EMBEDDING SUBGRAPHS UNDER EXTREMAL DEGREE CONDITIONS

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Abstract. Let G and H be graphs on p vertices. Let $\Delta(H)$ denote the maximum degree of the vertices of H, and let G^c denote the complement of G. We consider the problem of giving a sufficient condition, in terms of p, $\Delta(H)$, and $\Delta(G^c)$, for H to be a subgraph of G. We begin by summarizing known results, and we give two classes of graphs (one class is new) that show that the conjectured sufficient condition

$$(\Delta(H) + 1)(\Delta(G^c) + 1) \le p+1$$

is best possible, if true. For the special case when H has $[p/3]$ triangular components, we prove* the conjecture and show that these two classes of graphs G^c making the conjecture best possible are the only such extremal graphs. This improves a result of Corrádi and Hajnal.

In this paper, we consider simple graphs, and we follow the notation of Harary [12], with only a few exceptions.

The complement of a graph G is denoted G^c. Letting $\deg_G(v)$ denote the degree of v in G, we have

$$\deg_G(v) + \deg_{G^c}(v) = p - 1.$$

Also, the maximum degree $\Delta(G^c)$ of G^c and the minimum degree $\delta(G)$ of the vertices of G satisfy

$$\Delta(G^c) + \delta(G) = p - 1.$$

For two graphs G and H, each on p vertices, an embedding of H into G is a bijection

$$\pi: V(H) \to V(G)$$

*The proof of Theorem 6 will be published separately. It also appears in our dissertation, done at the Ohio State University under Professor Neil Robertson.

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that maps edges of H to edges of G. If such an embedding exists, we say that H is a <u>subgraph</u> of G. It is frequently useful in what follows to remember that H is a subgraph of G if and only if H and G^c can be constructed on the same set of vertices so that $E(H) \cap E(G^c) = \emptyset$. Also, H is a subgraph of G if and only if G^c is a subgraph of H^c.

We consider the following problem:

<u>Let G,H be graphs on p vertices, with $\Delta(H)$ "small". What lower bound on $\delta(G)$ (upper bound on $\Delta(G^c)$), in terms of $\Delta(H)$ and p, ensures that H is a subgraph of G?</u>

We shall first discuss the relevant results in the literature and their relationship to a conjectured lower bound on $\delta(G)$. We exhibit two classes of graphs that show that this conjecture is best possible. In the case $\Delta(H) = 2$, we state some partial results, and we discuss a tool used to obtain them: a generalization of the method of alternating paths.

An early result of the sort we wish to consider is:

<u>Theorem 1</u> (Dirac [8]). If G is a graph on $p \geq 3$ vertices, and if

(1) $\delta(G) \geq p/2$,

then G is hamiltonian.

In Dirac's Theorem, H is a cycle on p vertices, and $\Delta(H) = 2$. It has been generalized:

<u>Theorem 2</u> (Bondy [2]). If G satisfies (1), then either every cycle of girth at most p is a subgraph of G, or G is the complete bipartite graph $K_{p/2,p/2}$, and G contains no cycles of odd girth.

In Bondy's Theorem, H consists of a single cycle of girth $g \leq p$ and $p - g$ isolated vertices. Of course, $\Delta(H) = 2$. Actually, Bondy's hypothesis was more general: G is hamiltonian and $|E(G)| \geq p^2/4$. However, we are concerned here with conditions in terms of $\delta(G)$.

Theorems 1 and 2 raise the question: Given a positive integer h, for what real number $c_h < 1$ does

(2) $\delta(G) \geq c_h p$

guarantee that <u>any</u> graph H, with $\Delta(H) \leq h$, is a subgraph of G? Of course, both G and H have p vertices.

In [4], we showed that

$$c_h = 1 - \frac{1}{2h(h+1)}$$

could be substituted in (1), and we remarked in a footnote that this could be improved to

$$c_h = 1 - \frac{1}{2h}.$$

This is stated more precisely in the following theorem [5], also obtained independently by Sauer and Spencer [14]:

Theorem 3 If G and H are graphs on p vertices, and if

(3) $2\Delta(H)\Delta(G^c) \le p-1$,

then H is a subgraph of G.

There are conditions for H to be a subgraph of G that are based on the cardinality of $E(G)$, instead of $\delta(G)$. Some have been given by Erdös and Stone [10], Sauer and Spencer [14], and Bollobás and Eldridge [1]. Numerous others are listed by Erdös [9].

Except when $\Delta(H)$ or $\Delta(G^c)$ is 1, the inequality (3) is not best possible. The following, however, seems reasonable:

Conjecture For graphs G and H on p vertices, if

(4) $(\Delta(H) + 1)(\Delta(G^c) + 1) \le p+1$,

then H is a subgraph of G.

This conjecture would give the coefficient

(5) $c_h = 1 - \frac{1}{h+1}$

in (1).

We give two classes of examples to show that this conjecture, if true, is best possible.

Let g,h be positive integers satisfying

(6) $(h+1)(g+1) = p+2$.

The graph H is said to be in class $C_1(h)$ when $\Delta(H) = h$, $|V(H)| = p$, and when H has g components isomorphic to K_{h+1}; H is in class $C_2(h)$ when H has $g-1$ components isomorphic to K_{h+1} and one component isomorphic to $K_{h,h}$, where h is odd and $|V(H)| = p$.

For any p and odd h satisfying (6), there is only one graph $H \in C_2(h)$. It is regular of degree h. If h is even, $C_2(h)$ is empty. However, any $H \in C_1(h)$ has

$$p - g(h+1) = h - 1$$

vertices outside of the g components isomorphic to K_{h+1}. On these $h-1$ vertices, there is no restriction on the existence of edges.

If

$$H \in C_1(h) \cup C_2(h)$$

and if

$$G^c \in C_1(g) \cup C_2(g),$$

where p,h,g satisfy (6), then H is not a subgraph of G, unless both $H \in C_2(h)$ and $G \in C_2(g)$. By (6), these graphs barely violate (4). Therefore, (2.4) is best possible if the conjecture holds.

There are special cases for which the conjecture has been proved. By Theorem 3, the case $\Delta(H) = 1$ is one of them. There are others, of a similar nature:

<u>Theorem 4</u> (Corrádi and Hajnal [7]). If H has $[p/3]$ triangular components and if

$$(7) \qquad \delta(G) \geq \frac{2p-1}{3},$$

then H is a subgraph of G.

The inequality (7) is equivalent to (4) when $\Delta(H) = 2$. By the examples above, with $H \in C_1(2)$, Theorem 4 is best possible.

<u>Theorem 5</u> (Hajnal and Szemerédi [11]) Let $h \geq 2$ be an integer. If H has $[p/(h+1)]$ components isomorphic to K_{h+1}, and if

$$(8) \qquad \delta(G) \geq \frac{hp-1}{h+1},$$

then H is a subgraph of G.

As before, (8) is equivalent to (4), and (8) gives the coefficient c_h of (5).

The examples above, with $H \in C_1(h)$, show that Theorem 5 is best possible.

In each case, the authors of Theorems 4 and 5 gave $C_1(g)$ as a class of graphs which showed the respective theorems to be best possible. The class $C_2(g)$ is new.

We have made some progress ([6] and [7]) in the case $\Delta(H) = 2$. First, we state the results, whose proofs are quite long, and finally we describe the tool used in obtaining these results.

<u>Theorem 6</u> If H has $[p/3]$ components isomorphic to K_3, and if

$$\delta(G) \geq \frac{2p-2}{3},$$

then either H is a subgraph of G, or $H \in C_1(2)$, $G^c \in C_1(\frac{p-1}{3}) \cup C_2(\frac{p-1}{3})$ and H is not a subgraph of G.

This characterizes the graphs that make Theorem 4 best possible.

<u>Theorem 7</u> Suppose $\Delta(H) = 2$. There is a function f satisfying $f(p) = O(p^{1/3})$, such that if

$$\delta(G) \geq \frac{2p}{3} + f(p),$$

then H is a subgraph of G.

By the examples of Theorem 6, the coefficient $\frac{2}{3}$ is best possible. Recently, we have obtained $f(p) = O(1)$. The details will appear at a later date.

The method we use is a generalization of the concept of alternating paths used in matching theory. There is also another general method, due to Bondy and Chvátal [3], that may be applied to similar questions concerning subgraphs or other problems. However, we shall only describe here our "method of alternating chains".

Given a mapping (not necessarily an embedding)

$$\pi: V(H) \to V(G),$$

we may regard H and G as being two graphs with the same vertex set. For instance, $w \in V(H)$ is identified with $\pi(w) \in V(G)$. Define, for each $v \in V(G)$, the <u>neighborhood</u> $N_G(v)$ of v in G to be the set of vertices adjacent in G to v. For each $v \in V(G)$, define $N_H(v) \subseteq V(G)$ to be the set of vertices adjacent in H to v, where $V(H)$ and $V(G)$ are identified by π. In other words, $N_H(v) = \pi N(\pi^{-1}(v))$, where $N(w)$ is the neighborhood of $w \in V(H)$ in H.

An alternating chain is a sequence

$$v_0, v_1, \cdots, v_n$$

of distinct vertices of G having the properties

(i) $N_H(v_n) \subseteq N_G(v_0)$;

(ii) $N_H(v_i) \subseteq N_G(v_{i+1})$ for $i = 0,1,\ldots,n-1$;

(iii) $v_i \in N_H(v_j)$ for no $i,j \in \{0,1,\ldots,n\}$.

Let α be the permutation $(v_0\ v_1 \cdots v_n)$. Note that $\alpha\pi$ embeds into G every edge of H incident with $v_0, v_1, \ldots,$ or v_n. Also, except for these edges, $\alpha\pi$ agrees with π. Thus, if we can find an alternating chain in which an unembedded edge of H is moved by α, then $\alpha\pi$ embeds more edges of H into G than does π. We repeat this process until the mapping π is altered so that all edges are embedded in G.

When $\Delta(H) = 1$, this amounts to the method of alternating paths commonly used with matchings.

In the proof of Theorem 3, one considers all transpositions $(v_0\ v)$, where v_0 is incident with an unembedded edge of H, and v roams over all the vertices. When $v \notin N_H(v_0)$, the sequence v_0, v may be an alternating chain. It is easy to show, given (3), that for some $v \in V(G)$, $(v_0\ v)\pi$ maps more edges of H to $E(G)$ than does π. This was done in [5] and [14].

It is routine to show that if the inequality (3) is relaxed to

$$2\Delta(H)\Delta(G^c) \le p,$$

then either H is a subgraph of G or $H \in C_1(1)$, $G^c \in C_1(\frac{p}{2}) \cup C_2(\frac{p}{2})$, or vice versa.

Although we used alternating chains in a portion of the proof of Theorem 6, their main benefit was in proving Theorem 7. Here as many as three cycles α, β, and γ (each associated with an alternating chain in G + an edge), must be found, such that $\alpha\beta\gamma\pi$ embeds one more edge of H into G than does π. The details of this are in [5] or [6].

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