Reprinted from the AMERCAN MATHEMATICAL MONTHLY
Vol. 77, No. 1, January, 1970

CONCERNING THE ITERATED ϕ FUNCTION

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Define a set of functions ϕ^n mapping integers greater than 1 into the natural numbers, such that $\phi^0(a)=a$ and $\phi^{n+1}(a)=\phi(\phi^n(a))$, where ϕ is Euler's function. Furthermore, define the class $C(a)$ of a to be the integer n such that $\phi^n(a)=2$, and let M be the set of natural numbers with the least value of any member in their respective classes. Harold Shapiro made the conjecture in the last paragraph of his article [2] that each element of M was prime, and W. H. Mills [1] exhibited several composite members of M. The purpose of this note is to extend their results concerning the factorization of elements of M.

H. Shapiro gave a theorem (Theorem 15, [2]) which stated essentially that if S is the set of numbers in class n that are less than 2^{n+1}, then the factors of an element of S are in S. This is also true for odd elements of M.

THEOREM 1. *If m is an odd element of M, the factors of m are in M.*

Proof. It is sufficient to show that an arbitrary factor is in M. If $m\in M$ is prime, then the theorem is obvious. Otherwise, if $m\in M$ is odd, then so are its factors. Let $m=ab$. By Theorem 1 of [2], $C(m)=C(a)+C(b)$.

If $b\notin M$ then there must be a lower number $c\in M$ in the same class as b. However, by Theorem 1 of [2], we see that $C(m)=C(a)+C(b)=C(ac)$ where $ac<m$, contradicting the definition of M. Thus, $b\in M$, proving the theorem.

As a consequence of Theorem 7 of [2], we can make a similar statement for even members of M: in this case, the only prime factor of an element of M is 2, which is in M. However, 2 is the only known even member of M.

The following theorem shows that the existence of finitely many primes in M is equivalent to an apparently weaker condition:

THEOREM 2. *There are finitely many primes in M if and only if there are finitely many odd numbers in M.*

Proof. If there are infinitely many primes in M, then obviously there are infinitely many odd numbers. Conversely, if an odd prime $p\in M$ then $p^\alpha \in M$ for at most a finite number of α. This follows since $p^\alpha \in M$ implies

$$2^{\alpha C(p)} = 2^{C(p^\alpha)} < p^\alpha < 2^{\alpha C(p)+1}$$

or

$$2^{C(p)} < p < 2^{C(p)+1/\alpha}$$

which is false for α sufficiently large. Thus, if M contains only finitely many odd primes then it contains only finitely many odd prime powers; and by Theorem 1, only finitely many odd integers.

As a corollary, Theorem 2 implies that S contains infinitely many primes if and only if it contains infinitely many odd numbers. This follows because for any $m\in M$ there are finitely many $s\in S$ such that $m \leqq s < 2^{C(s)+1}$, so if M contains finitely many primes, so does S; the converse follows from $M\subset S$.

The questions of whether or not M and S contain infinitely many odd integers remain open. In all likelihood the smallest integer of each class is odd.

References

1. W. H. Mills, Iteration of the ϕ function, this MONTHLY, 50 (1943) 547–549.

2. Harold Shapiro, An arithmetic function arising from the ϕ function, this MONTHLY, 50 (1943) 18–30.