Table for Combinatorial Numbers and Associated Identities: Table 2

From the seven unpublished manuscripts of H. W. Gould
Edited and Compiled by Jocelyn Quaintance

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1 Bernoulli Numbers $\mathcal{B}_n$

Remark 1.1 *Throughout this chapter, we assume n and p are nonnegative integers. We assume x is a real or complex number. Furthermore, for any real x, we let $[x]$ denote the floor of x.*

1.1 Generating Function Definition of $\mathcal{B}_n$

$$\sum_{k=0}^{\infty} \mathcal{B}_k \frac{x^k}{k!} = \frac{x}{e^x - 1}, \qquad |x| < 2\pi \tag{1.1}$$

1.2 Alternative Definition for $\mathcal{B}_n$

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} n^{p+1-k} \mathcal{B}_k, \qquad n \geq 1 \tag{1.2}$$

1.3 Explicit Formulas of $\mathcal{B}_n$

$$\mathcal{B}_p = \sum_{j=0}^{p} \frac{1}{j+1} \sum_{k=0}^{j} (-1)^k \binom{j}{k} k^p \tag{1.3}$$

$$\mathcal{B}_n = \frac{2}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k+1} B_{k,k}^{n+1} \sum_{j=1}^{k} \frac{1}{j}, \quad \text{where } B_{k,k}^{n} = \sum_{j=0}^{k} (-1)^k \binom{k}{j} (k-j)^n \tag{1.4}$$

$$\mathcal{B}_n = n! \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{B_{k,k}^{n+k}}{(n+k)!} \tag{1.5}$$

1.3.1 Alternative Formulations of Equation (1.3)

$$\mathcal{B}_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} B_{k,k}^{n} \tag{1.6}$$

$$\mathcal{B}_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \sum_{j=0}^{n} \binom{j-1}{n-k} A_{j,n}, \text{ where } A_{j,n} = \sum_{k=0}^{j} (-1)^k \binom{n+1}{k} (j-k)^n \tag{1.7}$$

$$\mathcal{B}_n = (-1)^n \sum_{k=1}^{n} A_{k,n} \frac{(-1)^{k-1}}{(n+1)\binom{n}{k-1}}, \qquad n \geq 1 \tag{1.8}$$

$$\mathcal{B}_n = \sum_{k=1}^{n} (-1)^k \frac{A_{k,n}}{(n+1)\binom{n}{k}}, \qquad n \geq 1 \tag{1.9}$$

1.4 Vandiver's Formulas for $\mathcal{B}_n$

Remark 1.2 *The formulas in this section are the work of H. S. Vandiver. The pertinent papers are "On generalizations of the numbers of Bernoulli and Euler", Proc. of the National Academy of Sciences, Vol. 23, 1937, pp. 555-559 (also see Proc. of the National Academy of Sciences, Vol. 25, 1939, pp. 197-201), and "Explicit expressions for generalized Bernoulli numbers", Duke Math. Journal, Vol. 8, No. 3, Sept. 1941, pp. 575-584.*

$$\mathcal{B}_n = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{1}{k+1} \sum_{j=0}^{k} j^n \tag{1.10}$$

$$\mathcal{B}_n = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{1}{k+1} \sum_{j=0}^{k+1} j^n, \qquad n \geq 1 \tag{1.11}$$

$$\mathcal{B}_n = (-1)^n \frac{n!}{n+1} + \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{k+1} \sum_{j=0}^{k} j^n, \qquad n \geq 1 \tag{1.12}$$

1.5 Properties of $\mathcal{B}_n$

1.5.1 Recursive Relation

$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k = (-1)^n \mathcal{B}_n \tag{1.13}$$

1.5.2 Parity Properties

$$\mathcal{B}_{2n} = \sum_{k=0}^{2n} \frac{(-1)^k}{k+1} B_{k,k}^{2n} \tag{1.14}$$

$$\mathcal{B}_{2n+1} = \sum_{k=0}^{2n+1} \frac{(-1)^k}{k+1} B_{k,k}^{2n+1} = 0, \qquad n \geq 1 \tag{1.15}$$

$$\mathcal{B}_{2n} = \sum_{k=0}^{2n} (-1)^k \frac{A_{k,2n}}{(2n+1)\binom{2n}{k}} \tag{1.16}$$

$$\mathcal{B}_{2n+1} = \sum_{k=0}^{2n+1} (-1)^k \frac{A_{k,2n+1}}{(2n+2)\binom{2n+1}{k}} = 0, \qquad n \geq 1 \tag{1.17}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \mathcal{B}_k = \mathcal{B}_{2n} \tag{1.18}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} \mathcal{B}_k = 0, \qquad n \geq 1 \tag{1.19}$$

$$\sum_{k=0}^{n-1} \binom{2n}{2k} \mathcal{B}_{2k} = n, \qquad n \geq 1 \tag{1.20}$$

$$\sum_{k=0}^{n} \binom{2n+1}{2k} \mathcal{B}_{2k} = n + \frac{1}{2}, \qquad n \geq 1 \tag{1.21}$$

$$\sum_{k=1}^{n} \binom{2n}{2k-1} \mathcal{B}_{2k} = \frac{1}{2} - \mathcal{B}_{2n}, \qquad n \geq 1 \tag{1.22}$$

$$\sum_{k=1}^{n} \binom{2n+1}{2k-1} \mathcal{B}_{2k} = \frac{1}{2}, \qquad n \geq 1 \tag{1.23}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \mathcal{B}_k = \mathcal{B}_{2n} + 2n, \qquad n \geq 1 \tag{1.24}$$

2 Bernoulli Polynomials $\mathcal{B}_n(x)$

Remark 2.1 *Throughout this chapter, we assume n, r, and p are nonnegative integers. We assume x, y, a, b, and t are real or complex numbers. Furthermore, for any real x, we let $[x]$ denote the floor of x.*

2.1 Definition of $\mathcal{B}_n(x)$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{B}_k(x) = \frac{te^{xt}}{e^t - 1}, \qquad |t| < 2\pi \tag{2.1}$$

2.1.1 Relationship to $\mathcal{B}_n$

$$\mathcal{B}_n = \mathcal{B}_n(0) \tag{2.2}$$

2.2 Alternative Definitions of $\mathcal{B}_n(x)$

$$\sum_{k=0}^{p-1} k^n = \frac{\mathcal{B}_{n+1}(p) - \mathcal{B}_{n+1}}{n+1}, \qquad n, p \geq 1 \tag{2.3}$$

$$\mathcal{B}_{n+1}(x) = (n+1) \sum_{k=0}^{n} \binom{x}{k+1} B_{k,k}^{n} + \mathcal{B}_{n+1}, \text{ where } B_{k,k}^{n} = \sum_{j=0}^{k} (-1)^k \binom{k}{j} (k-j)^n \tag{2.4}$$

$$\mathcal{B}_{2n+1}(x) = (2n+1) \sum_{k=0}^{2n} \binom{x}{k+1} B_{k,k}^{2n}, \qquad n \geq 1 \tag{2.5}$$

2.3 Explicit Formulas for $\mathcal{B}_n(x)$

$$\mathcal{B}_n(x) = \sum_{j=0}^{n} \frac{1}{j+1} \sum_{k=0}^{j} (-1)^k \binom{j}{k} (x+k)^n \tag{2.6}$$

$$\boldsymbol{\mathcal{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \mathcal{B}_k} \tag{2.7}$$

2.3.1 Application of Equation (2.7)

$$\mathcal{B}_{n-j} = \frac{n}{\binom{n}{j}} \sum_{k=j-1}^{n-1} C_j^{k+1} B_{k,k}^{n-1}, \text{ where } 1 \le j \le n, \text{and } \binom{x}{n} = \sum_{k=0}^{n} C_k^n x^k \tag{2.8}$$

2.4 Properties of $\mathcal{B}_n(x)$

2.4.1 Shift Property

$$\boldsymbol{\mathcal{B}_n(1-x) = (-1)^n \mathcal{B}_n(x)} \tag{2.9}$$

Applications of Equation (2.9)

$$\mathcal{B}_n(1) = (-1)^n \mathcal{B}_n \tag{2.10}$$

$$\mathcal{B}_n = (-1)^n \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k^2} B_{k,k}^{n+1} \tag{2.11}$$

2.4.2 Addition Property

$$\boldsymbol{\mathcal{B}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \mathcal{B}_k(y)} \tag{2.12}$$

Application of Equation (2.12)

$$(-1)^n \mathcal{B}_n(y-1) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathcal{B}_k(y) \tag{2.13}$$

2.4.3 Appell Derivative Property

$$\frac{d}{dx}\mathcal{B}_n(x) = \begin{cases} n\mathcal{B}_{n-1}(x), & n \geq 1 \\ 0, & n = 0 \end{cases} \tag{2.14}$$

Applications of Equation (2.14)

Remark 2.2 *Recall that $D_x^r f(x)$ is the r^{th} derivative of $f(x)$ with respect to x.*

$$D_x^r \mathcal{B}_n(ax) = r!\binom{n}{r} a^r B_{n-r}(ax) \tag{2.15}$$

$$D_x \mathcal{B}_n(ax+b) = a \cdot n \cdot \mathcal{B}_{n-1}(ax+b) \tag{2.16}$$

2.4.4 Integration of $\mathcal{B}_n(x)$

$$\int \mathcal{B}_n(x)\,dx = \frac{1}{n+1}\mathcal{B}_{n+1}(x) + C \tag{2.17}$$

Applications of Equation (2.17)

$$\int \mathcal{B}_n(ax)\,dx = \frac{1}{a(n+1)}\mathcal{B}_{n+1}(ax) + C, \qquad a \neq 0 \tag{2.18}$$

$$\int_0^1 \mathcal{B}_n(x)\,dx = \begin{cases} 0, & n \geq 1 \\ 1, & n = 0 \end{cases} \tag{2.19}$$

2.4.5 Other Integrals Involving $\mathcal{B}_n(x)$

$$\int_0^1 x^j \mathcal{B}_n(x)\,dx = \frac{(-1)^n}{\binom{n+j}{n}} \sum_{k=0}^{n} \binom{n+j}{n-k} \frac{\mathcal{B}_{n-k}}{k+j+1} \tag{2.20}$$

$$\int_0^1 x^j \mathcal{B}_n(x)\,dx = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{\mathcal{B}_{n-k}}{\binom{k+j}{k}(k+j+1)} \tag{2.21}$$

$$\int_0^1 x^j \mathcal{B}_n(x)\,dx = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathcal{B}_{n-k}}{k+j+1} \tag{2.22}$$

$$\int_0^1 x\mathcal{B}_n(x)\,dx = (-1)^{n+1}\frac{\mathcal{B}_{n+1}}{n+1} \tag{2.23}$$

$$\int_0^1 \mathcal{B}_n(x)\mathcal{B}_r(x) = (-1)^{r-1}\frac{\mathcal{B}_{r+n}}{\binom{r+n}{n}} = (-1)^{n-1}\frac{\mathcal{B}_{r+n}}{\binom{r+n}{n}},\ r \geq 1,\ n \geq 1 \tag{2.24}$$

2.4.6 Convolution Properties

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathcal{B}_k(x)\mathcal{B}_{n-k}(x)=(-1)^{n-1}(n-1)\mathcal{B}_n \tag{2.25}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathcal{B}_k(x)\mathcal{B}_{n-k}(x)=(1-n)\mathcal{B}_n \tag{2.26}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathcal{B}_k\mathcal{B}_{n-k}=(1-n)\mathcal{B}_n \tag{2.27}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}\mathcal{B}_{2k}\mathcal{B}_{2n-2k}=(1-2n)\mathcal{B}_{2n},\qquad n\geq 2 \tag{2.28}$$

$$\sum_{k=1}^{n-1}\binom{2n}{2k}\mathcal{B}_{2k}\mathcal{B}_{2n-2k}=-(2n+1)\mathcal{B}_{2n},\qquad n\geq 2 \tag{2.29}$$

2.4.7 A Binomial Expansion

Remark 2.3 *In the following identity, we let $\mathcal{B}(x)$ denote the Bernoulli Polynomial, and assume*

$$\binom{\mathcal{B}(x)}{r}\equiv\sum_{k=0}^{r}C_k^r\mathcal{B}_k(x). \tag{2.30}$$

Then,

$$\binom{\mathcal{B}(x)+n}{n}=\sum_{k=0}^{n}(-1)^k\binom{n+x}{n-k}\frac{1}{k+1}. \tag{2.31}$$

Also,

$$\binom{\mathcal{B}+n}{n}=\frac{1}{n+1}. \tag{2.32}$$

2.5 Formulas Involving n^{th} Differences

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathcal{B}_j(k) = (-1)^n\sum_{k=0}^{j}\binom{j}{k}\mathcal{B}_{j-k}B_{n,n}^{k} \tag{2.33}$$

$$\Delta_{x,1}\mathcal{B}_n(x) \equiv \mathcal{B}_n(x+1) - \mathcal{B}_n(x) = nx^{n-1},\ \ n \geq 0,\ \ x \neq 0 \tag{2.34}$$

2.5.1 Applications of Equation (2.34)

$$x^n = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\mathcal{B}_k(x) \tag{2.35}$$

$$\sum_{k=0}^{r}(-1)^k k^{n-1} = \frac{1}{n}\left((-1)^r\mathcal{B}_n(r+1) - \mathcal{B}_n + 2\sum_{k=0}^{r}(-1)^{k-1}\mathcal{B}_n(k)\right),\ \ n \geq 1 \tag{2.36}$$

2.6 Polynomial Expansions Involving $\mathcal{B}_n(x)$

Remark 2.4 *Throughout this section, we assume $f(x)$ is a polynomial of degree n, namely,*

$$f(x) = \sum_{k=0}^{n} a_k x^k. \tag{2.37}$$

2.6.1 Basic Expansion Formulas

Remark 2.5 *The following expansion is equivalent to the formula given by Charles Jordan on Page 248 of Calculus of Finite Differences, Chelsea Publishing, New. York, 1947.*

$$\boldsymbol{f(x) = \sum_{j=0}^{n}\mathcal{B}_j(x)\sum_{k=j}^{n}\binom{k+1}{j}\frac{D^k f(0)}{(k+1)!}} \tag{2.38}$$

$$f(x) = \sum_{k=0}^{n}\mathcal{B}_k(x)C_k,\ \ \text{where } C_0 = \int_0^1 f(x)\,dx,\ \ C_k = \frac{1}{k!}\Delta_{x,1}D^{k-1}f(x)|_{x=0} \tag{2.39}$$

2.6.2 Raabe's Theorem

Remark 2.6 *The identities in this section are found on Page 252 of Charles Jordan's* *Calculus of Finite Differences.*

$$\mathcal{B}_n(x) = r^{n-1} \sum_{k=0}^{r-1} \mathcal{B}_n\left(\frac{x+k}{r}\right), \qquad r \geq 1 \tag{2.40}$$

$$\mathcal{B}_n(rx) = r^{n-1} \sum_{k=0}^{r-1} \mathcal{B}_n\left(x+\frac{k}{r}\right), \qquad r \geq 1 \tag{2.41}$$

Applications of Equation (2.40)

$$\mathcal{B}_n = r^{n-1} \sum_{k=0}^{r-1} \mathcal{B}_n\left(\frac{k}{r}\right), \qquad r \geq 1 \tag{2.42}$$

$$\mathcal{B}_n\left(\frac{1}{2}\right) = -\left(1-\frac{1}{2^{n-1}}\right)\mathcal{B}_n \tag{2.43}$$

$$\mathcal{B}_n\left(\frac{1}{3}\right) + \mathcal{B}_n\left(\frac{2}{3}\right) = -\left(1-\frac{1}{3^{n-1}}\right)\mathcal{B}_n \tag{2.44}$$

2.6.3 Generalizations of Equation (2.39)

Remark 2.7 *Two excellent reference for the formulas found in this subsection are Konrad Knopp's* *Theorie und Anwendung der unendlichen Reihen, fourth edition, Berlin, 1947, and N. E. Nörlund's* *Vorlesungen über Differenzenrechung, Berlin, 1924 (Chelsea Reprint, New York 1954).*

Let w be a nonzero real or complex number. Then,

$$f(x+wz) = \frac{1}{w}\int_x^{x+w} f(t)\,dt + \sum_{k=1}^{n} \frac{w^k}{k!}\mathcal{B}_k(z)\cdot \Delta_{x,w} D_x^{k-1} f(x), \tag{2.45}$$

$$\text{where } \Delta_{x,w} f(x) \equiv \frac{f(x+w)-f(x)}{w}.$$

Applications of Equation (2.45)

$$f(x+z) = \int_x^{x+1} f(t)\,dt + \sum_{k=1}^{n} \frac{\mathcal{B}_k(z)}{k!}\cdot \Delta_{x,1} D_x^{k-1} f(x) \tag{2.46}$$

$$D_x f(x+z) = \sum_{k=0}^{n-1} \frac{\mathcal{B}_k(z)}{k!}\cdot \Delta_{x,1} D_x^{k} f(x) \tag{2.47}$$

2.6.4 Euler-Maclaurin Formula

Remark 2.8 *For this subsection, we define $\tilde{\mathcal{B}}_n(x)$ to be the periodic real valued function which agrees with $\mathcal{B}_n(x)$ on the interval $0 \le x < 1$. Furthermore, we* do not *require $f(x)$ be a polynomial, only that $f(x)$ be sufficiently smooth.*

Euler-Maclaurin Formula

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac{f(0)+f(n)}{2} + \sum_{k=1}^{j} \frac{\mathcal{B}_{2k}}{(2k)!}\left(D_x^{2k-1} f(x)|_{x=n} - D_x^{2k-1} f(x)|_{x=0}\right) + R_j, \tag{2.48}$$

where,

$$R_j = \frac{1}{(2j+1)!}\int_0^n \tilde{\mathcal{B}}_{2j+1}(x) \cdot D_x^{2j+1} f(x)\,dx \tag{2.49}$$

Applications of Equation (2.48)

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \log\sqrt{2\pi} + \sum_{k=1}^{j} \frac{\mathcal{B}_{2k}}{2k(2k-1)z^{2k-1}} - \frac{1}{2j+1}\int_0^\infty \frac{\tilde{\mathcal{B}}_{2j+1}(x)}{(z+x)^{2j+1}}\,dx \tag{2.50}$$

$$\log n! = \left(n + \frac{1}{2}\right)\log n - n + \log\sqrt{2\pi} + \sum_{k=1}^{j} \frac{\mathcal{B}_{2k}}{2k(2k-1)n^{2k-1}} - \frac{1}{2j+1}\int_0^\infty \frac{\tilde{\mathcal{B}}_{2j+1}(x)}{x^{2j+1}}\,dx \tag{2.51}$$

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \frac{1}{2} + \frac{1}{2n} + \sum_{k=1}^{j} \frac{\mathcal{B}_{2k}}{2k}\left(1 - \frac{1}{n^{2k}}\right) - \int_1^n \frac{\tilde{\mathcal{B}}_{2j+1}(x)}{x^{2j+2}}\,dx, \qquad n \ge 1 \tag{2.52}$$

3 Eulerian Polynomials $E_n(x)$ and Eulerian Numbers E_n

Remark 3.1 *Throughout this chapter, we assume n, r, m, and p are nonnegative integers. We assume x, y, and t are real or complex numbers. Furthermore, for any real x, we let $[x]$ denote the floor of x.*

3.1 Definition of $E_n(x)$

$$\sum_{k=0}^{\infty} E_k(x)\frac{t^k}{k!} = \frac{2e^{xt}}{e^t+1} \tag{3.1}$$

3.1.1 Relationships Between $E_n(x)$ and $\mathcal{B}_n(x)$

$$E_{n-1}(x) = \frac{2^n}{n}\left(\mathcal{B}_n\left(\frac{x+1}{2}\right) - \mathcal{B}_n\left(\frac{x}{2}\right)\right), \qquad n \geq 1 \tag{3.2}$$

$$E_{n-1}(x) = \frac{2}{n}\left(\mathcal{B}_n(x) - 2^n\mathcal{B}_n\left(\frac{x}{2}\right)\right), \qquad n \geq 1 \tag{3.3}$$

3.2 Properties of $E_n(x)$

3.2.1 Appell Derivative Property

Remark 3.2 *Throughout this chapter, we let $D_x^r f(x)$ denote the r^{th} derivative of $f(x)$.*

$$D_x E_n(x) = \begin{cases} nE_{n-1}(x), & n \geq 1 \\ 0, & n = 0 \end{cases} \tag{3.4}$$

Applications of Equation (3.4)

$$D_x^r E_n(x) = r!\binom{n}{r}E_{n-r}(x) \tag{3.5}$$

$$\int E_n(x)\,dx = \frac{E_{n+1}(x)}{n+1} + C \tag{3.6}$$

$$\int_a^b E_n(x)\,dx = \frac{E_{n+1}(b) - E_{n+1}(a)}{n+1} \tag{3.7}$$

$$\int_0^1 E_n(x)\,dx = \frac{(-1)^{n+1}-1}{n+1}E_{n+1}(0) \tag{3.8}$$

3.2.2 Addition Property

$$\boldsymbol{E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^k E_{n-k}(y)} \tag{3.9}$$

Applications of Equation (3.9)

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k E_{n-k}(0) \tag{3.10}$$

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \frac{2(1-2^{k+1})}{k+1} \mathcal{B}_{k+1} \tag{3.11}$$

3.2.3 Shift Property

$$\boldsymbol{E_n(x) = (-1)^n E_n(1-x)} \tag{3.12}$$

3.2.4 Difference Equation

$$\boldsymbol{E_n(x) + E_n(x+1) = 2x^n} \tag{3.13}$$

Application of Equation (3.13)

$$\sum_{k=0}^{n} (-1)^k k^p = \frac{(-1)^n E_p(n+1) + E_p(0)}{2} \tag{3.14}$$

3.2.5 An Integral Property

$$\int_0^1 E_m(x) E_n(x)\, dx = -\frac{(-1)^m + (-1)^n}{m+n+1} \cdot \frac{E_{m+n+1}(0)}{\binom{m+n}{n}} \tag{3.15}$$

Application of Equation (3.15)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} E_k(x) E_{n-k}(x) = \left((-1)^{n+1} - 1\right) E_{n+1}(0) \tag{3.16}$$

3.3 Explict Formula for $E_n(x)$

$$\boldsymbol{E_n(x) = \sum_{j=0}^{n} \frac{1}{2^j} \sum_{k=0}^{j} (-1)^k \binom{j}{k} (x+k)^n} \tag{3.17}$$

3.3.1 Calculations of $E_n(0)$ and $\mathcal{B}_n$

$$E_n(0) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+1}{n-k} \sum_{j=0}^{k} (-1)^j j^n, \qquad n \geq 1 \tag{3.18}$$

$$E_n(0) = \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k A_{k,n}, \text{ where } A_{k,n} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n \tag{3.19}$$

$$E_n(0) = \frac{1}{2^n} \sum_{k=1}^{n} (-1)^k A_{k,n}, \qquad n \geq 1 \tag{3.20}$$

$$E_n(0) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} B_{k,k}^n \text{ where } B_{k,k}^n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \tag{3.21}$$

$$\mathcal{B}_{n+1} = \frac{n+1}{(2^{n+1}-1)} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{2^{k+1}} B_{k,k}^n \tag{3.22}$$

$$\mathcal{B}_{n+1} = \frac{n+1}{2^{n+1}(2^{n+1}-1)} \sum_{k=0}^{n} (-1)^{k+1} A_{k,n} \tag{3.23}$$

3.4 Definition of E_n

$$\frac{1}{\cos z} = \sum_{k=0}^{\infty} E_k \frac{i^k z^k}{k!}, \qquad i \equiv \sqrt{-1}, \qquad |z| < \frac{\pi}{2} \tag{3.24}$$

3.4.1 Relationship Between E_n and $E_n(x)$

$$E_n = 2^n E_n\left(\frac{1}{2}\right) \tag{3.25}$$

3.5 Properties of E_n

3.5.1 Parity Properties

$$E_{2n+1} = 0 \tag{3.26}$$

$$E_0 = 1 \tag{3.27}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = \begin{cases} 0, & n \geq 1 \\ 1, & n = 0 \end{cases} \tag{3.28}$$

$$E_{2n} = 1 + \sum_{k=1}^{n} \binom{2n}{2k-1} \frac{2^{2k}(1-2^{2k})}{2k} \mathcal{B}_{2k}, \qquad n \geq 1 \tag{3.29}$$

3.5.2 Explicit Formula for $E_n(x)$

$$\boldsymbol{E_n(z) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(z - \frac{1}{2}\right)^{n-k}} \tag{3.30}$$

3.5.3 Eulerian Numbers in Various Trigometric Expansions

$$\pi \sec \pi z = \pi \sum_{k=0}^{\infty} (-1)^k \frac{E_{2k}}{(2k)!} \pi^{2k} z^{2k}, \qquad |z| < \frac{1}{2} \tag{3.31}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = (-1)^n \frac{\pi^{2n+1} E_{2n}}{2^{2n+2}(2n)!} \tag{3.32}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{(-1)^n \pi^{2n+1}}{(2n!)} \sum_{j=0}^{2n} \frac{1}{2^{2n+2+j}} \sum_{k=0}^{j} (-1)^j \binom{j}{k} (2k+1)^{2n} \tag{3.33}$$

4 Generalized Bernoulli Polynomials $\mathcal{B}_k^{(n)}(x)$

Remark 4.1 *Throughout this chapter, we assume n, r, m, and p are nonnegative integers. We assume x, y, a, b, and t are real or complex numbers. Furthermore, for any real x, we let $[x]$ denote the floor of x.*

4.1 Definition of $\mathcal{B}_k^{(n)}(x)$

$$\sum_{k=0}^{\infty} \mathcal{B}_k^{(n)}(x) \frac{z^k}{k!} = \frac{z^n e^{xz}}{(e^z - 1)^n}, \qquad |z| < 2\pi \tag{4.1}$$

$$\mathcal{B}_k(x) = \mathcal{B}_k^{(1)}(x), \qquad \mathcal{B}_k^{(1)}(0) = \mathcal{B}_k(0) = \mathcal{B}_k \tag{4.2}$$

4.1.1 The Reciprocal Generating Function of Equation (4.1)

$$\frac{(e^x - 1)^n}{e^{tx} x^n} = n! \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{B_{n,n}^{j+n}}{\binom{j+n}{j}} t^{k-j}, \text{ where } B_{k,k}^n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \tag{4.3}$$

4.1.2 Generating Function for $\mathcal{B}_k^{(k-n+1)}(t)$: Formula 2247 Volume 5

$$\frac{(1+x)^{t-1} x^n}{(\ln(1+x))^n} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \mathcal{B}_k^{(k-n+1)}(t) \tag{4.4}$$

4.2 Derivative Definition of $\mathcal{B}_k^{(n)}(x)$

Remark 4.2 *Throughout this section, and the remainder of this chapter, we let $D_x^r f(x)$ denote the r^{th} derivative of $f(x)$ with respect to x.*

$$\mathcal{B}_k^{(n+1)}(z+1) = k! D_z^{n-k} \binom{z}{n} \tag{4.5}$$

4.2.1 Applications of Equation (4.5)

$$C^n_{n-k} = \frac{1}{k!(n-k)!}\mathcal{B}^{(n+1)}_k(1), \quad \text{where } \binom{x}{n} = \sum_{k=0}^{n} C^n_k x^k \tag{4.6}$$

$$\mathcal{B}^{(n+1)}_n(z+1) = n!\binom{z}{n} \tag{4.7}$$

$$\mathcal{B}^{(n+1)}_n(z) = \prod_{k=1}^{n}(z-k), \qquad n \geq 1 \tag{4.8}$$

4.3 Generalized Bernoulli Numbers $\mathcal{B}^{(n)}_k \equiv \mathcal{B}^{(n)}_k(0)$

4.3.1 Relationships with C^n_{n-k}

$$\boldsymbol{\mathcal{B}^{(n)}_k = \frac{n!}{\binom{n-1}{k}} C^n_{n-k}} \tag{4.9}$$

$$n!k!C^{n+k}_n = \frac{n}{n+k}\mathcal{B}^{(n+k)}_k \tag{4.10}$$

4.3.2 Exponential Generating Function

$$\left(\frac{\ln(1+x)}{x}\right)^n = n\sum_{k=0}^{\infty}\frac{x^k}{k!}\frac{\mathcal{B}^{(n+k)}_k}{(n+k)}, \qquad n \geq 1, \qquad |x| < 1 \tag{4.11}$$

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{k=0}^{\infty}\frac{x^k}{k!}\mathcal{B}^{(k)}_k, \qquad |x| < 1 \tag{4.12}$$

4.3.3 Relationships with $B^{k+n}_{n,n}$

$$\boldsymbol{\mathcal{B}^{(n)}_k = \sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}\frac{B^{k+jn}_{jn,jn}}{(jn)!\binom{k+jn}{k}}} \tag{4.13}$$

$$\mathcal{B}^{(-n)}_k = \frac{k!}{(k+n)!}B^{k+n}_{n,n} \tag{4.14}$$

$$(-1)^n \frac{(n+j)!}{j!}\mathcal{B}_j^{(-n)} = \sum_{k=0}^{n}(-1)^k \binom{n}{k} k^{n+j} \tag{4.15}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} k^{n+j} = (-1)^n \frac{(n+j)!}{j!}\mathcal{B}_j^{(-n)} = (-1)^n B_{n,n}^{n+j} \tag{4.16}$$

4.3.4 A Generalization of Generalized Bernoulli Numbers $\mathcal{B}_k^{(z)}$

$$\mathcal{B}_k^{(z)} = \sum_{j=0}^{k}(-1)^j \binom{z+k}{k-j}\binom{z+j-1}{j}\frac{B_{j,j}^{j+k}}{j!\binom{j+k}{j}} \tag{4.17}$$

Applications of Equation (4.17)

$$\mathcal{B}_k^{(n)} = \sum_{j=0}^{k}(-1)^j \binom{n+k}{k-j}\binom{n+j-1}{j}\frac{B_{j,j}^{j+k}}{j!\binom{j+k}{j}} \tag{4.18}$$

$$\mathcal{B}_k^{(-n)} = \sum_{j=0}^{k}(-1)^j \binom{k-n}{k-j}\binom{-n+j-1}{j}\frac{B_{j,j}^{j+k}}{j!\binom{j+k}{j}} \tag{4.19}$$

$$\mathcal{B}_k^{(-z)} = \sum_{j=0}^{k} \binom{k-z}{k-j}\binom{z}{j}\frac{B_{j,j}^{j+k}}{j!\binom{j+k}{j}} \tag{4.20}$$

$$\left(\frac{x}{e^x-1}\right)^z = \sum_{k=0}^{\infty}\frac{x^k}{k!}\mathcal{B}_k^{(z)}, \qquad |x| < 2\pi \tag{4.21}$$

$$\frac{t^z e^{xt}}{(e^t-1)^z} = \sum_{k=0}^{\infty}\mathcal{B}_k^{(z)}(x)\frac{t^k}{k!}, \qquad |t| < 2\pi \tag{4.22}$$

Restatement of a Theorem of Schläfli's

$$\mathcal{B}_k^{(z)} = \sum_{j=0}^{k}\binom{-z}{j}\binom{k+z}{k-j}\mathcal{B}_k^{(-j)} \tag{4.23}$$

4.4 Properties of $\boldsymbol{\mathcal{B}_k^{(n)}(x)}$

4.4.1 Addition Property

$$\boldsymbol{\mathcal{B}_k^{(n)}(x+y) = \sum_{j=0}^{k} \binom{k}{j} \mathcal{B}_j^{(n)}(x) y^{k-j}} \tag{4.24}$$

Application of Equation (4.24)

$$\mathcal{B}_j^{(n+1)}(x+1) = \sum_{k=0}^{j} \binom{j}{k} x^{j-k} \sum_{J=0}^{k} \binom{k}{J} \sum_{r=0}^{J} (-1)^r \binom{J+1}{r+1} \frac{B_{rn+r,rn+r}^{J+rn+r}}{(rn+r)!\binom{J+rn+r}{J}} \tag{4.25}$$

4.4.2 Appell Derivative Property

$$\boldsymbol{D_x \mathcal{B}_k^{(n)}(x) = k\mathcal{B}_{k-1}^{(n)}(x), \qquad k \geq 1} \tag{4.26}$$

Application of Equation (4.26)

$$D_x^j \mathcal{B}_k^{(n)}(x) = \frac{k!}{(k-j)!} \mathcal{B}_{k-j}^{(n)}(x) \tag{4.27}$$

4.4.3 n^{th} Difference Formula

$$\boldsymbol{\Delta_{x,1} \mathcal{B}_k^{(n)}(x) = \begin{cases} k\mathcal{B}_{k-1}^{(n-1)}(x), & k \geq 1 \\ 0, & k = 0 \end{cases}}, \tag{4.28}$$

where $\Delta_{x,w} f(x) \equiv \dfrac{f(x+w) - f(x)}{w}$

Applications of Equation (4.28)

$$\Delta_{x,1}^r \mathcal{B}_k^{(n)}(x) = r! \binom{k}{r} \mathcal{B}_{k-r}^{(n-r)}(x) \tag{4.29}$$

$$\Delta_{x,1} D_x^k \binom{x}{n} = \frac{1}{(n-k-1)!} \mathcal{B}_{n-k-1}^{(n)}(x+1) \tag{4.30}$$

Newton Expansion

$$\mathcal{B}_k^{(z)}(x+y) = \sum_{j=0}^{k} \binom{x}{j} \binom{k}{j} j! \mathcal{B}_{k-j}^{(z-j)}(y) \tag{4.31}$$

4.4.4 Difference Relation

$$\boldsymbol{k\mathcal{B}_k^{(n)}(t) = n\mathcal{B}_k^{(n)}(t) + tk\mathcal{B}_{k-1}^{(n)}(t) - n\mathcal{B}_k^{(n+1)}(t+1), \qquad k \geq 1} \tag{4.32}$$

Applications of Equation (4.32)

$$\mathcal{B}_k^{(n+1)}(t) = \left(1 - \frac{k}{n}\right)\mathcal{B}_k^{(n)}(t) + (t-n)\frac{k}{n}\mathcal{B}_{k-1}^{(n)}(t), \qquad k \geq 1, \;\; n \geq 1 \tag{4.33}$$

$$\mathcal{B}_n^{(n+1)}(x) = (x-n)\mathcal{B}_{n-1}^{(n)}(x), \qquad n \geq 1 \tag{4.34}$$

$$\mathcal{B}_n^{(n+1)}(x) = \prod_{k=0}^{j}(x-n-k) \cdot \mathcal{B}_{n-j-1}^{(n-j)}(x) \tag{4.35}$$

$$\mathcal{B}_k^{(n+1)}(1) = \left(1 - \frac{k}{n}\right)\mathcal{B}_k^{(n)}, \qquad n \geq 1 \tag{4.36}$$

4.4.5 An Integral Involving $\mathcal{B}_k^{(n)}(t)$

$$\boldsymbol{\int_x^{x+1} \binom{z}{n} dz = \frac{1}{n!}\mathcal{B}_n^{(n)}(x+1)} \tag{4.37}$$

Applications of Equation (4.37)

$$\sum_{k=0}^{n} \binom{x}{n-k} \frac{1}{k!}\mathcal{B}_k^{(k)}(z+1) = \frac{1}{n!}\mathcal{B}_n^{(n)}(z+x+1) \tag{4.38}$$

$$(-1)^n \mathcal{B}_n^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!}\mathcal{B}_k^{(k)} \tag{4.39}$$

4.4.6 Convolution Property

$$\boldsymbol{\mathcal{B}_k^{(z+w)}(x+y) = \sum_{j=0}^{k} \binom{k}{j} \mathcal{B}_j^{(z)}(x)\mathcal{B}_{k-j}^{(w)}(y)} \tag{4.40}$$

Applications of Equation (4.40)

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} \mathcal{B}_j^{(z)}(x)\mathcal{B}_{k-j}^{(-z)}(y) \tag{4.41}$$

$$x^k = \sum_{j=0}^{k} \binom{k}{j} \mathcal{B}_{k-j}^{(n)}(x) \frac{B_{n,n}^{j+n}}{n! \binom{j+n}{j}} \tag{4.42}$$

$$y^k = \sum_{j=0}^{k} \binom{k}{j} \mathcal{B}_{k-j}^{(-n)} \frac{n!}{\binom{n-1}{j}} C_{n-j}^{n} \tag{4.43}$$

4.5 Polynomial Expansions Using Generalized Bernoulli and Euler Polynomials

Remark 4.3 *The identities of this section are found in Chapters 6 and 9 of N. E. Nörlund's* <u>*Vorlesungen über Differenzenrechnung*</u>*, Berlin, 1924 (reprinted by Chelsea Publ. Co., New York, 1954).*

Remark 4.4 *Throughout this section, we assume, unless otherwise stated, that $f(x)$ is a polynomial of degree m, namely, $f(x) = \sum_{k=0}^{m} a_k x^k$.*

4.5.1 Definition of General Euler Polynomials $E_k^{(n)}(t)$

$$\sum_{k=0}^{\infty} E_k^{(n)}(t) \frac{x^k}{k!} = \frac{2^n e^{tx}}{(e^x + 1)^n} \tag{4.44}$$

4.5.2 Generalized Bernoulli and Euler Expansions

Remark 4.5 *Throughout this section, we will use the averaging operator $\bigtriangledown_{x,w} f(x) \equiv \frac{f(x+w)+f(x)}{2}$.*

$$D_t^n f(t)|_{t=x+z} \equiv f^{(n)}(x+z) = \sum_{k=0}^{m} \mathcal{B}_k^{(n)}(z) \frac{1}{k!} \Delta_{x,1}^n D_x^k f(x) \tag{4.45}$$

$$f^{(n)}(x+z) = \sum_{k=0}^{m} E_k^{(n)}(z) \frac{1}{k!} \bigtriangledown_{x,1}^n D_x^k f(x), \tag{4.46}$$

where $f(x)$ is a polynomial of degree $n+m$

4.5.3 Newton Series Involving $\mathcal{B}_k^{(k+2)}$

Recall that the Newton Series for $f(x+y)$ is

$$f(x+y) = \sum_{k=0}^{m} \binom{\frac{x}{z}}{k} z^k \Delta_{y,z}^k f(y). \tag{4.47}$$

Alternate Form of the Newton Series

$$f(x+y) = \sum_{k=0}^{m} \binom{\frac{y-z}{z}}{k} z^k \Delta_{x,z}^k f(x+z) \tag{4.48}$$

Derivative Applications of Equation (4.48)

$$D_y f(x+y) = \sum_{k=1}^{m} \binom{\frac{y-z}{z}}{k} z^k \sum_{j=1}^{k} \frac{1}{y-jz} \Delta_{x,z}^k f(x+z), \qquad m \geq 1 \tag{4.49}$$

$$f'(x) = \sum_{k=0}^{m-1} (-1)^k z^k \sum_{j=0}^{k} \frac{1}{j+1} \Delta_{x,z}^{k+1} f(x+z), \qquad m \geq 1 \tag{4.50}$$

$$f'(x) = \sum_{k=0}^{m-1} \frac{z^k}{k!} \mathcal{B}_k^{(k+2)} \Delta_{x,z}^{k+1} f(x+z), \qquad m \geq 1 \tag{4.51}$$

$$f'(x) = \sum_{k=0}^{m-1} (-1)^k \frac{z^k}{k+1} \Delta_{x,z}^{k+1} f(x), \qquad m \geq 1 \tag{4.52}$$

$$D_x^j f(x) = \sum_{k=0}^{m-j} \frac{z^k}{k!} \mathcal{B}_k^{(k+j+1)} \Delta_{x,z}^{k+j} f(x+z) \tag{4.53}$$

$$D_x^j f(x) = j \sum_{k=0}^{m-j} \frac{z^k}{k+j} \frac{1}{k!} \mathcal{B}_k^{(k+j)} \Delta_{x,z}^{k+j} f(x), \qquad j \geq 1 \tag{4.54}$$

Remark 4.6 *For the remaining identities of this section, we <u>do not</u> require that $f(x)$ be a polynomial.*

$$\frac{1}{x^r} = \sum_{k=0}^{\infty} (-1)^k \frac{\binom{r-1+k}{k}}{\binom{x+r+k}{r+k}} \frac{1}{(r+k)!} \mathcal{B}_k^{(r+k)}, \qquad Re(x) > 0, \;\; r \geq 1 \tag{4.55}$$

$$\frac{1}{(x-y)^{n+1}} = \sum_{k=n}^{\infty} \binom{k}{n} \frac{\mathcal{B}_{k-n}^{(k+1)}(y+k)}{x(x+1)(x+2)...(x+k)}, \qquad Re(x) > Re(y) \tag{4.56}$$

$$\frac{1}{z} \int_z^{x+z} f(t)\, dt = \sum_{k=0}^{\infty} z^k \frac{1}{k!} \mathcal{B}_k^{(k)} \Delta_{x,z}^k f(x+z) \tag{4.57}$$

$$\frac{1}{z} \int_z^{x+z} f(t)\, dt = f(x) + \frac{z}{2} \Delta_{x,z} f(x) - \sum_{k=2}^{\infty} \frac{z^k}{(k-1)k!} \mathcal{B}_k^{(k-1)} \Delta_{x,z}^k f(x) \tag{4.58}$$

$$\log\left(1 + \frac{1}{x}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{\mathcal{B}_k^{(k)}}{k! \binom{x+k}{k}}, \qquad Re(x) > 0 \tag{4.59}$$

$$\log\left(1 + \frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x(x+1)} - \sum_{k=2}^{\infty} (-1)^k \frac{\mathcal{B}_k^{(k-1)}}{(k-1)k!x\binom{x+k}{k}}, \qquad Re(x) > 0 \tag{4.60}$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log x - \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{\mathcal{B}_k^{(k)}}{k \cdot k! \binom{x+k}{k}}, \qquad Re(x) > 0 \tag{4.61}$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log x - \frac{1}{2x} - \sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)} \frac{\mathcal{B}_k^{(k-1)}}{k! \binom{x+k-1}{k}}, \qquad Re(x) > 0 \tag{4.62}$$

$$D_x^n \left(\frac{\Gamma'(x)}{\Gamma(x)}\right) = (-1)^{n-1}(n-1)! \sum_{k=0}^{\infty} (-1)^k \binom{k+n-1}{k} \cdot \frac{\mathcal{B}_k^{(k+n+1)}}{(x+1)(x+2)...(x+k+n)}, \qquad Re(x) > 0 \tag{4.63}$$

$$D_x^n \left(\frac{\Gamma'(x)}{\Gamma(x)}\right) = (-1)^{n-1} n! \sum_{k=0}^{\infty} (-1)^k \binom{k+n-1}{k} \cdot \frac{\mathcal{B}_k^{(k+n)}}{x(x+1)(x+2)...(x+k+n)(k+n)}, \qquad Re(x) > 0 \tag{4.64}$$

4.5.4 Functional Expansions Involving Generalized Bernoulli Polynomials

Remark 4.7 *For the identities of this section, we do not require $f(x)$ to be a polynomial.*

$$f(x+y)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{x}{x+bn}\mathcal{B}_n^{(n+1)}\left(\frac{x+bn}{z}+1\right)\Delta_{t,z}^{n}f(t)|_{t=y-bn} \tag{4.65}$$

$$f(x+y)-f(y)=x\sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}\mathcal{B}_n^{(n+1)}\left(\frac{x+bn+b}{z}\right)\Delta_{t,z}^{n+1}f(t)|_{t=y-bn-b} \tag{4.66}$$

$$f'(y)=\sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}\mathcal{B}_n^{(n+1)}\left(\frac{bn+b}{z}\right)\Delta_{t,z}^{n+1}f(t)|_{t=y-bn-b} \tag{4.67}$$

5 Catalan Numbers c_n

Remark 5.1 *Throughout this chapter, we assume n is a nonnegative integer.*

5.1 Definition of c_n

$$\boldsymbol{c_n=\frac{1}{n+1}\binom{2n}{n}} \tag{5.1}$$

5.2 Shifted Catalan Numbers a_n

$$a_n=\frac{1}{2}\binom{2n}{n}\frac{1}{2n-1}=\frac{1}{2n-1}\binom{2n-1}{n},\qquad n\geq 1 \tag{5.2}$$

5.2.1 Properties of a_n

$$\boldsymbol{a_n=c_{n-1},\qquad n\geq 1} \tag{5.3}$$

$$a_n=\sum_{k=1}^{n-1}a_ka_{n-k},\qquad n\geq 2,\ a_1=a_2=1 \tag{5.4}$$

Remark 5.2 *Recall that for x real, $[x]$ denotes the floor of x. We also define $a_0\equiv -1$*

$$\sum_{k=1}^{n} a_k a_{n-k} = 0, \qquad n \geq 2 \tag{5.5}$$

$$\sum_{k=0}^{n} a_k a_{n-k} = -a_n, \qquad n \geq 2 \tag{5.6}$$

$$\sum_{k=0}^{n} a_k a_{n-k} = 2 \sum_{k=0}^{[\frac{n-1}{2}]} a_k a_{n-k} + \frac{1+(-1)^n}{2} a_{[\frac{n}{2}]}^2 \tag{5.7}$$

$$\sum_{k=1}^{n-1} a_k a_{2n-k} = \frac{1}{2}\left(a_{2n} - a_n^2\right), \qquad n \geq 1 \tag{5.8}$$

5.2.2 Generating Function

$$\sum_{k=1}^{\infty} a_k z^k = \sum_{k=1}^{\infty} \binom{2k}{k} \frac{z^k}{2k-1} = 1 - (1-4z)^{\frac{1}{2}}, \qquad |z| < \frac{1}{4} \tag{5.9}$$

6 Fibonacci Numbers F_n

Remark 6.1 *Throughout this chapter, we assume n is a nonnegative integer.*

6.1 Recursive Definition for f_n

$$\boldsymbol{F_n = F_{n-1} + F_{n-2}, \qquad n \geq 3, \qquad F_1 = F_2 = 1,\ F_0 = 0} \tag{6.1}$$

6.2 Properties of F_n

6.2.1 Summation Formulas

$$\sum_{k=1}^{n} F_n = F_{n+2} - 1, \qquad n \geq 1 \tag{6.2}$$

$$\sum_{k=1}^{n} F_k^2 = F_n \cdot F_{n+1}, \qquad n \geq 1 \tag{6.3}$$

6.2.2 Power Recurrences

$$F_{n+1}^2 - F_{n-2}^2 = 4F_nF_{n-1}, \qquad n \geq 3 \tag{6.4}$$

$$F_{2n+1} = F_n^2 + F_{n+1}^2, \qquad n \geq 1 \tag{6.5}$$

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3, \qquad n \geq 2 \tag{6.6}$$

6.2.3 Determinant Property

$$F_nF_{n+3} - F_{n+1}F_{n+2} = (-1)^{n-1}, \qquad n \geq 1 \tag{6.7}$$

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \qquad n \geq 2 \tag{6.8}$$

6.3 Explicit Formulas for F_n

$$\boldsymbol{F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \qquad n \geq 1} \tag{6.9}$$

$$F_{2n+1} = \sum_{k=0}^{n} \binom{n+k}{2k} \tag{6.10}$$

6.4 Limit Calculations Involving F_n

Remark 6.2 *Throughout this section, we assume r, a, and b are nonnegative integers.*

$$\lim_{n\to\infty} \frac{F_n}{F_{n+r}} \equiv u_r = \frac{(-1)^r}{2}\left(F_{r-1} - \sqrt{5}F_r + F_{r+1}\right) \tag{6.11}$$

$$\lim_{n\to\infty} \frac{F_n}{F_{n+r}} \equiv u_r = \frac{(-1)^r}{2}\left(2F_{r-1} - (\sqrt{5}-1)F_r\right) \tag{6.12}$$

6.4.1 Properties of u_r

$$u_a \cdot u_b = u_{a+b} \tag{6.13}$$

$$u_1^r = u_r \tag{6.14}$$

$$u_r + u_{r+1} = u_{r-1}, \qquad r \geq 1 \tag{6.15}$$

$$\sum_{k=2}^{r} u_k = 1 - u_{r-1}, \qquad r \geq 2 \tag{6.16}$$

$$\sum_{k=2}^{r} u_1^k = \frac{u_1^2}{1-u_1}\left(1 - u_1^{r-1}\right), \qquad r \geq 2 \tag{6.17}$$

6.5 Shifted Fibonacci Numbers f_n

6.5.1 Recursive Formula

$$f_{n+1} = f_n + f_{n-1}, \qquad f_0 = f_1 = 1 \tag{6.18}$$

$$f_n = F_{n+1} \tag{6.19}$$

6.5.2 Explicit Formula

Remark 6.3 *Let x be a real number. Recall that $[x]$ denotes the floor of x.*

$$\boldsymbol{f_n = \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k}} \tag{6.20}$$