adding a new vertex v and new edges vx and vy. We say that the resulting (x, y)-path xvy in $G * xy$ corresponds to $xy \in E(G)$.

Lemma 1 Let G be a graph and let $e \in E(G)$. If $G * e$ satisfies (a) of the Theorem, then G satisfies (a); if $G * e$ satisfies (b), then G satisfies (b).

Proof: Let G be a graph, let $xy \in E(G)$, and let xvy be the path in $G * xy$ that corresponds to xy.

Suppose $G * xy$ satisfies (a). Then $G * xy$ has a connected subgraph H with a set S of odd-degree vertices of H, where $|S| \leq 2k$. Also, $1 \leq d_H(v) \leq 2$. If $d_H(v) = 2$, then S is also the set of odd-degree vertices of the subgraph $G[E(H - v) \cup \{xy\}]$ in G. If $d_H(v) = 1$, then the set of odd-degree vertices of the subgraph $G[E(H - v)]$ is either $(S - v) \triangle \{x\}$ or $(S - v) \triangle \{y\}$, where $\triangle$ is the symmetric difference. In each case, G satisfies (a).

If $G*xy$ satisfies (b), then a similar argument shows that G satisfies (b). □

Lemma 2 [2] Let S be an even set of vertices of a tree H. There is a forest Γ in H such that for all $v \in V(H)$, $d_\Gamma(v)$ is odd if and only if $v \in S$.

Proof: Let $S = \{v_1, v_2, ..., v_{2s}\}$. Since H is connected, H contains a set S of paths, say $P_1, P_2, ..., P_s$, that connect the $2s$ vertices of S in disjoint pairs, where each P_i $(1 \leq i \leq s)$ is a (v_{2i}, v_{2i-1})-path in H. Define Γ to be the subgraph of H induced by edges lying in an odd number of the paths $P_1, P_2, ..., P_s$. Since H is a tree, Γ is a forest, and the definition of Γ implies that $d_\Gamma(v)$ is odd if and only if $v \in S$. □

Proof of Theorem: First, suppose, by way of contradiction, that (a) and (b) both hold for some connected graph G satisfying (3). By (a), G has a spanning connected subgraph H with at most $2k$ vertices of odd degree. By (b), G can be contracted onto a tree G' of order $2k+2$, where all vertices of G' have odd degree. Define

$$E_0 = E(G').$$

Since G' is a tree of order $2k+2$,

$$| E_0 | = 2k + 1,$$

and by the definition of contractions, E_0 is a set of $2k+1$ cut-edges of G. Since H is a spanning connected subgraph of G, each cut-edge of G lies in $E(H)$, and so $E_0 \subset E(H)$. Therefore, $H - E_0$ has components $H_1, H_2, \ldots, H_{2k+2}$. Since each H_i $(1 \leq i \leq 2k+2)$ of $H - E_0$ has an even number of vertices whose degree in H_i is odd, and since H_i is incident in H with an odd number of edges of E_0, an odd number of vertices of $V(H_i)$ have odd degree in H. Therefore, H has at least $2k+2$ vertices of odd degree, contrary to its definition. Hence, (a) and (b) cannot both hold.

Let $k \geq 0$ and let G_0 be a connected graph with $F(G_0) \leq 2k+1$, i.e., a graph satisfying the hypothesis of the theorem. Let $G_0, G_1, G_2, ..., G_t, G_{t+1}, ...$ be a sequence of graphs such that for all $i \leq 0$,

(4) $$G_{i+1} = G_i * e \quad (e \in E(G_i))$$

and such that

(5) $$F(G_t) \leq 2k + 1.$$

Given that sequence, we claim that there is a maximum value of t such that (5) holds. By the definition of F (or by (1)), for all $i \geq 0$,

(6) $$F(G_i) \leq F(G_{i+1}),$$

and so the values of $F(G_i)$ are nondecreasing. Also, by the definition of F (or by (1)),

(7) If $vw \in E(G)$ and if $d(v) = 2$, then $F(G * vw) > F(G)$.

Since each subdivision $G_i \to G_{i+1}$ replaces an edge $xy \in E(G_i)$ with two edges vx and vy satisfying the hypothesis of (7), equality can hold in (6) only finitely many times, for any sequence $G_0, G_1, G_2, \ldots.$ satisfying (4). Therefore, we can assume, as claimed, that G_t is the last term in the sequence $G_1, \ldots, G_t, \ldots$ such that (5) holds. Using the definition of F, we obtain

$$F(G_{i+1}) - F(G_i) \leq 1,$$

by a straightforward argument, and hence the bound of (5) is attained with equality:

$$F(G_t) = 2k + 1.$$

Let E' be a minimum set of edges in G_t^c, the complement of G_t, such that $G_t + E'$ has two edge-disjoint spanning trees, say T and U. By the definition of $F(G_t)$,

$$F(G_t) = \mid E' \mid.$$

By way of contradiction, suppose that there is an edge

(8) $$e \in E(G_t) - E(T \cup U).$$

Then T and U are edge-disjoint spanning trees of $(G - e) + E'$, and so

$$F(G_t - e) \leq \mid E' \mid = F(G_t),$$

and since $F(G_t - e) \geq F(G_t)$ holds trivially, we have

$$F(G_t - e) = F(G_t).$$

Merely by the definition of F, we have $F(G_t - e) \geq F(G_t * e)$, and in combination with the prior equality and with (4) and (6), this gives

$$F(G_t) = F(G_t - e) \geq F(G_t * e) = F(G_{t+1}) \geq F(G_t)$$

and so

$$F(G_{t+1}) = F(G_t).$$

But the maximality of t implies that G_t has no such edge e. Hence, no edge e exists satisfying (8), and so

$$a(G_t) = 2.$$

By Lemma 1. if G_{i+1} satisfies (a) or (b), then so does G_i, $(0 \leq i \leq t-1)$, and thus if G_t satisfies (a) or (b), then so does G_0. Thus it suffices to prove (a) and (b) for a graph $G = G_t$ that satisfies

$$F(G) = 2k+1 \tag{9}$$

and

$$a(G) \leq 2. \tag{10}$$

We shall assume, by way of contradiction that

$$\text{Both } (a) \text{ and } (b) \text{ fail.} \tag{11}$$

By (10), (2) holds, where n denotes the order of G, and since G is connected, G has a spanning tree T such that the subgraph

$$U = G - E(T)$$

is a forest. By (2) and (9),

$$\omega(U) = n - \mid E(U) \mid = n - \mid E(G) - E(T) \mid$$

$$= 2n - 1 - \mid E(G) \mid = F(G) + 1 = 2k+2. \tag{12}$$

We shall denote the $2k+2$ components of U by $U_1, U_2, ..., U_{2k+2}$. Since any graph has an even number of odd-degree vertices, $\omega_{odd}(U)$, the number of R-odd components of U, is even.

Lemma 3 For any spanning tree T of G, if the graph $U = G - E(T)$ is a forest, then $\omega_{odd}(U) = 2k+2 = \omega(U)$.

Proof: The last equation of Lemma 3 is (12). Pick a set $S \subseteq R$ such that

$$\mid S \cap V(U_i) \mid = 2\lfloor \frac{1}{2} \mid R \cap V(U_i) \mid \rfloor, \tag{13}$$

whenever $1 \leq i \leq 2k+2$. Since $\mid S \cap V(U_i) \mid$ is thus even for all $i \leq 2k+2$, Lemma 2 implies that U contains a forest Γ such that S is the set of odd-degree vertices of Γ. Hence, $G - E(\Gamma)$ is a spanning subgraph of G, with $R - S$ as its set of odd-degree vertices. Since $\Gamma \subseteq U$ implies $T \subseteq G - E(\Gamma)$, the subgraph $G - E(\Gamma)$ is connected. By (13), $R - S$ has no vertex in any R-even component of U, and $R - S$ has just one vertex in each R-odd component of U. By (12),

$$\mid R - S \mid = \omega_{odd}(U) \leq \omega(U) = 2k+2,$$

and equality must hold, as stated in Lemma 3, for otherwise $\omega_{odd}(U) \leq 2k$, since $\omega_{odd}(U)$ must be even, in which case $G - E(\Gamma)$ satisfies (a), contrary to (11). □

Proof of Theorem, continued By Lemma 3, we can assume henceforth that

(14) $$\omega_{odd}(U) = 2k + 2.$$

Define E_0 to be the set of edges of T that join vertices in distinct components of U. Since G is connected, (12) gives

(15) $$| E_0 | \geq \omega(U) - 1 = 2k + 1,$$

and

(16) $$\omega(G - E_0) = 2k + 2.$$

First, suppose that (15) holds with equality. Form the graph G' from G by contracting each subgraph $G[V(U_i)]$ $(1 \leq i \leq 2k+2)$ to a distinct vertex. Then $E(G') = E_0$. G' is connected, and equality in (15) and (16) gives

$$| E_0 | = 2k + 1 = \omega(G - E_0) - 1 = | V(G') | -1.$$

Therefore, G' is a tree. By (14) each component of U is R-odd, and so (b) of the theorem follows, contrary to (11). Therefore,

(17) $$| E_0 | > 2k + 1.$$

We shall alter T and U to form a spanning tree T_0 and spanning forest $U_0 = G - E(T_0)$, such that $\omega_{odd}(U_0) < 2k + 2$, and hence such that U_0 and T_0 violate Lemma 3.

By (17) and (16),

$$| E_0 | \geq 2k + 2 = \omega(G - E_0),$$

and so some edges of E_0 are not cut-edges of G. Hence, we can index the components $U_1, U_2, ..., U_{2k+2}$ of the forest $U = G - E(T)$ such that $| V(U_1) |$ is minimized, subject to the conditions that T is a spanning tree and that G has a cycle C such that

(18) $$E_0 \cap E(C) \neq \emptyset \text{ and } V(U_1) \cap V(C) \neq \emptyset.$$

Lemma 4 $T[V(U_1)]$ is connected.

Proof: Suppose not. Since U_1 is connected, there is an edge $e \in E(U_1)$ such that e joins different components of $T[V(U_1)]$, and $T[V(U_1)]+e$ is acyclic. Hence, the unique cycle C' of $T + e$ has an edge $e' \in E(C') \cap E_0$. Therefore, the pair

$$T' = T + e - e', \quad U' = U - e + e'$$

are edge-disjoint spanning forests of G with

$$\omega(T') = 1, \quad \omega(U') = \omega(U) = 2k + 2.$$

Since the component of $U_1 - e$ not incident with e' is smaller than U_1, either it contradicts the minimality of $|V(U_1)|$, or we have $\omega_{odd}(U') < 2k+2$, contrary to Lemma 3. □

Let $e_1, e_2, ..., e_r$ denote the edges of E_0 incident with $V(U_1)$, and let v_i be the end of e_i not in $V(U_1)$, for $1 \leq i \leq r$. Denote by V_i $(1 \leq i \leq r)$ the set of vertices of the component of $T - V(U_1)$ containing v_i. We claim that

Lemma 5 For all $i \leq r$,

(19) $$[V_i, V(U_1)] = \{e_i\};$$

thus the sets $V_1, V_2, ..., V_r$ are disjoint.

Proof: Since $T[V(U_1)]$ is connected, by Lemma 4, since $T[V_i]$ is connected, by definition, and since T is acyclic, it is not possible for two edges of T to join V_i and $V(U_1)$. By the definitions of U and U_1, no edge of U joins V_i and $V(U_1)$. Therefore, (19) follows, by the definition of e_i and since $E(G) = E(U) \cup E(T)$. By the definition of $V_1, V_2, ..., V_r$, and by (18), these sets are disjoint. □

Lemma 6 For any i and j satisfying $1 \leq i < j \leq r$, the subgraph

$$T_{ij} = T[V(U_1)] \cup T[V_i] \cup T[V_j] \cup \{e_i, e_j\}$$

is a subtree of T, and if $w_i \in V_i$ and $w_j \in V_j$, then the (w_i, w_j)-path in T_{ij} contains e_i and e_j.

Proof: Lemma 6 is a direct consequence of Lemmas 4 and 5. □

Case 1 Suppose that some component of U contains at least two vertices of $\{v_1, v_2, ..., v_r\}$.

Define U' to be the smallest subforest of $U - U_1$ that contains each existing (v_i, v_j)-path in U, where $1 \leq i < j \leq r$. By the minimality of U',

(20) Each endvertex of U' is in $\{v_1, v_2, ..., v_r\}$.

Any edge of $E(U') \cap [V_i, V_j]$ $(1 \leq i < j \leq r)$ shall be called a linking edge. By the condition of Case 1, and by the definition of U', there is a (v_s, v_t)-path $P(s,t)$ in U', for some distinct integers s and t, and $E((P(s,t))$ must contain at least one linking edge. Define the graph U'' to be the smallest forest in U'

that contains all linking edges and all paths in U' that connect distinct linking edges. Since U' has a linking edge,

$$(21) \qquad E(U'') \neq \emptyset,$$

and by the minimality of U'', each edge of U'' incident with an endvertex of U'' is a linking edge.

Hence, by (21) and since U'' is a forest, there is a linking edge $xy \in E(U'')$ such that

$$(22) \qquad x \text{ is an endvertex of } U''.$$

Since xy is a linking edge, we may suppose without loss of generality that

$$(23) \qquad x \in V_1, \quad y \in V_2.$$

Since xy is a linking edge, (20) implies that there are distinct integers s and t, each at most r, such that x precedes y on a (v_s, v_t)-path $P(s,t)$ of U'. Let U_x denote the component of $U - xy$ containing x. Since x is an endvertex of U'', by (22), U_x contains no linking edge. This fact, the existence of $P(s,t)$ and (23) together imply $s = 1$ and

$$(24) \qquad V(U_x) \cap \{v_1, v_2, ..., v_r\} = \{v_1\}.$$

Let U^* denote the component of $U - xy$ such that

$$(25) \qquad v_2 \in V(U^*).$$

By (24), v_2 is not in U_x, and so (25) implies that U_x and U^* are distinct components of $U - xy$. Thus,

$$(26) \qquad U^* \neq U_x.$$

By (12) and (14), all $2k+2$ components of U are R-odd, and hence exactly one component of $U - xy$ is R-even. Therefore, by (26), either U_x or U^* (or both) is an R-odd component of $U - xy$, and since U_1 is also R-odd, (19),(24), and (25) imply that there is some $i \in \{1,2\}$ such that the component of

$$U_0 = U - xy + e_i$$

containing e_i is R-even. Thus,

$$(27) \qquad \omega_{odd}(U_0) < \omega(U_0) = 2k + 2.$$

By (23) and Lemma 6, there is an (x,y)-path P_{12} in the tree T, and $e_1, e_2 \in E(P_{12})$. Hence, e_1 and e_2 are in the unique cycle of $T + xy$, and so for either value of $i \in \{1,2\}$,

$$T_0 = T + xy - e_i$$

is a spanning tree of G. But $U_0 = G - E(T_0)$, and so (27) shows that the tree T_0 and the forest U_0 contradict Lemma 3. The contradiction concludes Case 1.

Case 2 Suppose that no component of U contains more than one vertex of $\{v_1, v_2, ..., v_r\}$.

By (18), G has a cycle C that contains some edges of E_0 and at least one vertex of U_1. It follows from Lemma 5 that C must contain an edge $w_s w_t \in [V_s, V_t]$ for some integers s and t such that $1 \leq s < t \leq r$. By Lemma 6, T contains a (w_s, w_t)-path through e_s and e_t, and so we may assume that C is the unique cycle in $T + w_s w_t$.

Since $w_s w_t \in E(U)$, (12) and (14) imply that $U - w_s w_t$ contains $2k+2$ odd components and one even component. Hence, by the hypothesis of Case 2, there is an $i \in \{s, t\}$ such that v_i is in an odd component U_{odd}, say, of $U - w_s w_t$, and by definition, v_i is not in $V(U_1)$. Therefore, $U_1 \cup U_{odd} \cup \{e_i\}$ is an even component of

$$U_0 = U - w_s w_t + e_i,$$

and U_0 is a forest with $2k+2$ components, and with

$$\omega_{odd}(U_0) < \omega(U_0) = 2k+2. \tag{28}$$

Define

$$T_0 = T + w_s w_t - e_i.$$

Since e_i is an edge of the unique cycle C of $T + w_s w_t$, T_0 is a spanning tree. Also, $T_0 = G - E(U_0)$. By (28), U_0 and T_0 contradict Lemma 3. This completes Case 2 and proves the theorem. □

Corollary 1 (Catlin [2]) If $F(G) \leq 1$, then either G has a spanning closed trail, or G has exactly one cut-edge.

Corollary 2 (Jaeger [3]) If $F(G) = 0$, then G has a spanning closed trail.

Corollary 3 If $F(G) \leq 3$, then either G has a spanning trail (open or closed), or G is contractible to $K_{1,3}$ or $\omega(G) = 2$.

Corollaries 1 and 3 are immediate consequences of the main theorem and Euler's Theorem [1]. Corollary 2 is a special case of Corollary 1.

Conjecture If G is a 2-edge-connected graph with $F(G) \leq 2$, then either G has a spanning closed trail or G is contractible to $K_{2,t}$ for some odd $t \geq 3$.

We can prove this conjecture for simple graphs of girth at most four.

The main theorem is best-possible in a certain sense. Let G' be a tree of order $2k+2$, such that every vertex of G' has odd degree. Let $t \geq 3$ be an odd natural number. For any $vw \in E(G')$, define $G'(vw,t)$ to be the graph obtained from $(G'-vw) \cup K_{2,t}$ by identifying the two degree t vertices of $K_{2,t}$ with v and w, respectively, of $G'-vw$. Since $a(K_{2,t}) = 2$ and since $a(G') = 1$, it follows that $G'(vw,t)$ has arboricity 2, and thus satisfies (2). Hence,

$$\begin{aligned} F(G'(vw,t)) &= 2\,|V(G'(vw,t)|-2-|E(G'(vw,t)| \\ &= 2(|V(G')|+|V(K_{2,t})|-2)-2-|E(G'-vw)\cup K_{2,t}| \\ &= F(G')+1 = 2k+2. \end{aligned}$$

Since G' violates (a), it is easy to check that $G'(vw,t)$ also violates (a); and since there are only $|E(G'-vw)| = 2k+1$ cut-edges of $G'(vw,t)$, (b) also fails for $G'(vw,t)$.

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