JOURNAL OF COMBINATORIAL THEORY, Series B **26**, 268–274 (1979)

Hajós' Graph-Coloring Conjecture: Variations and Counterexamples

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Received June 7, 1978

For each integer $n \geqslant 7$, we exhibit graphs of chromatic number n that contain no subdivided K_n as a subgraph. However, we show that a graph with chromatic number 4 contains as a subgraph a subdivided K_4 in which each triangle of the K_4 is subdivided to form an odd cycle.

1. INTRODUCTION

In this paper, unless otherwise stated, we follow the notation of [4]. A *subdivided* K_n is a graph obtained by replacing edges $\{x, y\}$ of the complete graph K_n with $x - y$ paths. We refer to the vertices where such paths meet as *nodes* of the subdivided K_n. A node has degree $n - 1$.

The Hajós conjecture asserts that a graph with chromatic number n has a subdivided K_n as a subgraph.

For $n = 1$ and 2 this is trivial, and for $n = 3$, it is clear, because a 3-chromatic graph contains an odd cycle, which is a subdivided K_3. The case $n = 4$ of the conjecture was proved by Dirac [1].

2. COUNTEREXAMPLES

Let $\Sigma(G)$, the *subdivision number* of a graph G, denote the largest integer n such that G contains a subdividison of K_n as a subgraph. The Hajós conjecture asserts that $\Sigma(G) \geqslant \chi(G)$. Let $L(G)$, the *line graph* of G, be the graph with vertices $V(L(G)) = E(G)$, where $e, e' \in E(G)$ are adjacent in $L(G)$ whenever e and e' are incident at a vertex of G.

Let ${}_kG$ denote the multigraph obtained by replacing each edge $\{x, y\}$ of G with k edges joining x and y.

We first consider the case where G is an odd cycle.

PROPOSITION 1. *For all $k \geqslant 1$, if $n \geqslant 2$, then*

$$\Sigma(L({}_kC_{2n+1})) = 2k + 1.$$

0095-8956/79/020268-07$02.00/0

Proof. Among any $2k + 2$ vertices of $L({}_kC_{2n+1})$, at least two will be separated from each other by a cutset of $2k$ vertices. Hence, no subdivision of K_{2k+2} is contained in $L({}_kC_{2n+1})$.

On the other hand, the $2n$ edges of ${}_kC_{2n+1}$ incident with a given vertex form a clique K_{2n} in the line graph. This K_{2n} is contained in subdivided K_{2n+1}'s in $L({}_kC_{2n+1})$. This determines the subdivision number.

PROPOSITION 2. *For all positive k and n,*

$$\chi(L({}_kC_{2n+1})) = 2k + \lceil k/n.\rceil$$

Proof. Observe that $L(C_{2n+1}^k)$ has $(2n + 1)k$ vertices, and at most n lie in a given color class. Hence, $\chi(L({}_kC_{2n+1})) \geqslant (2n + 1)k/n$. While equality may be verified, we shall not need it as we show below that $L({}_kC_{2n+1})$ is a counterexample to Hajós' conjecture.

THEOREM 1. *For any integer $n \geqslant 2$, if $G = L({}_kC_{2n+1})$, then*

$$\inf_{k \geqslant 1} \Sigma(G)/\chi(G) = \frac{2n}{2n+1}.$$

Proof. Immediate, from Propositions 1 and 2.

A *class* 2 graph G is defined to be a graph having line-chromatic number $\chi'(G) = \Delta(G) + 1$. Such a graph is *critical* if the removal of any edge decreases $\chi'(G)$. Trivially, $\chi'(G) = \chi(L(G))$.

Conjecture. For any critical class 2 graph G, there is a natural number N such that if $k \geqslant N$ then

$$\Sigma(L({}_kG)) < \chi(L({}_kG)).$$

In other words, we conjecture that $L({}_kG)$ is a counterexample to Hajós' conjecture. The Petersen graph G is a noncritical class 2 graph for which such an integer N does not exist. Jakobsen [5] surveys some recent results concerning multigraphs G with $\chi'(G)$ significantly larger than $\Delta(G)$. The graphs ${}_kG$ discussed above are such multigraphs.

More counterexamples may be formed as follows: if v_1, v_2 are nonadjacent vertices in $L({}_kC_5)$, then $\Sigma(L({}_kC_5) - \{v_1, v_2\}) = 2k$ and $\chi(L({}_kC_5) - \{v_1, v_2\}) = \lceil (5k - 1)/2\rceil$. Thus, Hajós' conjecture fails for $L({}_3C_5) - \{v_1, v_2\}$, which has chromatic number 7, and subdivision number 6. If the Hajós conjecture fails for G, then since $\Sigma(G + v) = \Sigma(G) + 1$ and $\chi(G + v) = \chi(G) + 1$, the conjecture fails for $G + v$. Therefore, for any $n \geqslant 7$, there is a graph of chromatic number n for which Hajós' conjecture is false. The conjecture remains unsettled for $n = 5$ and 6.

Hajós' construction (see [3]), in which two n-critical graphs G and H (n-critical in the sense that the removal of an edge lowers the chromatic number to $n - 1$) are combined to form a single larger n-critical graph, is useful for creating even more counterexamples to his conjecture. Define an n-critical graph to be *n-minimal* if it has no proper subgraph that is a subdivision of an n-critical graph. Hajós' conjecture is that K_n is the only n-minimal graph. Let G and H be two n-critical graph, where $G_1, G_2, \ldots, G_s$ are the n-minimal graphs, subdivisions of which are subgraphs of G, and where $H_1, H_2, \ldots, H_t$ are the n-minimal graphs, subdivisions of which are subgraphs of H. Then $G_1, G_2, \ldots, G_s, H_1, \ldots, H_t$ are precisely the graphs which appear, subdivided, as subgraphs of the n-critical graph obtained from G and H when they are combined by Hajós' construction. It follows that for any finite set of n-minimal graphs, by repeated application of Hajós' construction, one can construct infinitely many n-critical graphs containing as subgraphs subdivisions of these, and no other, n-minimal graphs.

In [9], Tutte surveys some of the major problems on the chromatic number, including Hajós' conjecture, and he discusses their interrelation. We know of no counterexamples to the weaker conjecture of Hadwiger [2].

3. A Stronger Result for $n = 4$

The case $n = 3$ of Hajós' conjecture can be strengthened to assert that the subdivided K_3 must be an *odd* cycle. This is the well-known characterization of graphs G with chromatic number $\chi(G) \geqslant 3$. Thus, it is natural to ask if a similar strengthening of the conjecture is valid for any $n \geqslant 4$.

Toft [8] has conjectured that any 4-chromatic graph has a subdivided K_4 in which each of the six edges of the K_4 is subdivided to form a path of odd length. This is stronger than our theorem below. He asks other similar questions in [8], also. Zeidl [10] showed that any vertex of a 4-chromatic graph lies in some subdivided K_4 that contains an odd circuit. Indeed, Ore [6] showed that to merely find a subdivided K_4 in a graph G one only requires that $\delta(G) \geqslant 3$, and so it is not surprising that the stronger hypothesis $\chi(G) \geqslant 4$ gives stronger conclusions.

An *oddly subdivided* K_n is a subdivided K_n in which each triangle of the K_n is subdivided to form an odd cycle.

It is easy to verify that if three of the four triangles in a K_4 are subdivided to form odd cycles, then the fourth triangle is also subdivided to form an odd cycle.

THEOREM 2. *A graph with chromatic number* 4 *contains an oddly subdivided* K_4.

Proof. We assume that the graph G is 4-critical, i.e., $\chi(G - e) = 3$ for any edge e in the 4-chromatic graph G. Also, we assume that G is a 4-critical counterexample to the theorem with the minimum possible number of points.

It is known (see, e.g., [4], p. 141) that a 4-critical graph G must be 2-connected. If there are two vertices, x and y, whose removal disconnects G, then they are not adjacent, and $G - \{x, y\}$ has two components G_1 and G_2 such that x and y have the same color in any 3-coloring of $G - G_1$, and x and y have different colors in a 3-coloring of $G - G_2$. Let G' denote the graph obtained by adding the edge $\{x, y\}$ to $G - G_1$. Thus, $\chi(G') = 4$, and since we have assumed that G is the smallest counterexample, G' has an oddly subdivided K_4. Hence, $G + \{x, y\}$ has this same oddly subdivided K_4 as a subgraph in $(G - G_1) + \{x, y\}$. We only need to consider the case where the edge $\{x, y\} \in E(G')\backslash E(G)$ is part of this subdivided K_4. Since both vertices of attachment (x and y) of G_1 to $G - G_1$ have the same color in a 3-coloring of $G - G_1$, we see that if G_1 is bipartite, then $\chi(G) = 3$. Hence, G_1 has an odd cycle C. By a generalization of Menger's Theorem, there are disjoint paths P_{ax} and P_{by} from $a, b \in V(C)$ to x and to y, respectively. The edge $\{x, y\}$ of the oddly subdivided K_4 may be replaced by an odd $x - y$ path in $G - G_2$ containing P_{ax}, P_{by}, and the appropriate $a - b$ path in C. Hence, if G is not 3-connected, then G contains an oddly subdivided K_4.

Throughout the remainder of the proof, which we divide into three cases, we assume that G is 3-connected.

Case I. Every vertex of G lies in two or more triangles, and there is a pair of triangles that share an edge.

Suppose that G contains a wheel as a subgraph. If the wheel has the form $C_n + x$, n odd, then $G = C_n + x$, since $\chi(C_n + x) = 4$ and G is 4-critical. But then G contains an oddly subdivided K_4. If the wheel has the form $C_n + x$, n even, then we claim that G is not 4-critical. Since G is 4-critical, $\chi(G - e) = 3$, for any edge $e \in C_n$. In this 3-coloring of $G - e$, the colors assigned to $C_n - e$ alternate between the two colors not used on x. Hence the ends of e have different colors, and so $\chi(G) = 3$, a contradiction that shows that n is not even. Toft [7] credits this observation that n cannot be even to *M.* Simonovits.

Hence no subgraph of G is a wheel. Let P be a maximum path such that each vertex of P is adjacent to a vertex x of G. Denote the vertices of P by $y_0, y_1, \ldots, y_n$. Since two triangles of G share an edge and since P is maximum, $n \geqslant 2$.

By the condition of Case 1, y_0 lies in a second triangle $\{v, w, y_0\}$, where possibly $v = y_1$. Since G has no wheel,

$$\{v, w\} \cap \{y_2, \ldots, y_n\} = \varnothing,$$

and since P is maximum, $x \notin \{v, w\}$.

If $y_1 \notin \{v, w\}$, then we have triangles $\{v, w, y_0\}$, $\{x, y_0, y_1\}$, and $\{x, y_1, y_2\}$ for distinct v, w, x, y_0, y_1, y_2. A complete subgraph in a 4-critical graph cannot be a cutset (see e.g., [2], p. 141, Corollary 12.24), and so there is a $\{v, w\} - y_2$ path in $G - \{x, y_0, y_1\}$. Such a path can be extended to form an odd $y_0 - y_2$ path P' in $G - \{x, y_1\}$. The union of the path P' and the triangles $\{x, y_0, y_1\}$ and $\{x, y_1, y_2\}$ forms the desired oddly subdivided K_4.

If, however, $y_1 = v$, then each vertex of the path w, y_0, x, y_2 is adjacent to v, and since P is a maximum path whose vertices are all adjacent to the same vertex, $n \geqslant 3$. Thus, we have triangles $\{w, y_0, y_1\}$, $\{x, y_1, y_2\}$, and $\{x, y_2, y_3\}$, for distinct w, x, y_0, y_1, y_2, y_3. We can proceed as in the previous paragraph to construct a $\{w, y_0\} - y_3$ path (and hence an odd $y_1 - y_3$ path) in $G - \{x, y_1, y_2\}$, thus forming an oddly subdivided K_4 in G.

Case II. Every vertex of G lies in two or more triangles, but no two triangles share an edge.

Suppose first that for any odd cycle C and any vertex $x \in V(G)\backslash V(C)$, x is adjacent to a vertex of C. Let $\{a, b, c\}$ and $\{c, d, e\}$ be two triangles with a common vertex c. There is a second triangle $\{e, f, g\}$ at e. If $f \in \{a, b\}$ or $g \in \{a, b\}$, then $\{e, f, g\}$ overlaps another triangle in an edge (for instance, if $f = a$, then $\{e, f, g\}$ and $\{e, f, c\}$ share the edge $\{e, f\}$), contrary to the condition of Case II. Hence, $\{f, g\} \cap \{a, b\} = \varnothing$. We have assumed that for an odd cycle $C = \{a, b, c\}$ and a vertex f (or g), f (resp., g) is adjacent to a vertex of C. Since no two triangles share an edge, $\{f, c\}, \{g, c\} \notin E(G)$, and we cannot have both $\{a, f\}, \{a, g\} \in E(G)$ nor both $\{b, f\}, \{b, g\} \in E(G)$, for the same reason. Thus, either $\{a, f\}, \{b, g\} \in E(G)$ or $\{a, g\}, \{b, f\} \in E(G)$. Without loss of generality, assume that $\{a, f\}, \{b, g\} \in E(G)$. Then an oddly subdivided K_4 is formed by the cycle $C = \{a, b, c\}$ and the even arcs (c, d, e), (a, f, e), and (b, g, e).

Next, suppose that C is an odd cycle and $x \in V(G)$, such that x is adjacent to no vertex of C. Then two triangles $\{v, w, x\}$ and $\{x, y, z\}$ containing x share no vertex with C. Since G is 3-connected, there are three disjoint paths from C to $\{v, w, x, y, z\}$.

Subcase IIA. Two of these disjoint paths terminate at v and w, respectively. (This is equivalent to the case where two paths terminate at y and z.) Let $a, b \in V(C)$, where a is the start of the C-v path P_{av} and b is the start of the C-w path P_{bw}. There are two a-b paths in C: one is odd and one is even. We choose the one which, together with P_{av} and P_{bw}, forms an even v-w path P_{vw} through part of C.

If $\{v, w, x\}$ were a cutset, then G would not be 4-critical, since a three-point cutset in a 4-critical graph cannot form a complete subgraph (see, e.g., [4], p. 141, Corollary 12.24). Hence, there is a path P in $G - \{v, w, x\}$

from a vertex u of $P_{vw} - \{v, w\}$ to either y or z (to z, say). Thus, we have an oddly subdivided K_4, with nodes u, v, w, x, formed by the cycle $\{v, w, x\}$, the paths along P_{vw} from v and w to u, and the x-u path along P and with the edge $\{x, z\}$ or the path (x, y, z). The choice of the edge $\{x, z\}$ or the path (x, y, z) is determined by the requirement that the two cycles of the subdivided K_4 that share the path P must be odd.

Subcase IIB. There are three disjoint paths P_{av}, P_{bx}, and P_{cy} from the odd cycle C to the overlapping triangles $\{v, w, x\}$, $\{x, y, z\}$, where the ends of each path are denoted by its two subscripts. (After appropriate relabelling, any three disjoint paths from C to the two triangles are either an instance of Subcase IIA or of subcase IIB.)

We form an oddly subdivided K_4 with nodes a, b, c, and x. We use the odd cycle C, the paths P_{av}, P_{bx}, P_{cy}, and either edges ($\{v, x\}$ or $\{y, x\}$) or paths ((v, w, x) or (y, z, x)), such that the lengths of the corresponding subdivided triangles are odd.

Case III. There is a vertex x lying in at most one triangle. If x lies in a triangle, denote one of its remaining two vertices by w. Denote each of the remaining vertices adjacent in G to x by $v_1, v_2, \ldots, v_s$, where $\{v_1, w, x\}$ is the triangle, if a triangle containing x exists. Of course, $\{v_1, v_2, \ldots, v_s\}$ are pairwise nonadjacent.

Define G_x to be the graph obtained from G by deleting x and identifying all vertices $\{v_1, v_2, \ldots, v_s\}$ as a single vertex v. If $\chi(G_x) \leqslant 3$, then there is a 3-coloring of $G - x$ in which the vertices adjacent to x in G are colored in at most 2 colors ($\{v_1, \ldots, v_s\} \in V(G)$ are assigned the color of $v \in V(G_x)$, and $w \in V(G)$ is assigned the color of $w \in V(G_x)$), and so $\chi(G) = 3$, a contradiction. Thus $\chi(G_x) \geqslant 4$.

Since $\chi(G_x) \geqslant 4$ and since the theorem is assumed to be true for graphs smaller than G, G_x contains an oddly subdivided K_4, say H. If $v \notin V(H)$, then H is an oddly subdivided K_4 of G. Therefore, suppose $v \in V(H)$ in G_x.

In H, v is adjacent to either two or three other vertices. Denote the neighborhood $N_H(v)$ of v in H by $\{z_i, z_j\}$ or by $\{z_i, z_j, z_k\}$, accordingly.

If all vertices of $N_H(v)$ are adjacent in G to the same vertex $v_h \in \{v_1, \ldots, v_s\}$, then H is a subgraph of G, with v_h in place of v.

If all vertices (z_i, z_j, and possibly z_k) of $N_H(v)$ are adjacent in G to different vertices $v_i, v_j, v_k \in \{v_1, \ldots, v_s\}$, respectively, then we replace the edges $\{v, z_i\}$, $\{v, z_j\}$, $\{v, z_k\}$ of H in G_x by paths (x, v_i, z_i), (x, v_j, z_j), and (x, v_k, z_k), which, together with the remaining edges of H, form a larger oddly subdivided K_4 in G. (Of course, when $N_H(v) = \{z_i, z_j\}$, the references in the previous sentence to edges and paths containing z_k are deleted.)

Finally, if two vertices $z_i, z_j \in N_H(v)$ are adjacent to $v_h \in \{v_1, \ldots, v_s\}$, and if $z_k \in N_H(v)$ is adjacent to $v_k \neq v_h$, then an oddly subdivided K_4 is formed in G by the edges of H together with a path (v_h, x, v_k) in G in place of the vertex $v \in V(G_x)$.

Since an oddly subdivided K_4 was formed in each case, the theorem is proved.

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Printed in Belgium