HOMOMORPHISMS AS A GENERALIZATION OF GRAPH COLORING

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1. Introduction. We shall use the notation of Bondy and Murty [3], with minor modifications. A graph homeomorphic to H will be denoted a TH; for example, if G has a subgraph that is a subdivision of K_4, then we say that G is a TK_4.

Given a graph H, a homomorphism of G into H is a mapping $\theta : G \to H$ such that if $xy \in E(G)$ then $\theta(x)\theta(y) \in E(H)$. For example, an n-coloring of G is a homomorphism of G into K_n. Following the terminology for graph-coloring, we shall say that if $\theta : G \to H$ is a homomorphism, then θ is an H-coloring of G; and G is H-critical if G has no H-coloring, but for any $e \in E(G)$, $G - e$ has an H-coloring. We shall investigate H-critical graphs.

One known result on homomorphisms relates the chromatic and achromatic numbers. It is the Homomorphism Interpolation Theorem [6]. Minor changes in its proof yield the following slightly more general result: If G has an H-coloring and if there is a homomorphism of G onto $H + K_n$ then for any m such that $0 \le m \le n$, there is a homomorphism of G onto $H + K_m$.

A node in a graph G is a vertex not divalent. A path in G, joining two nodes, is an arc if its internal vertices are divalent. We denote the set of nodes of G by $N(G)$. For $x,y \in N(G)$, we use $A(x,y)$ to denote an arc joining nodes x and y in G. If the length of $A(x,y)$ is s, i.e., if $|E(A(x,y))| = s$, we may also denote it by $A_s(x,y)$. Given $u,v \in V(G)$, the graph $G + A_s(u,v)$ is obtained from G by attaching an arc of length s at u and v. The subgraph obtained from G by removing all edges and internal vertices of $A(u,v)$ is denoted $G - A(u,v)$. Note that $u,v \in V(G - A(u,v))$. Given a set $Y \subseteq N(G)$, the symbol $G - [Y]$ denotes the subgraph of G obtained by removing the nodes of Y, and the edges and internal vertices of all arcs incident with at least one vertex of Y.

2. Some general results. A graph G' is uniquely H-colorable if for any H-colorings θ_1, θ_2 of G' there is an automorphism $\emptyset$ of

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H such that $\emptyset\theta_1 = \theta_2$. Thus, $\emptyset\theta_1$ is also an H-coloring of G_1 .

<u>Proposition 1</u>. If G is H-critical, then G cannot be separated by a uniquely H-colorable subgraph G' .

<u>Proof</u>: Suppose that G is a union of proper subgraphs G_1 and G_2 , where $G_1 \cap G_2 = G'$ is uniquely H-colorable. Then there are H-colorings

$$\theta_1 : G_1 \to H \ , \quad \theta_2 : G_2 \to H$$

of G_1 and G_2 , since G is H-critical. Since G' is uniquely H-colorable, there is an automorphism $\emptyset$ of H such that $\emptyset\theta_1 = \theta_2$ on $V(G')$. Hence, $\emptyset\theta_1$ is an H-coloring of G_1 that can be combined with the H-coloring θ_2 of G_2 to form an H-coloring of G . This contradiction implies that G' does not exist.

<u>Proposition 2</u>. Let m be fixed. Suppose that for any subset

$$S = \{x_1, x_2, \ldots x_m\} \subseteq V(H)$$

there is a vertex $x_0 \in V(H - S)$ such that $x_0 x_i \in E(H)$ for $i = 1,2,3, \ldots ,m$. If G is an H-critical graph, then $\delta(G) \geq m + 1$.

<u>Proof</u>: Suppose by way of contradiction that in G

$$\deg(v) = k \leq m \ ,$$

where m and H satisfy the condition of the Proposition. If G is H-critical, then there is an H-coloring θ of $G - v$. Let $w_1, w_2, \ldots ,w_k$ be the vertices adjacent to v in G . By the condition on m and H , there is a vertex $x_0 \in V(H)$ adjacent to $\theta(w_1)$, $\theta(w_2),\ldots,\theta(w_m)$. Define $\theta(v) = x_0$, to obtain an H-coloring of G and a contradiction.

Note that for many graphs, the bound $\delta(G) \geq 2$ following from Proposition 1 gives a better bound on $\delta(G)$ than does Proposition 2.

<u>Proposition 3</u>. Suppose that any two vertices of H lie in an odd cycle of length at most $2m + 1$. Any arc of an H-critical graph has length at most $2m - 1$.

<u>Proof</u>: Let A be an arc of length at least 2m in an H-critical graph G , and suppose H satisfies the condition of the proposition. Let x,y be the ends of A . Since G is H-critical, there is a homomorphism θ that maps into H everything of G except A , but including x and y . By the condition of the proposition, $\theta(x),\theta(y) \in V(H)$ lie in an odd cycle C in H . The cycle C consists of an odd

path P_1 and an even path P_2 from $\theta(x)$ to $\theta(y)$, each of length less than $2m + 1$, and hence no longer than A . The mapping θ can be extended to A by mapping A onto P_1 if A has odd length, and onto P_2 if A has even length. Hence, G has an H-coloring, a contradiction. Thus, no arc of G has length more than $2m - 1$.

It may happen that for different graphs H_1 and H_2 , the statement

(1) G is H_1-colorable if and only if G is H_2-colorable

holds for any graph G . A simple example is when both H_1 and H_2 are bipartite. Then, for $i = 1,2$, G is H_i-colorable if and only if G is bipartite.

Theorem 1. The statement (1) holds for any graph G if and only if there is a graph K such that both

(i) H_1 and H_2 are both K-colorable; and

(ii) H_1 and H_2 both have subgraphs isomorphic to K .

Proof: Suppose that H_1 and H_2 both have the subgraph K , and suppose that for $i = 1,2$, $\theta_i : H_i \to K$ is a K-coloring of H_i . Let $\theta : G \to H_1$ be an H_1-coloring of G . Then $\theta_1\theta$ is a K-coloring of G . Since K is a subgraph of H_2 also, G has an H_2-coloring. Since the converse part of (1) is similarly proved, (1) holds.

Conversely, suppose that (1) holds for any graph G . Let K be a minimum subgraph of H_1 such that H_1 is K-colorable. By (1), with $G = K$, K has an H_2-coloring $\theta_2 : K \to H_2$. By way of contradiction, suppose θ_2 is not one-one. By (1), with $G = \theta_2[K]$, since $\theta_2[K]$ has an H_2-coloring (it is a subgraph of H_2) , it must have an H_1-coloring θ_1 and the composition $\theta_1\theta_2$ maps K to a subgraph in H_1 that is smaller than K , since θ_2 is not an injection. Since this contradicts the minimality of K , θ_2 must be an injection and K must be a subgraph of H_2 also. Now, let $\theta_K : H_1 \to K$ be a K-coloring of H_1 . By (1), with $G = H_2$, there is an H_1-coloring $\theta_1 : H_2 \to H_1$ of H_2 . Then $\theta_K\theta_1$ is a K-coloring of H_2 , and so H_2 is K-colorable. This completes the proof.

Theorem 1 asserts that when H-colorings are studied, it is sufficient to consider only those graphs H that have no proper subgraph K , where G has a K-coloring.

3. Homomorphisms into odd cycles. Obviously, the smallest C_{2k+1}-critical graphs ($k \geq 2$) are the odd cycles C_{2m+1}, where $1 \leq m < k$. Aside from these trivial examples, the smallest C_{2k+1}-critical graphs are subdivisions of K_4. A class of such graphs occurs in Theorem 2: for k odd, when the six edges of K_4 are each subdivided into arcs of length k, a C_{2k+1}-critical graph is formed. The smallest TK_4's that are C_{2k+1}-critical have $4k$ vertices, $4k+2$ edges, and girth $2k+1$.

Another class of small C_{2k+1}-critical graphs is formed by combining at most $2k-1$ copies of C_{2k+1} in a cyclic formation, such that each has two vertices of attachment to the rest of the graph. For $k = 2$, the only such example is the graph R of Figure 1, in which three copies of C_5 are arranged in a cyclic formation.

The set of all graphs having C_{2k+1}-colorings is denoted $\mathcal{B}_{2k+1}$. We shall occasionally express the homomorphism

$$\theta : G \to C_{2k+1}$$

in the alternative form

$$\theta : G \to Z_{2k+1}$$

where residue classes $a,b \in Z_{2k+1}$ are regarded as adjacent when $a-b \in \{1,-1\} \pmod{2k+1}$.

We generalize the case of Brooks' Theorem for graphs of maximum degree 3 to a condition for a homomorphism into an odd cycle.

Theorem 2. If a connected graph G has $\Delta(G) = 3$, and if every arc has length at least k, then either $G \in \mathcal{B}_{2k+1}$, or k is odd and G is obtained by subdividing each edge of a K_4 to form an arc of length k.

Proof: Let G be the smallest counterexample. Let $r = [k/2]$.

1. Suppose that G has an arc $A(v,y)$ of length at least $k+1$. Denote the other four arcs incident with $\{v,y\}$ by $A(v,w)$, $A(u,v)$, $A(x,y)$, $A(y,z)$. Let

$$P_v = A(u,v) \cup A(v,w) \ ; \ P_y = A(x,y) \cup A(y,z) \ .$$

Let $s = |E(P_v)|$, $t = |E(P_y)|$. Since G is the smallest counterexample,

$$H = G - [v,y] + A_{s-2r}(u,w) + A_{t-2r}(x,z)$$

has a C_{2k+1}-coloring θ. Since $|E(P_v)| - 2r = |E(A_{s-2r}(u,w)|$, θ can be extended from $G - [v,y]$ to P_v so that $\theta(v)$ has any one of at least $r+1$ values. Likewise, θ can be independently extended from $G - [v,y]$ to P_y so that $\theta(y)$ has any one of at least $r+1$ values. Since

$$|E(A(v,y))| + (r+1) + (r+1) \geq 2(k+1) > 2k+1 ,$$

θ can be further extended to $A(v,y)$, and thus $G \in \mathscr{S}_{2k+1}$.

We can assume henceforth that all arcs of G have length k. If k is even, then G is bipartite, and so $G \in \mathscr{S}_{2k+1}$. Therefore, assume that k is odd.

2. Suppose that G contains length k arcs $A(v,y_i)$, $A(x_i,y_i)$, $A(y_i,z_i)$, for $k = 1,2,3$, where y_1,y_2,y_3 are distinct and $\{y_1,y_2,y_3\} \cap \{x_1,x_2,x_3,z_1,z_2,z_3\} = \emptyset$. Let

$$H = G - [v,y_1,y_2,y_3] + \bigcup_{i=1}^{3} A_{2k-2r}(x_i,z_i) .$$

Since G is the smallest counterexample, H has a C_{2k+1}-coloring θ. Note that θ can be extended from $G - [v,y_1,y_2,y_3]$ to $G - [v]$ such that for each i, $\theta(y_i)$ has at least $r+1$ possible values. Extending θ further along $A_k(v,y_i)$ of length k, we see that the $r+1$ possible values of $\theta(y_i)$ permit $k+r+1$ possible values in $G - [v] + A_k(v,y_i)$. Hence, each $A(v,y_i)$ precludes $(2k+1) - (k+r+1) = k-r$ values of $\theta(v)$; and since $i \in \{1,2,3\}$, a total of

$$3(k-r) \leq 3(\frac{k+1}{2}) < 2k+1$$

values of $\theta(v)$ are precluded, if $k > 1$. Hence, if $k > 1$, then $G \in \mathscr{S}_{2k+1}$; and if $k = 1$, then $G \in \mathscr{S}_{2k+1}$ follows from a case of Brooks' Theorem [4].

3. Suppose that G contains length k arcs $A(v_1,v_2)$, $A(v_2,v_3)$, $A(v_1,v_3)$, and for all $i \in \{1,2,3\}$, $A(v_i,y_i)$, where the 3 y_i's are distinct. Let

$$H = G - [v_1,v_2,v_3] + A_k(y_1,y_2) .$$

Since G is the smallest counterexample, H has a C_{2k+1}-coloring $\theta : H \to Z_{2k+1}$. Let P denote the path $A(y_1,v_1) \cup A(v_1,v_2) \cup A(v_2,y_2)$ on $3k$ edges.

Define $S = \{1,3,5,\ldots,k\}$. Without loss of generality, $\theta(y_1) \equiv 0 \pmod{2k+1}$ and $\theta(y_2) \in S$. Since all arcs of G have length k , k odd, we can extend θ to P such that $\theta(v_2) - \theta(v_1) \equiv k$ and $(\theta(v_1),\theta(v_2))$ is any chosen member of

$$T = \{(-k,0),(-k+2,2),\ldots,(-1,k-1),(1,k+1) \pmod{2k+1}\} .$$

Let $(m-k,m) \in T$ and let $j \in \{m+1,m+3,\ldots,m+k\}$. Then $j-m \in S$ and

$$m - k - j \equiv m + k + 1 - j \pmod{2k+1} \in S .$$

Set $\theta(v_3) = j$. Then $\theta(v_3) - \theta(v_2) = j-m \in S$, and $\theta(v_1) - \theta(v_3) = (m-k) - j \in S$, and therefore, θ can be extended to $A(v_1,v_3) \cup A(v_2,v_3)$, for any choice of j and m . By selecting m appropriately, we can choose any of $k+1$ values for $j = \theta(v_3)$, in extensions of θ to $G - A(v_3,y_3)$. Among extensions of θ from H to $G - [v_1,v_2]$, since $|E(A(v_3,y_3))| = k$, there are $k+1$ possible values for $\theta(v_3)$. Since $(k+1) + (k+1) > 2k+1$, these two sets intersect, and so $G \in \mathcal{S}_{2k+1}$.

4. Suppose G contains length k arcs $A(u,v)$, $A(v,w)$, $A(v,x)$, $A(w,x)$, $A(w,y)$, $A(x,y)$, and $A(y,z)$, where $\{u,z\} \cap \{v,y\} = \emptyset$. Since G is the smallest counterexample, each component of $G - A(u,v) - A(y,z)$ has a C_{2k+1}-coloring. Since $|E(A(u,v))| + |E(A(y,z))| = 2k$, these colorings can be adjusted to form a C_{2k+1}-coloring of G .

5. The only remaining possibility is that G has a subdivided K_4 in which each edge of the K_4 is subdivided to form an arc of length k , k odd. Since G is connected and $\Delta(G) = 3$, this is all of G . This completes the proof.

Dirac [5] has given an inequality relating the number of edges and vertices in a chromatically critical graph. The following generalizes the special case of Dirac's Theorem for chromatically 4-critical graphs:

<u>Conjecture</u>. If G is C_{2k+1}-critical, then

$$(4k-1)|E(G)| \geq (4k+1)|V(G)| - 4k + 2 .$$

This conjecture, if proved, would imply the previous theorem of this paper.

4. <u>Results on C_5-critical graphs</u>. In a paper to be published separately, we prove the results stated below on C_5-critical graphs having no topological K_4 .

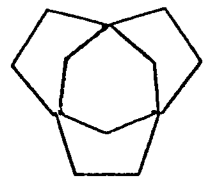

Figure 1

Let R be the graph of Figure 1. We refer to the three graphs of Figure 2 as <u>incremental subgraphs</u> of G when they occur as subgraphs of G with two vertices of attachment $\{x,y\}$.

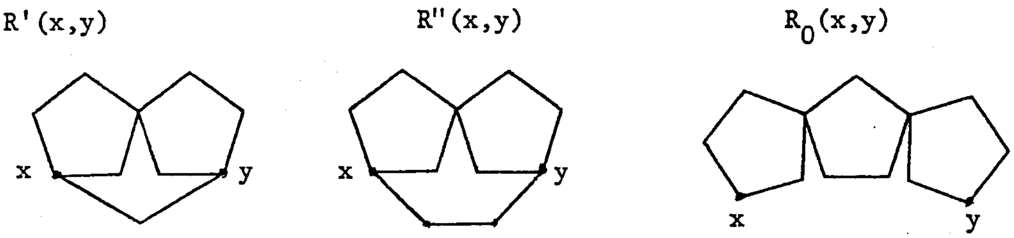

Figure 2: The incremental subgraphs

<u>Theorem 3</u>. If G is a C_5-critical graph with no TK_4 subgraph, and if G is neither K_3 nor R, then G contains two edge-disjoint incremental subgraphs.

<u>Theorem 4</u>. The graph G is C_5-critical and has no TK_4 subgraph if and only if G is obtained from K_3 by repeated applications of the following three operations:

1. The replacement of an arc $A_3(x,y)$ by $R_0(x,y)$;
2. The replacement of an edge xy by $R''(x,y)$;
3. The replacement of a vertex u by nonadjacent vertices x,y, the joining of every neighbor of u to exactly one of $\{x,y\}$, and the addition of $R'(x,y)$.

In each operation, the distinguished vertices $\{x,y\}$ of the incremental subgraph are identified with the corresponding vertices $\{x,y\}$ in the graph.

For example, R is obtained from K_3 by a single application of any of the three operations of Theorem 4.

<u>Corollary</u>. If G is C_5-critical and has no TK_4 subgraph, then

$$3|E(G)| + 3 = 4|V(G)| ,$$

and

$$|V(G)| \equiv 3 \pmod 9) .$$

Proof: Any operation of Theorem 4 adds nine vertices and twelve edges to a graph, and the construction begins with K_3 .

Theorem 5. If a graph G is obtained from R by repeated applications of the three operations of Theorem 4, then G is R-colorable.

Vestergombi [9] has given a necessary and sufficient condition for G to have a C_5-coloring.

5. Other results. In a joint paper with M. Albertson and L. Gibbons (this proceedings,[2]) we have given a number of other references to related work. That paper includes a statement of Vestergombi's result and a recent theorem of Albertson and Collins [1] giving necessary conditions for the existence of an H-coloring, when H is vertex transitive.

Pavol Hell has studied homomorphisms for over a decade. Much of his work takes the approach of category theory (see, e.g., [7]). An exception is [8].

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