Combinatorial Numbers and Associated Identities: Table 1: Stirling Numbers

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1 Notational Conventions for Table 1

Throughout this table, we assume n and k are nonnegative integers. We let B_n denote the n^{th} Bell number. This is a fairly standard notation for the Bell numbers. However, there are many notations for Stirling numbers of the first and second kinds. The following table lists equivalent notations for Stirling numbers of the second kind.

Notation	Author	Source
$S(n,k)$	John Riordan	*Combinatorial Identities*, 1968
$S(n,k)$	L. Carlitz	numerous papers, mainly that of 1971
$S_2(k,n-k)$	H. W. Gould	various papers circa 1956
$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	Donald Knuth	*Art of Computer Programming*
S_k^n	G. Pólya	*Notes on Combinatorics*,1978
$\left[C_{n-k}^w(k)\right]$	J. G. Hagen	*Combinationen mit Wiederholungen*, 1891
$\mathfrak{C}_{k+1}^{n-k}$	Niels Nielsen	1906
$\mathfrak{S}_n^k$	Charles Jordan	1939
$\mathbb{S}_n^{(k)}$	Karl Goldberg and Tomlinson Fort	Bureau of Standards, 1959
$\frac{1}{k!}\Delta^k 0^n$	Differences of zero	actuarial work
$\prod_n^{(k)}$	Goldberg, Leighton, Newman, Zuckerman	1976
$\frac{1}{k!}B_{k,k}^n$	H. W. Gould	private notation of the seven notebooks

Table 1: Equivalent notations for $S(n,k)$, a Stirling number of the second kind

As implied by the last line of Table 1, all of the identities in this volume will use Gould's original notation $B^n_{k,k}$. The reader is urged to remember that $\mathbf{B^n_{k,k} = k!S(n,k)}$.

The next table lists equivalent notations for Stirling numbers of the first kind.

Notation	Author	Source
$s(n,k)$	John Riordan	*Combinatorial Identities*, 1968
$S_1(n,k)$	L. Carlitz	numerous papers, mainly that of 1971
$(-1)^{n-k}S_1(n-1,n-k)$	H. W. Gould	various papers circa 1956
$(-1)^{n-k}\left[\begin{matrix} n \\ k \end{matrix}\right]$	Donald Knuth	*Art of Computer Programming*
$(-1)^{n-k}\mathcal{S}^n_k$	G. Pólya	*Notes on Combinatorics*,1978
$(-1)^{n-k}\left[C_{n-k}(n-1)\right]$	J. G. Hagen	*Combinationen ohne Wiederholungen*, 1891
$(-1)^{n-k}C^{n-k}_n$	Niels Nielsen	1906
S^k_n	Charles Jordan	1939
$S^{(k)}_n$	Karl Goldberg	Bureau of Standards, 1959
$(-1)^{n-k}I^{(k)}_n$	Goldberg, Leighton, Newman, Zuckerman	1976
$n!C^n_k$	H. W. Gould	private notation of the seven notebooks

Table 2: Equivalent notations for $s(n,k)$, a Stirling number of the second kind

As implied by the last line of Table 2, all of the identities in this volume will use Gould's original notation C^n_k. The reader is urged to remember that $\mathbf{C^n_k = \frac{s(n,k)}{n!}}$.

2 Stirling Numbers of the Second Kind $\mathrm{B^n_{k,k} = k!S(n,k)}$

Remark 2.1 *Throughout this chapter, we assume r and j are nonnegative integers. We assume x, y, and z are real or complex numbers. We also let $[x]$ denote the floor of x for any real x.*

2.1 Basis Definition for $\mathrm{B^n_{k,k}}$

$$\boldsymbol{x^n = \sum_{k=0}^{n} \binom{x}{k} B^n_{k,k}} \tag{2.1}$$

2.1.1 Applications of Equation (2.1)

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} B^n_{k,k} = \begin{cases} 0, & n \geq 2 \\ 1, & n = 1 \end{cases} \tag{2.2}$$

$$\sum_{k=2}^{n} \frac{(-1)^k}{k} B_{k,k}^{n} \sum_{j=1}^{k-1} \frac{1}{j} = \begin{cases} 0, & n \geq 3 \\ 1, & n = 2 \end{cases} \tag{2.3}$$

$$\sum_{k=0}^{n} (-1)^k B_{k,k}^{n} = (-1)^n \tag{2.4}$$

$$\sum_{k=0}^{n} (-1)^k \binom{x+k-1}{k} B_{k,k}^{n} = (-1)^n x^n \tag{2.5}$$

$$\sum_{k=0}^{n} (-1)^k k B_{k,k}^{n} = (-1)^n (2^n - 1) \tag{2.6}$$

$$\sum_{k=0}^{n} (-1)^k \binom{2k}{k} \frac{1}{2^{2k}} B_{k,k}^{n} = \frac{(-1)^n}{2^n} \tag{2.7}$$

$$B_{j,j}^{n} = \sum_{k=j}^{n} (-1)^{n-k} \binom{k-1}{j-1} B_{k,k}^{n} \tag{2.8}$$

2.2 Explicit Formulas for $\mathrm{B}_{\mathbf{k},\mathbf{k}}^{\mathbf{n}}$

$$\boldsymbol{B_{k,k}^{n} = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n, \qquad 0^0 = 1} \tag{2.9}$$

$$\boldsymbol{B_{k,k}^{n} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n} \tag{2.10}$$

2.3 Applications of Equation (2.9)

2.3.1 Evaluations of $\mathbf{\Delta_h^n x^r}$

$$\Delta_h^n x^r \equiv \frac{(-1)^n}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+kh)^r = \sum_{k=0}^{r} \binom{r}{k} x^{r-k} h^{k-n} B_{n,n}^{k} \tag{2.11}$$

$$\Delta_1^n x^r |_{x=0} = B_{n,n}^{r} \tag{2.12}$$

2.3.2 Evaluation of $\sum_{\mathbf{k=0}}^{\mathbf{n}}(-\mathbf{1})^{\mathbf{k}}\binom{\mathbf{x}}{\mathbf{k}}\mathbf{k}^{\mathbf{r}}$

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}k^r=(-1)^n\sum_{k=0}^{r}\binom{x-k-1}{n-k}\binom{x}{k}B_{k,k}^r,\qquad r\geq 1,\qquad n\geq 1 \tag{2.13}$$

Applications of Equation (2.13)

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}x^r=-x\sum_{k=0}^{r-1}\binom{r-1}{k}\sum_{j=0}^{n-1}(-1)^j\binom{x-1}{j}j^k,\qquad r\geq 1,\qquad n\geq 1 \tag{2.14}$$

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}k^2=(-1)^n x\binom{x-2}{n-1}+(-1)^n x(x-1)\binom{x-3}{n-2},\qquad n\geq 2 \tag{2.15}$$

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}k^3=(-1)^n x\binom{x-2}{n-1}+3(-1)^n x(x-1)\binom{x-3}{n-2}$$
$$+(-1)^n x(x-1)(x-2)\binom{x-4}{n-3},\qquad n\geq 3 \tag{2.16}$$

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}k^4=(-1)^n x\binom{x-2}{n-1}+7(-1)^n x(x-1)\binom{x-3}{n-2}$$
$$+6(-1)^n x(x-1)(x-2)\binom{x-4}{n-3}+(-1)^n x(x-1)(x-2)(x-3)\binom{x-5}{n-4},\ n\geq 4 \tag{2.17}$$

$$\sum_{k=0}^{n}\binom{x+k}{k}k^r=\sum_{k=0}^{r}\binom{x+k}{k}\binom{x+n+1}{n-k}B_{k,k}^r \tag{2.18}$$

$$\sum_{k=0}^{n}\binom{x+k}{k}=\binom{n+x+1}{n} \tag{2.19}$$

$$\sum_{k=0}^{n}\binom{n+k}{k}k^r=\sum_{k=0}^{r}\binom{n+k}{k}\binom{2n+1}{n-k}B_{k,k}^r \tag{2.20}$$

$$\sum_{k=0}^{n}\binom{n+k}{k}=\binom{2n+1}{n} \tag{2.21}$$

$$\sum_{k=0}^{n} \binom{n+k}{k} k = (n+1)\binom{2n+1}{n-1}, \qquad n \geq 1 \tag{2.22}$$

$$\sum_{k=0}^{n} \binom{n+k}{k} k^2 = (n+1)\binom{2n+1}{n-1} + (n+2)(n+1)\binom{2n+1}{n-2}, \qquad n \geq 2 \tag{2.23}$$

$$\begin{aligned}\sum_{k=0}^{n} \binom{n+k}{k} k^3 = (n+1)\binom{2n+1}{n-1} + 3(n+2)(n+1)\binom{2n+1}{n-2} \\ + (n+3)(n+2)(n+1)\binom{2n+1}{n-3}, \qquad n \geq 3\end{aligned} \tag{2.24}$$

$$\sum_{k=0}^{n} k \frac{\binom{n}{k}}{\binom{2n-1}{k}} = \frac{2n}{n+1} \tag{2.25}$$

$$\sum_{k=0}^{n} \binom{2k}{k} \frac{k^r}{2^{2k}} = \frac{2n+1}{2^{2n}} \binom{2n}{n} \sum_{k=0}^{r} \binom{n}{k} \frac{1}{2k+1} B_{k,k}^{r} \tag{2.26}$$

Remark 2.2 *The following identity is Problem 4551, P.482, of the American Math. Monthly, 1953.*

$$\lim_{n\to\infty} n^{\alpha-k} \sum_{j=0}^{n-1} (-1)^j j^k \binom{\alpha}{j} = \frac{1}{(k-\alpha)\Gamma(-\alpha)}, \tag{2.27}$$

where α is real number which is not a nonnegative integer, and k is a positive integer.

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k} (z+yk)^r = \sum_{k=0}^{r} \binom{r}{k} z^{r-k} y^k \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{n-j} B_{j,j}^{k} \tag{2.28}$$

2.3.3 Evaluation of $\sum_{\mathbf{n=0}}^{\infty} \frac{\mathbf{n^r x^n}}{\mathbf{n!}}$

$$\boldsymbol{\sum_{n=0}^{\infty} \frac{n^r x^n}{n!} = e^x \sum_{k=1}^{r} \frac{x^k}{k!} B_{k,k}^{r}} \tag{2.29}$$

Applications of Equation (2.29)

$$\sum_{n=0}^{\infty} \frac{n^{r+1} x^n}{n!} = x \sum_{k=0}^{r} \binom{r}{k} \sum_{n=0}^{\infty} \frac{n^k x^k}{k!} \tag{2.30}$$

$$\sum_{k=0}^{\infty} \frac{k^r \cos kx}{k!} = e^{\cos x} \left(\cos(\sin x) \sum_{k=0}^{r} B_{k,k}^{r} \frac{\cos kx}{k!} - \sin(\sin x) \sum_{k=0}^{r} B_{k,k}^{r} \frac{\sin kx}{k!} \right) \tag{2.31}$$

$$\sum_{k=0}^{\infty} \frac{k^r \sin kx}{k!} = e^{\cos x} \left(\cos(\sin x) \sum_{k=0}^{r} B_{k,k}^{r} \frac{\sin kx}{k!} + \sin(\sin x) \sum_{k=0}^{r} B_{k,k}^{r} \frac{\cos kx}{k!} \right) \tag{2.32}$$

2.4 Grunert's Formula

Grunert's Formula
Let S be a function of x. Let $\frac{d^j S}{dx^j}$ denote the j^{th} derivative of S with respect to x. Then,

$$\boldsymbol{\left(x \frac{d}{dx} \right)^r S = \sum_{k=0}^{r} B_{k,k}^{r} \frac{x^k}{k!} \frac{d^k S}{dx^k}.} \tag{2.33}$$

2.4.1 Applications of Grunert's Formula

$$\sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^{n} B_{k,k}^{n} \frac{x^k}{(1-x)^{k+1}}, \qquad |x| < 1 \tag{2.34}$$

$$\sum_{k=0}^{n} k^r x^k = \sum_{k=0}^{r} B_{k,k}^{r} \sum_{j=0}^{n} \binom{j}{k} x^j \tag{2.35}$$

$$\sum_{k=0}^{n} k^r = \sum_{k=0}^{r} \binom{n+1}{k+1} B_{k,k}^{r} \tag{2.36}$$

$$\sum_{k=0}^{n} k^r = \sum_{k=0}^{r} (-1)^{r-k} \binom{n+k}{k+1} B_{k,k}^{r}, \qquad r \geq 1 \tag{2.37}$$

$$\sum_{k=0}^{n} (-1)^k = \frac{1 + (-1)^n}{2} \tag{2.38}$$

$$\sum_{k=0}^{n} (-1)^k k = (-1)^n \left[\frac{n+1}{2} \right] = \frac{(2n+1)(-1)^n - 1}{4} \tag{2.39}$$

$$\sum_{k=0}^{n}(-1)^k k^2 = (-1)^n \cdot \frac{n^2+n}{2} \tag{2.40}$$

$$\sum_{k=0}^{n}(-1)^k k^3 = \frac{(4n^3+6n^2-1)(-1)^n+1}{8} \tag{2.41}$$

$$\sum_{k=0}^{n}(-1)^k k^4 = (-1)^n \cdot \frac{n^4+2n^3-n}{2} \tag{2.42}$$

$$\sum_{k=0}^{n}(-1)^{k+1} k^r = \sum_{k=0}^{r}\left(\binom{n+1}{k+1} - 2^{r+1}\binom{[\frac{n}{2}]+1}{k+1}\right) B_{k,k}^r \tag{2.43}$$

$$\sum_{k=0}^{n}\binom{n}{k} k^r x^k = (1+x)^n \sum_{k=0}^{r}\binom{n}{k}\frac{x^k}{(1+x)^k} B_{k,k}^r, \qquad x \neq -1 \tag{2.44}$$

$$\sum_{k=0}^{n}\binom{n}{k} k^r = 2^n \sum_{k=0}^{r}\frac{1}{2^k}\binom{n}{k} B_{k,k}^r \tag{2.45}$$

$$\sum_{k=0}^{n}\binom{n}{k} k^3 = 2^n\left(\frac{n}{2}+\frac{3n(n-1)}{4}+\frac{n(n-1)(n-2)}{8}\right) \tag{2.46}$$

$$\sum_{k=0}^{n}\binom{n}{k} k^4 = 2^n\left(\frac{n}{2}+\frac{7n(n-1)}{4}+\frac{6n(n-1)(n-2)}{8}+\frac{n(n-1)(n-2)(n-3)}{16}\right) \tag{2.47}$$

$$\sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)^r (nx)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{r} x^k n^k \sin\left(nx+\frac{k\pi}{2}\right) B_{k,k}^r \tag{2.48}$$

$$\sum_{k=0}^{\infty}(-1)^k \frac{(2k)^r (nx)^{2k}}{(2k)!} = \sum_{k=0}^{r} x^k n^k \cos\left(nx+\frac{k\pi}{2}\right) B_{k,k}^r \tag{2.49}$$

2.5 Properties of $\mathrm{B}^{\mathbf{n}}_{\mathbf{k,k}}$

2.5.1 Recurrence Formulas for $\mathrm{B}^{\mathbf{n}}_{\mathbf{k,k}}$

$$B^n_{j,j} = (-1)^j \sum_{k=0}^{n} (-1)^k \binom{n}{k} j^{n-k} B^k_{j,j} \tag{2.50}$$

$$\boldsymbol{B^{n+1}_{j+1,j+1} = (j+1)\left(B^n_{j+1,j+1} + B^n_{j,j}\right)} \tag{2.51}$$

Remark 2.3 *For the definition of $\Delta_{j,1}$, we refer the reader to Volume 2, Book 1, Chapter 8, Equation (8.1).*

$$(-1)^{j+1} B^{n+1}_{j+1,j+1} = (j+1)\left[(-1)^{j+1} B^n_{j+1,j+1} - (-1)^j B^n_{j,j}\right] = \left[(j+1)\Delta_{j,1}\right](-1)^j B^n_{j,j} \tag{2.52}$$

$$\left[\prod_{k=1}^{r} (j+k)\Delta_{j,1}\right](-1)^j B^n_{j,j} = (-1)^{j+r} B^{n+r}_{j+r,j+r}, \qquad r \geq 1 \tag{2.53}$$

$$\sum_{j=a+1}^{r} \frac{(-1)^j}{j} B^n_{j,j} = (-1)^r B^{n-1}_{r,r} - (-1)^a B^{n-1}_{a,a}, \qquad n \geq 1, \qquad a \geq 0 \tag{2.54}$$

$$B^n_{j,j} = \sum_{k=1}^{j} \binom{j}{k} k! B^{n-k}_{j-k+1,j-k+1}, \qquad j \geq 1, \qquad n \neq j \tag{2.55}$$

2.5.2 Convolution Property for $\mathrm{B}^{\mathbf{n}}_{\mathbf{k,k}}$

$$\boldsymbol{\sum_{k=0}^{n} \binom{n}{k} B^k_{r,r} B^{n-k}_{j,j} = B^n_{r+j,r+j}} \tag{2.56}$$

Applications of Equation (2.56)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (z-k)^{n+j} = \sum_{k=0}^{j} \binom{z-n}{k} B^{n+j}_{n+k,n+k} \tag{2.57}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (z-k)^{n+1} = \frac{2z-n}{2}(n+1)! \tag{2.58}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (z-k)^{n+2} = \frac{3n^2+n+12z^2-12zn}{24}(n+2)! \tag{2.59}$$

2.6 Expansions of $\left(\frac{\mathbf{e^x-1}}{\mathbf{x}}\right)^{\mathbf{n}}$

Remark 2.4 *In this section, we assume $D_x^n f(x)$ is the n^{th} derivative of $f(x)$ with respect to x.*

$$\left(\frac{e^x-1}{x}\right)^n = \sum_{k=0}^{\infty} B_{n,n}^{k+n} \frac{x^k}{(k+n)!} \tag{2.60}$$

$$D_x^n \left(\frac{e^x-1}{x}\right)^k |_{x=0} = \frac{n!}{(n+k)!} B_{k,k}^{n+k} \tag{2.61}$$

$$D_x^n \left(\frac{x}{e^x-1}\right)^k |_{x=0} = k\binom{k+n}{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{j!\binom{n+j}{j}(k+j)} B_{j,j}^{n+j} \tag{2.62}$$

2.7 Polynomial Expansions from $\mathrm{B}_{\mathbf{k},\mathbf{k}}^{\mathbf{n}}$

Remark 2.5 *Throughout this section, we assume $f(x)$ is a polynomial of degree n.*

$$\boldsymbol{f(x) = \sum_{k=0}^{n} \binom{x}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(k-j)} \tag{2.63}$$

2.7.1 Applications of Equation (2.63)

$$\binom{x+n}{n} = \sum_{k=0}^{n} \binom{x}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k-j+n}{n} \tag{2.64}$$

$$\binom{mx}{n} = \sum_{k=0}^{n} \binom{x}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{mk-mj}{n}, \text{ where } m \text{ is a complex number} \tag{2.65}$$

$$\binom{x}{n}^r = \sum_{k=0}^{nr} \binom{x}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k-j}{n}^r \tag{2.66}$$

$$\sum_{k=0}^{n} \binom{x}{k}^r = \sum_{k=0}^{nr} \binom{x}{k} \sum_{j=0}^{k} (-1)^k \binom{k}{j} \sum_{p=[\frac{k+r-1}{r}]}^{n} \binom{k-j}{p}^r \tag{2.67}$$

$$\sum_{k=0}^{n}\binom{n}{k}^3=\sum_{k=0}^{3n}\binom{n}{k}\sum_{j=0}^{k}(-1)^k\binom{k}{j}\sum_{p=[\frac{k+2}{3}]}^{n}\binom{k-j}{p}^3 \tag{2.68}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2=\sum_{k=0}^{2n}\binom{n}{k}\sum_{j=0}^{k}(-1)^k\binom{k}{j}\sum_{p=[\frac{k+1}{2}]}^{n}\binom{k-j}{p}^2=\binom{2n}{n} \tag{2.69}$$

2.8 Polynomial Series via $\mathrm{B}_{\mathbf{k,k}}^{\mathbf{n}}$

Remark 2.6 *Throughout this section, we assume $f(x)$ is a polynomial of degree n.*

$$\boldsymbol{\sum_{k=0}^{\infty}\frac{x^k}{k!}f(k)=e^x\sum_{k=0}^{n}\frac{x^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}f(k-j)} \tag{2.70}$$

2.8.1 Applications of Equation (2.70)

$$\sum_{k=0}^{\infty}\binom{k+n}{n}\frac{x^k}{k!}=e^x\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{k!} \tag{2.71}$$

Dobinski's Formula

$$\sum_{k=0}^{\infty}(-1)^k\frac{k^n}{k!}=\frac{1}{e}\sum_{k=0}^{n}(-1)^kB_{k,k}^n \tag{2.72}$$

3 Bell Numbers $\mathrm{B}_{\mathbf{n}}$

Remark 3.1 *Throughout this chapter, we assume r and j are nonnegative integers. We assume x, y, and z are real (complex) numbers. We also let $[x]$ denote the floor of x for any real x.*

3.1 Definition of $\mathrm{B}_{\mathbf{n}}$

$$\boldsymbol{B_n=\sum_{k=0}^{n}\frac{B_{k,k}^n}{k!}} \tag{3.1}$$

Stirling's Formula

$$B_n=\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!},\qquad n\geq 1 \tag{3.2}$$

3.2 Properties of Bell Numbers

3.2.1 Bell Number Recurrences

$$\boldsymbol{B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \qquad B_0 \equiv 1} \tag{3.3}$$

$$B_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B_{k+1} \tag{3.4}$$

3.2.2 Exponential Generating Function

$$\boldsymbol{e^{e^x - 1} = \sum_{k=0}^{\infty} \frac{x^k}{k!} B_k} \tag{3.5}$$

3.3 Schäfli's Modified Bell Number Recurrence

Remark 3.2 *The identities of this section are found in L. Schläfli's "On a generalization given by Laplace of Lagrange's Theorem", Quarterly Journal of Pure and Applied Mathematics, Vol. 2 (1858), pp. 24-31. The reader is also referred to Oliver A. Gross's, "Preferential Arrangements", American Math. Monthly, Vol. 69 (1962), pp. 4-8.*

Schläfli's Recurrence

$$A_n = \sum_{k=0}^{n-1} \binom{n}{k} A_k, \qquad n \geq 1, \qquad A_0 \equiv 1 \tag{3.6}$$

3.3.1 Alternative Forms of Equation (3.6)

$$2A_n = \sum_{k=0}^{n} \binom{n}{k} A_k, \qquad n \geq 1 \tag{3.7}$$

$$A_n = \sum_{k=0}^{n} B_{k,k}^{n} \tag{3.8}$$

3.3.2 Exponential Generating Function

Let $A_n(x) = \sum_{k=0}^{n} B_{k,k}^n x^k$. *Then,*

$$\frac{1}{1 - x(e^t - 1)} = \sum_{k=0}^{\infty} A_k(x) \frac{t^k}{k!}. \tag{3.9}$$

$$\frac{1}{2 - e^t} = \sum_{k=0}^{\infty} A_n \frac{t^n}{n!} \tag{3.10}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{k}{r} x^{k-r} B_{k,k}^n = \frac{(e^t - 1)^r}{(1 - (e^t - 1)x)^{r+1}} \tag{3.11}$$

3.4 Dobinski Numbers $\mathrm{D_n}$

3.4.1 Definition of $\mathrm{D_n}$

$$D_n = \sum_{k=0}^{n} (-1)^k \frac{B_{k,k}^n}{k!} \tag{3.12}$$

Dobinski's Formula

$$D_n = e \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!} \tag{3.13}$$

3.4.2 Properties of Dobinski Numbers

Recurrence Relation

$$(-1)^{n+1} D_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} D_{k+1}, \qquad D_0 \equiv 1 \tag{3.14}$$

Exponential Generating Function

$$\frac{1}{e^{e^x - 1}} = \sum_{k=0}^{\infty} \frac{x^k}{k!} D_k \tag{3.15}$$

3.5 Functional Bell Number Recurrence

Remark 3.3 *In this section, we assume f is a real or complex valued funtion over the nonnegative integers. We also assume p is a positive integer.*

$$\sum_{k=1}^{p}\binom{n+1}{k}kf(k-1)=(n+1)\sum_{k=0}^{p}\binom{n}{k}f(k)-(p+1)\binom{n+1}{p+1}f(p) \tag{3.16}$$

4 Shifted Stirling Numbers of the Second Kind $\mathrm{A_{j,n}}$

Remark 4.1 *Throughout this chapter, we assume r and j are nonnegative integers. We assume x, y, and z are real or complex numbers. We also let $[x]$ denote the floor of x for any real x.*

4.1 Definition of $\mathrm{A_{j,n}}$

$$\boldsymbol{x^n=\sum_{j=0}^{n}\binom{x+j-1}{n}A_{j,n}} \tag{4.1}$$

4.2 Explicit Formulas for $\mathrm{A_{j,n}}$

$$\boldsymbol{A_{j,n}=\sum_{k=0}^{j}(-1)^k\binom{n+1}{k}(j-k)^n} \tag{4.2}$$

$$\boldsymbol{A_{j,n}=\sum_{k=0}^{j}(-1)^{j+k}\binom{n+1}{j-k}k^n} \tag{4.3}$$

4.3 Relationships Between $\mathrm{A_{j,n}}$ and $\mathrm{B^n_{j,j}}$

$$B^n_{j,j}=\sum_{k=0}^{n}\binom{k-1}{n-j}A_{k,n}, \qquad 0\le j\le n \tag{4.4}$$

$$A_{j,n}=\sum_{r=0}^{n}(-1)^{j+r}\binom{n}{r}j^{n-r}\sum_{k=0}^{r}\binom{n-k}{j-k}\binom{n+1}{k}B^r_{k,k} \tag{4.5}$$

$$A_{j,n}=\sum_{k=0}^{n}(-1)^{k+j}\binom{n-k}{n-j}B^n_{k,k} \tag{4.6}$$

4.3.1 Applications of Equation (4.4)

$$\sum_{j=0}^{n} A_{j,n} = n! \tag{4.7}$$

Remark 4.2 *The following two identities are found in Robert Stalley's "A Generalization of the Geometric Series", American Math. Monthly, May 1949, Vol. 56, No. 5, pp. 325-327*

$$\sum_{k=1}^{\infty} k^n x^k = \sum_{k=1}^{n} \frac{x^k}{(1-x)^{n+1}} A_{k,n}, \qquad |x| < 1,\ n \geq 1 \tag{4.8}$$

$$\sum_{k=1}^{\infty} k^n x^k = \sum_{j=1}^{n} \sum_{k=1}^{j} (-1)^{k+1} \binom{n+1}{k-1} (j-k+1)^n \frac{x^j}{(1-x)^{n+1}}, \qquad |x| < 1,\ n \geq 1 \tag{4.9}$$

Remark 4.3 *The following identity can be found in T. M. Apostol's "On the Lerch Zeta Function", Pacific Journal of Mathematics, Vol. 1, No. 2, June 1951, pp. 161-167.*

$$\frac{z}{ye^z - 1} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{k+1}}{k!(y-1)^{k+1}} \sum_{j=0}^{k} y^j A_{j,k}, \qquad |y| < 1,\ \ |ye^z| < 1 \tag{4.10}$$

4.4 Properties of $\mathbf{A_{j,n}}$

4.4.1 Indice Equivalence Property

$$A_{j,n} - A_{n-j+1,n} = \begin{cases} 0, & n \geq 1 \\ (-1)^j, & n = 0,\ j = 0,1 \end{cases} \tag{4.11}$$

4.4.2 Recurrence Relation

$$A_{j,n+1} = (n+2-j)A_{j-1,n} + jA_{j,n}, \qquad j \geq 1 \tag{4.12}$$

4.4.3 Grunert's Formula

Let S be a real or complex valued funtion over the set of nonnegative integers. Let $\frac{d^k S}{dx^k}$ denote the k^{th} derivative of S with respect to x. Then,

$$\left(x\frac{d}{dx}\right)^n S = \sum_{j=0}^{n} A_{j,n} \sum_{k=0}^{n} \binom{j-1}{n-k} \frac{x^k}{k!} \frac{d^k S}{dx^k} \tag{4.13}$$

4.4.4 Evaluation of $\sum_{k=0}^{n} k^r$

$$\sum_{k=0}^{n} k^r = \sum_{j=0}^{r} \binom{n+j}{r+1} A_{j,r}, \qquad n \geq 0, \;\; r \geq 1 \tag{4.14}$$

5 Worpitzky/Nielsen Numbers $B_{r,q}^{n}$

Remark 5.1 *Throughout this chapter, we assume r, q, m, and j are nonnegative integers. We assume x, y, and z are real or complex numbers. We also let $[x]$ denote the floor of x for any real x.*

5.1 Definition of $B_{r,q}^{n}$

$$\boldsymbol{B_{r,q}^{n} = \sum_{k=0}^{r} (-1)^k \binom{q}{k} (r-k)^n} \tag{5.1}$$

5.1.1 Connection to $A_{j,n}$

$$\boldsymbol{A_{j,n} = B_{j,n+1}^{n} = \sum_{k=0}^{j} (-1)^k \binom{n+1}{k} (j-k)^n} \tag{5.2}$$

5.2 Properties of $B_{r,q}^{n}$

5.2.1 Index Shift Property

$$B_{k,m+1}^{n} + (-1)^{m+n-1} B_{n-k+1,m+1}^{n} = \begin{cases} 0, & m \geq n \geq 1 \\ (-1)^k \binom{m+1}{k}, & m \geq n = 0 \end{cases} \tag{5.3}$$

5.2.2 Relationships to $B_{j,j}^{n}$

$$(-1)^{m+n} B_{j,j}^{n} = \sum_{k=0}^{m+1} \binom{k-1}{m-j} B_{k,m+1}^{n}, \qquad m \geq n \tag{5.4}$$

$$B_{j,j}^{n} = \sum_{k=0}^{m+1} \binom{m-k}{m-j} B_{k,m+1}^{n}, \qquad m \geq n \tag{5.5}$$

$$B^n_{k,m+1} = (-1)^k \sum_{j=0}^{m} (-1)^j \binom{m-j}{m-k} B^n_{j,j}, \qquad m \ge n \tag{5.6}$$

Applications of Equation (5.4)

Remark 5.2 *The following identity is a formula given by N. Nielsen in* *Traité élémentaire des nombres de Bernoulli, Paris, 1924, pp. 26-30.*

Nielsen's Formula

$$\boldsymbol{(-1)^{m+n} x^n = \sum_{k=0}^{m+1} \binom{x+k-1}{m} B^n_{k,m+1}, \qquad m \ge n} \tag{5.7}$$

Recurrence Relation

$$B^{n+1}_{k,m+1} = (m-k+1) B^n_{k-1,m} + k B^n_{k,m}, \qquad m \ge n+1 \ge 1 \tag{5.8}$$

5.2.3 Grunert's Formula

Let S be a real or complex valued funtion over the set of nonnegative integers. Let $\frac{d^j S}{dx^j}$ denote the j^{th} derivative of S with respect to x. Then,

$$\left(x \frac{d}{dx}\right)^n S = (-1)^{m+n} \sum_{k=0}^{m+1} B^n_{k,m+1} \sum_{j=0}^{n} \binom{k-1}{m-j} \frac{x^j}{j!} \frac{d^j S}{dx^j}, \qquad m \ge n \tag{5.9}$$

5.3 Polynomial Expansions from Nielsen's Formula

Remark 5.3 *Throughout this section, we assume $f(x)$ is an arbitrary polynomial of degree $\le m$, namely, $f(x) = \sum_{k=0}^{m} a_k x^k$.*

Nielsen's Polynomial Expansion

$$\boldsymbol{f(x) = (-1)^m \sum_{k=0}^{m} \binom{x+k-1}{m} \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} f(j-k)} \tag{5.10}$$

5.3.1 Applications of Equation (5.10)

Remark 5.4 *Throughout this subsection, we assume p is a positive integer.*

$$\binom{x}{r}^p = (-1)^{m+rp} \sum_{k=0}^{m} \binom{x+k-1}{m} \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} \binom{k-j+r-1}{r}^p, \ rp \le m \tag{5.11}$$

$$\binom{x}{r}^p = \sum_{k=1}^{rp-r+1} \binom{x+k-1}{rp} \sum_{j=0}^{k-1} (-1)^j \binom{rp+1}{j} \binom{k-j+r-1}{r}^p, \ r,p \ge 1 \tag{5.12}$$

$$\binom{x+n}{n} = (-1)^m \sum_{k=0}^{m} \binom{x+k-1}{m} \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} \binom{j-k+n}{n}, \qquad n \le m \tag{5.13}$$

$$\binom{x}{n} = (-1)^n \sum_{k=0}^{n} \binom{x-n+k-1}{n} \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} \binom{j-k+n}{n} \tag{5.14}$$

$$\binom{px}{n} = (-1)^m \sum_{k=0}^{m} \binom{x+k-1}{m} \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} \binom{pj-pk}{n}, \qquad n \le m \tag{5.15}$$

$$\binom{x}{n} = (-1)^n \sum_{k=0}^{n} \binom{x+k-1}{n} \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} \binom{j-k}{n} \tag{5.16}$$

An n^{th} Difference Application of Equation (5.10)
Assume $f(x)$ is a polynomial of degree $\le m$. Then,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) = (-1)^{m+n} \sum_{k=0}^{m} \binom{k-1}{m-n} \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} f(j-k). \tag{5.17}$$

Applications of Equation (5.17)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k}{r}^p = (-1)^{rp} \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} \binom{k-j+r-1}{r}^p, \qquad rp \le n \tag{5.18}$$

$$\sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} \binom{k-j+r-1}{r}^p = \begin{cases} 0, & rp < n \\ \frac{(rp)!}{r!^p}, & rp = n \end{cases} \tag{5.19}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{3n}{k}\binom{3n-k}{n}^3=\binom{3n}{2n}\binom{2n}{n} \tag{5.20}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{3n+1}{k}\binom{3n-k}{n}^3 k=(3n+1)\left(1-\binom{3n}{2n}\binom{2n}{n}\right) \tag{5.21}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3\frac{1}{\binom{3n}{k}^2}=\frac{\binom{2n}{n}}{\binom{3n}{2n}^2} \tag{5.22}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{3n-k}{n}^2=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{n+k}{n}^2=\binom{2n}{n} \tag{5.23}$$

5.4 Generalized Nielsen Expansions

Remark 5.5 *Throughout this section, we assume $f(x)$ is an arbitrary polynomial of degree n, namely, $f(x)=\sum_{k=0}^{n}a_kx^k$.*

Nielsen's First Generalized Polynomial Expansion: Assume $n\leq m$. Then,

$$\boldsymbol{f(x+y)=(-1)^m\sum_{k=0}^{m+1}\binom{x+k-1}{m}\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}f(j-k+y).} \tag{5.24}$$

Nielsen's Second Generalized Polynomial Expansion

$$\boldsymbol{f(x+y)=\sum_{k=0}^{n}\binom{x}{k}\sum_{j=0}^{k}(-1)^k\binom{k}{j}f(k-j+y)} \tag{5.25}$$

5.4.1 Applications of Equation (5.24) and (5.25)

Remark 5.6 *In this subsection, we assume p is a positive integer.*

$$f(x)=(-1)^m\sum_{k=0}^{m}(-1)^k\binom{m+1}{k}f(j-m+x-1),\qquad n\leq m \tag{5.26}$$

$$\binom{x+y}{r}^p = (-1)^m \sum_{k=0}^{m+1} \binom{x+k-1}{m} \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} \binom{j-k+y}{r}^p \tag{5.27}$$

$$\begin{aligned} \binom{x+y}{r}^p &= (-1)^{rp+m} \sum_{k=0}^{m+1} \binom{x+k-1}{m} \\ &\quad \cdot \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} \binom{k-j-y+r-1}{r}^p, \qquad rp \leq m \end{aligned} \tag{5.28}$$

$$\binom{x+y}{r}^p = \sum_{k=0}^{rp+1} \binom{x+k-1}{rp} \sum_{j=0}^{k} (-1)^j \binom{rp+1}{j} \binom{k-j-y+r-1}{r}^p \tag{5.29}$$

$$\binom{x-y}{r}^p = \sum_{k=0}^{rp+1} \binom{x+k-1}{rp} \sum_{j=0}^{k} (-1)^j \binom{rp+1}{j} \binom{k-j+y+r-1}{r}^p \tag{5.30}$$

$$\binom{1-y}{r}^p = \sum_{k=0}^{rp} (-1)^k \binom{rp+1}{k} \binom{rp-k+y+r-1}{r}^p \tag{5.31}$$

$$\sum_{k=0}^{rp} (-1)^k \binom{rp+1}{k} \binom{rp-k+1}{r}^p = 0, \qquad r \geq 1 \tag{5.32}$$

$$\sum_{k=0}^{rp-r+1} (-1)^k \binom{rp+1}{k} \binom{rp-k}{r}^p = 1 \tag{5.33}$$

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \binom{2n-k}{n}^2 = 1 \tag{5.34}$$

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{k} \binom{2n-k}{n}^2 \frac{1}{2n+1-k} = \frac{1}{2n+1} \tag{5.35}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\frac{1}{2n+1-k}=\frac{1}{(2n+1)\binom{2n}{n}} \tag{5.36}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{3n+1}{k}\binom{3n-k}{n}^2=1 \tag{5.37}$$

$$\sum_{k=0}^{3n}(-1)^k\binom{4n+1}{k}\binom{4n-k}{n}^2=1 \tag{5.38}$$

$$\binom{px+py}{n}=\sum_{k=0}^{n}\binom{x}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{pk-pj+py}{n} \tag{5.39}$$

$$\binom{px+py}{n}=(-1)^m\sum_{k=0}^{m+1}\binom{x+k-1}{m}\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{pj-pk+py}{n},\ \ n\le m \tag{5.40}$$

$$\begin{aligned}\binom{px+py}{n}&=(-1)^{m+n}\sum_{k=0}^{m+1}\binom{x+k-1}{m}\\&\cdot\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{pk-pj-py+n-1}{n},\qquad n\le m\end{aligned} \tag{5.41}$$

$$\binom{2x+2y}{n}=\sum_{k=0}^{n+1}\binom{x+k-1}{n}\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}\binom{2k-2j-2y+n-1}{n} \tag{5.42}$$

$$\binom{2y+2}{n}=\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{3n-2k-2y-1}{n} \tag{5.43}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{3n-2k+1}{n}=\begin{cases}1, & n=0\\ 0, & n\ge 1\end{cases} \tag{5.44}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{3n-2k-1}{n}=\binom{2}{n} \tag{5.45}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{2n-2k+y}{n}=\binom{n-y+1}{n} \tag{5.46}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{2n-2k}{n}=n+1 \tag{5.47}$$

Remark 5.7 *The following identity is the solution of B. C. Wong's Problem 3399 of The American Math. Monthly, Vol. 36, No. 10, December 1929.*

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n+1}{k}\binom{2n-2k}{n}=n+1 \tag{5.48}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{2n-2k+1}{n}=1 \tag{5.49}$$

Remark 5.8 *The following identity is the solution of B. C. Wong's Problem 3426 of The American Math. Monthly, May 1930.*

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n+1}{k}\binom{2n-2k-1}{n}=1, \qquad n\geq 1 \tag{5.50}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{2n-2k-1}{n}=\binom{n+2}{n} \tag{5.51}$$

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n+1}{k}\binom{2n-2k-1}{n}=\begin{cases}1, & n=0\\ \frac{n^2+n}{2}, & n\geq 1\end{cases} \tag{5.52}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{pn-pk+r}{n}=\binom{n-r+p-1}{n} \tag{5.53}$$

$$\sum_{k=0}^{[\frac{(p-1)n}{p}]} (-1)^k \binom{n+1}{k} \binom{pn-pk}{n} = \binom{n+p-1}{n} \tag{5.54}$$

$$\sum_{k=0}^{[\frac{2n}{3}]} (-1)^k \binom{n+1}{k} \binom{3n-3k}{n} = \binom{n+2}{n} \tag{5.55}$$

5.5 Nielsen Numbers $\beta_{\mathbf{p}}^{\mathbf{m,n}}(\alpha)$

Remark 5.9 *Throughout this section, we assume p is a nonnegative integer while α is a real or complex number.*

5.5.1 Definition of $\beta_{\mathbf{p}}^{\mathbf{m,n}}(\alpha)$

$$\boldsymbol{\beta_p^{m,n}(\alpha) = \sum_{k=0}^{p} (-1)^k \binom{m+1}{k} (\alpha + p - k)^n} \tag{5.56}$$

Remark 5.10 *This definition is found on Page 31 of Traité élémentaire des nombres de Bernoulli, by Niels Nielsen, Gauthier-Villars, Paris, 1923.*

5.5.2 Relationship to Worpitzky Numbers

$$\beta_p^{m,n}(0) = B_{p,m+1}^n \tag{5.57}$$

$$\beta_p^{m,n}(\alpha) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} B_{p,m+1}^k \tag{5.58}$$

5.5.3 Polynomial Expansions via Nielsen Numbers

Remark 5.11 *This identity is found on Page 28 of Traité élémentaire des nombres de Bernoulli by Niels Nielsen.*

$$(-1)^{m+n}(x-\alpha)^n = \sum_{k=0}^{m+1} \binom{x+k-1}{m} \beta_k^{m,n}(\alpha), \qquad m \geq n \tag{5.59}$$

6 Stirling Numbers of the First Kind $\mathbf{C_k^n = \frac{s(n,k)}{n!}}$

Remark 6.1 *Throughout this chapter, we assume r, q, m, and j are nonnegative integers. We assume x, y, and z are real or complex numbers. We also let $[x]$ denote the floor of x for any real x.*

6.1 Basis Definition of $\mathbf{C_k^n}$

$$\binom{x}{n} = \sum_{k=0}^{n} C_k^n x^k \tag{6.1}$$

6.1.1 Derivatives of $\binom{x}{n}$

Remark 6.2 *For this subsection, we let D_x^p denote the p^{th} derivative with respect to x.*

$$D_x^1 \binom{x}{n} = \binom{x}{n} \sum_{k=1}^{n} \frac{1}{k+x-n}, \qquad n \geq 1 \tag{6.2}$$

$$D_x^1 \binom{x}{n}|_{x=0} = \frac{(-1)^{n-1}}{n} = C_1^n, \qquad n \geq 1 \tag{6.3}$$

$$D_x^2 \binom{x}{n} = \binom{x}{n} \left[\left(\sum_{k=1}^{n} \frac{1}{k+x-n} \right)^2 - \sum_{k=1}^{n} \frac{1}{(k+x-n)^2} \right], \ n \geq 1 \tag{6.4}$$

$$D_x^2 \binom{x}{n}|_{x=0} = \frac{2}{n}(-1)^n \sum_{k=1}^{n-1} \frac{1}{k} = 2C_2^n, \qquad n \geq 2 \tag{6.5}$$

6.2 Properties of $\mathbf{C_k^n}$

6.2.1 Recurrence Formula

$$(n+1)C_k^{n+1} = C_{k-1}^n - nC_k^n, \qquad k \geq 1 \tag{6.6}$$

6.2.2 Convolution Formula

$$\sum_{r=0}^{n} C_j^r C_k^{n-r} = C_{k+j}^n \binom{k+j}{j} \tag{6.7}$$

6.2.3 Orthogonality Properties

$$\sum_{j=0}^{n} \boldsymbol{B^n_{j,j} C^j_k} = \binom{\mathbf{0}}{\boldsymbol{n-k}} \tag{6.8}$$

$$\sum_{j=0}^{n} \boldsymbol{C^n_j B^j_{k,k}} = \binom{\mathbf{0}}{\boldsymbol{n-k}} \tag{6.9}$$

Remark 6.3 *The following identity is a formula of Frank R. Olson from The American Math. Monthly, October 1956, Vol. 63, No. 8, p. 612.*

$$\sum_{j=0}^{n} C^r_j B^{n+j-1}_{k,k} = \begin{cases} 0, & k < r \\ r^{n-1}, & k = r \end{cases} \tag{6.10}$$

6.2.4 Inversion Formulas

Assume b_k *is independent of* n*, and* $a_n = \sum_{k=0}^{n} B^n_{k,k} b_k$*. Then,* $b_n = \sum_{k=0}^{n} C^n_k a_k$*.* (6.11)

Let $r \geq n$*. Assume* b_k *is independent of* n*, and* $a_n = \sum_{k=0}^{r} B^k_{n,n} b_k$*. Then,* $b_n = \sum_{k=0}^{r} C^k_n a_k$*.* (6.12)

6.2.5 Alternating Sum Formulas

$$\sum_{k=j}^{n} (-1)^k C^k_j = (-1)^n \sum_{k=j}^{n} (-1)^{k-j} \binom{k}{j} C^n_k \tag{6.13}$$

$$\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} (-1)^{n-k} k C^n_k, \qquad k \geq 1 \tag{6.14}$$

6.3 Functional Expansions Involving $\mathbf{C^n_k}$

6.3.1 Expansions of $\prod_{\mathbf{k=0}}^{\mathbf{n-1}}(\mathbf{1+kx})$

$$\prod_{k=0}^{n-1} (1+kx) = n! \sum_{k=0}^{n} (-1)^k C^n_{n-k} x^k, \qquad n \geq 1 \tag{6.15}$$

$$\frac{n!}{\prod_{k=1}^{n}(1-kx)} = \sum_{k=0}^{\infty} B^{k+n}_{n,n} x^k, \qquad n \geq 1, \qquad |x| < \frac{1}{n} \tag{6.16}$$

6.3.2 Expansions of $\binom{\mathbf{x+n}}{\mathbf{n}}$

$$\binom{x+n}{n} = \sum_{k=0}^{n} x^k \sum_{j=0}^{n} \binom{n}{j} C_k^j \equiv \sum_{k=0}^{n} q_k^n x^k \tag{6.17}$$

$$\textit{If } \binom{x+n}{n} = \sum_{k=0}^{n} q_k^n x^k, \;\; \textit{then, } C_k^r = \sum_{j=0}^{r} (-1)^{j+r} \binom{r}{j} q_k^j. \tag{6.18}$$

6.3.3 Expansions of $\binom{\mathbf{x}}{\mathbf{n}}$ Involving $\mathrm{B}^{\mathbf{x}}_{\mathbf{j,j}}$

Remark 6.4 *For this subsection, we define, for arbitrary real or complex x, the generalized Stirling number of the second kind $B^x_{j,j}$ as follows.*

$$B^x_{j,j} = \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} k^x \tag{6.19}$$

Note that Equation (6.19) implies

$$z^x = \sum_{k=0}^{\infty} B^x_{k,k} \binom{z}{k}. \tag{6.20}$$

$$\binom{x}{n} = \sum_{k=0}^{\infty} B^x_{k,k} \left(C_n^k + C_n^{k-1} \right) \tag{6.21}$$

$$x = 1 + \sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)} B^x_{k,k} \tag{6.22}$$

$$\sum_{k=0}^{n} C_k^n \binom{k}{j} = C_j^n + C_j^{n-1} \tag{6.23}$$

$$\sum_{k=1}^{n} k C_k^n = \frac{(-1)^n}{n(n-1)}, \qquad n \geq 2 \tag{6.24}$$

Two Orthogonality Relationships

$$\textit{Let } B^x_{j,j} = \sum_{k=0}^{\infty} \binom{x}{k} E^k_j, \ \textit{where } E^k_j = \sum_{r=0}^{j} (-1)^{r+j} \binom{j}{r} (r-1)^k. \tag{6.25}$$

Then,

$$\sum_{j=0}^{\infty} \left(C^j_n + C^{j-1}_n \right) E^k_j = \binom{0}{n-k}, \tag{6.26}$$

and

$$\sum_{j=0}^{\infty} E^j_n \left(C^k_j + C^{k-1}_j \right) = \binom{0}{n-k}. \tag{6.27}$$

6.3.4 Fractional Binomial Sum Expansions

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{(-1)^n}{\binom{x+n}{n}} \sum_{k=0}^{[\frac{n}{2}]} C^{n+1}_{2k+1} x^{2k} \tag{6.28}$$

$$\sum_{k=0}^{1} (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{1}{(x+1)} \tag{6.29}$$

$$\sum_{k=0}^{2} (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{x^2+2}{(x+2)(x+1)} \tag{6.30}$$

$$\sum_{k=0}^{3} (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{6x^2+6}{(x+3)(x+2)(x+1)} \tag{6.31}$$

$$\sum_{k=0}^{4} (-1)^k \frac{\binom{x}{k}}{\binom{x+k}{k}} = \frac{x^4+35x^2+24}{(x+4)(x+3)(x+2)(x+1)} \tag{6.32}$$

6.3.5 Expansion of $(1+z)^{\frac{1}{z}}$

Remark 6.5 *The following identity is found in " On the Expansion of* $(1+x)^{\frac{1}{x}}$ *in Ascending Powers of* x*", by Percival Frost, Quarterly Journal of Mathematics, London: Vol. 7, No. 28, Feb. 1866.*

$$(1+z)^{\frac{1}{z}} = \sum_{j=0}^{\infty} z^j \sum_{k=0}^{\infty} C_k^{k+j}, \qquad \textit{where} \tag{6.33}$$

$$C_{k-j}^{k} = \frac{(-1)^j}{k!} \sum_{r=0}^{j} \binom{j+k}{j-r} \binom{r-k}{j+r} \frac{(-1)^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} s^{r+j} \tag{6.34}$$

6.4 A Derivative Expansion Involving $\mathbf{C_k^n}$

Remark 6.6 *Throughout this subsection, we let* $D_x^n f(x)$ *denote the* n^{th} *derivative of* $f(x)$ *with respect to* x*.*

$$\boldsymbol{D_x^n f(\ln x) = \frac{n!}{x^n} \sum_{k=0}^{n} C_k^n D_x^k f(z),} \qquad \textit{where } \boldsymbol{z = \ln x} \tag{6.35}$$

6.4.1 Applications of Equation (6.35)

$$D_z^n (\ln z)^j = \frac{n!j!}{z^n} \sum_{k=0}^{j} \frac{1}{k!} (\ln z)^k C_{j-k}^n \tag{6.36}$$

$$D_z^n (\ln z)^2 = \frac{2(-1)^{n-1}(n-1)!}{z^n} \left(\ln z - \sum_{k=1}^{n-1} \frac{1}{k} \right), \qquad n \geq 2 \tag{6.37}$$

$$\sum_{k=0}^{n} B_{k,k}^n \frac{z^k}{k!} D_z^k (\ln z)^n = n! \tag{6.38}$$

6.4.2 Inversion of Equation (6.35)

$$C_j^n = \frac{(-1)^j z^n}{j!n!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} (\ln z)^{j-k} D_z^n (\ln z)^k \tag{6.39}$$

Applications of Equation (6.39)

$$\boldsymbol{C_j^n = \frac{1}{j!n!} D_z^n \left(\ln z\right)^j |_{z=1}} \tag{6.40}$$

$$(\ln(z+1))^j = j! \sum_{k=j}^{\infty} C_j^k z^k, \qquad |z| < 1 \tag{6.41}$$

$$(z+1)^x = \sum_{k=0}^{\infty} x^k \sum_{j=0}^{\infty} C_k^j z^j \tag{6.42}$$

$$(x+1)^x = \sum_{j=0}^{\infty} x^j \sum_{k=0}^{[\frac{j}{2}]} C_k^{j-k} \tag{6.43}$$

$$x^j = j! \sum_{k=0}^{\infty} \left(e^x - 1\right)^k C_j^k, \qquad |e^x| < 1 \tag{6.44}$$

6.5 Explicit Formulas for $\mathbf{C_k^n}$ using Bernoulli Polynomials and $\mathbf{B_{j,j}^n}$

Remark 6.7 *Let t be a real (complex) number. We define $B_k^{(n)}(t)$ as the general Bernoulli polynomial of order n and degree k by the generating function relation*

$$\sum_{k=0}^{\infty} B_k^{(n)}(t) \frac{x^k}{k!} = \frac{x^n e^{tx}}{(e^x - 1)^n}, \qquad |x| < 2\pi. \tag{6.45}$$

An excellent reference for properties of $B_k^{(n)}(t)$ is Calculus of Finite Differences by Charles Jordan, Second Edition, Chelsea Publishing, New York, 1947.

$$\boldsymbol{C_{n-k}^n = \frac{1}{k!(n-k)!} B_k^{(n+1)}(1)} \tag{6.46}$$

$$C_{n-k}^n = \frac{1}{k!(n-k)!} D_x^k \left(e^x \left(\frac{x}{e^x - 1} \right)^{n+1} \right) |_{x=0} \tag{6.47}$$

$$C_{n-k}^{n}=\frac{1}{k!(n-k)!}\sum_{j=0}^{k}\binom{k}{j}D_x^j\left(\frac{x}{e^x-1}\right)^{n+1}|_{x=0} \tag{6.48}$$

$$n!C_{n-k}^{n}=\binom{n-1}{k}\sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}\frac{k!}{(k+nj)!}B_{nj,nj}^{k+nj} \tag{6.49}$$

Inverse of Identity (6.49)

$$\frac{1}{n!}B_{n,n}^{k+n}=\binom{k+n}{k}\sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}\frac{(jn)!}{\binom{jn-1}{k}}C_{jn-k}^{jn} \tag{6.50}$$

$$C_{n-k}^{n}=\frac{1}{(n-k)!k!}\sum_{j=0}^{k}\binom{k}{j}\sum_{s=0}^{j}(-1)^s\binom{j+1}{s+1}\frac{j!}{(j+sn+s)!}B_{sn+s,sn+s}^{j+sn+s} \tag{6.51}$$

$$C_{n-k}^{n}=\frac{1}{(n-k)!k!}\sum_{j=0}^{k}\binom{k}{j}\sum_{s=0}^{j}(-1)^s\frac{j+1}{s+1}\binom{j}{s}\frac{1}{(sn+s)!\binom{j+sn+s}{j}}B_{sn+s,sn+s}^{j+sn+s} \tag{6.52}$$

$$\begin{aligned}C_{n-k}^{n}=&\frac{1}{(n-k)!k!}\sum_{j=0}^{k}\binom{k}{j}(n+1)\binom{n+1+j}{j}\\&\cdot\sum_{s=0}^{j}(-1)^s\binom{j}{s}\frac{1}{j!(n+1+s)\binom{j+s}{j}}B_{s,s}^{j+s}\end{aligned} \tag{6.53}$$

$$C_{n-k}^{n}=\frac{1}{(n-k)!k!}\sum_{j=0}^{k}\binom{k}{j}\binom{n+j}{j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}\frac{n+j+1}{n+s+1}\frac{1}{s!\binom{j+s}{s}}B_{s,s}^{j+s} \tag{6.54}$$

6.6 Schläfli's Formulas for $\mathbf{C_k^n}$

Remark 6.8 *The identities of this section can be found in the following two papers, both written by L. Schläfli: "Sur les coefficients du développement du produit* $1(1+x)(1+2x)...(1+(n-1)x)$ *suivant les puissances ascendantes de* x*", Jour. reine u. agnew. Math., Volume 43, 1852, pp. 1-22: "Ergänzung der Abhandlung über die Entwickelung des Prudukts* $1(1+x)(1+2x)...(1+(n-1)x) = \prod^n(x)$*", Jour. reine u. agnew. Math., Volume 67, 1867, pp. 179-182.*

$$C_{n-k}^n = \frac{(-1)^k}{n!}\sum_{j=0}^{k}\binom{k+n}{k-j}\binom{k-n}{k+j}\frac{1}{j!}B_{j,j}^{j+k} \tag{6.55}$$

$$C_{n-k}^n = \frac{1}{n!}\sum_{j=0}^{k}(-1)^j\binom{n+j-1}{n-k-1}\binom{n+k}{n+j}\frac{1}{j!}B_{j,j}^{j+k} \tag{6.56}$$

6.6.1 Inverse Relations of Equation (6.55)

$$B_{n-k,n-k}^n = (-1)^k(n-k)!\sum_{j=0}^{k}\binom{k-n}{k+j}\binom{k+n}{k-j}(k+j)!C_j^{k+j} \tag{6.57}$$

7 Appendix A: Contour Integral Formulas for Stirling Numbers

Remark 7.1 *Throughout this appendix, we assume* $i \equiv \sqrt{-1}$*. We also let* γ *denote a simple closed curve around the origin, namely, the unit circle.*

7.1 Contour Integrals for $\mathrm{B}^{\mathbf{n}}_{\mathbf{k},\mathbf{k}}$

$$B^n_{k,k} = \frac{n!}{2\pi i} \int_\gamma \frac{(e^z - 1)^k}{z^{n+1}} \, dz \tag{7.1}$$

$$B^n_{k,k} = \binom{n-1}{k-1} \frac{k!(n-k)!}{2\pi i} \int_\gamma \frac{e^z (e^z - 1)^{k-1}}{z^n} \, dz \tag{7.2}$$

7.2 Contour Integrals for $\mathrm{C}^{\mathbf{n}}_{\mathbf{k}}$

$$C^n_k = \frac{1}{2\pi i} \int_\gamma \binom{z}{n} \frac{1}{z^{k+1}} \, dz \tag{7.3}$$

$$C^n_k = \frac{1}{2\pi} \int_0^{2\pi} \binom{e^{it}}{n} e^{-ikt} \, dt \tag{7.4}$$

$$k! C^n_k = \frac{1}{2\pi i} \int_\gamma \frac{(\ln(z+1))^k}{z^{n+1}} \, dz \tag{7.5}$$

$$k! C^n_k = \frac{1}{2\pi i} \int_\gamma \frac{z^k e^z}{(e^z - 1)^{n+1}} \, dz \tag{7.6}$$

$$k! C^n_k = \frac{k}{2n\pi i} \int_\gamma \frac{z^{k-1}}{(e^z - 1)^n} \, dz \tag{7.7}$$

$$n! C^n_{n-k} = \binom{n-1}{k} \frac{k!}{2\pi i} \int_\gamma \frac{z^{n-k-1}}{(e^z - 1)^n} \, dz \tag{7.8}$$

7.2.1 Extension of Identity (7.7)

Remark 7.2 *In the following identity, assume α is a complex number with $\Re(\alpha) > 1$.*

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{z^{\alpha-1}}{(e^z-1)^n}\, dz = n \sum_{k=0}^{n} C_k^n \sum_{j=n}^{\infty} \frac{1}{j^{\alpha+1-k}} \tag{7.9}$$

8 Appendix B: Asymptotic Approximations for Stirling Numbers

Remark 8.1 *Many of the identities in this appendix are found in Charles Jordan's "On Stirling's numbers", Tohoku Math. Journal, Vol. 37, 1933, pp. 254-278.*

8.1 Approximations Involving $\mathbf{B^n_{k,k}}$

$$\lim_{n\to\infty} \frac{B_{k,k}^{n+1}}{B_{k,k}^{n}} = k \tag{8.1}$$

$$\lim_{n\to\infty} \frac{B_{n,n}^{n+k}}{(n+k)^{2k}} = \frac{1}{k!2^k} \tag{8.2}$$

8.2 Approximations Involving $\mathbf{C^n_k}$

$$|C_n^{n+k}| < \frac{e^{k+n}}{n!}, \qquad n \geq 1, \;\; k \geq 0 \tag{8.3}$$

$$\lim_{n\to\infty} \frac{n!|C_{n-k}^{n}|}{n^{2k}} = \frac{1}{k!2^k} \tag{8.4}$$

$$\lim_{n\to\infty} |\frac{C_{k+1}^{n}}{(\ln n + \gamma)C_k^n}| = \frac{1}{k}, \qquad \textit{where } \gamma \textit{ is Euler's constant} \tag{8.5}$$

Remark 8.2 *The following identity, due to H. W. Becker, is found in the American Math. Monthly, Vol. 50, 1943, Page 327.*

$$\lim_{r\to\infty} \frac{\Delta_{n,1}^{r}|(n+1)!C_k^{n+1}|}{|(n+r+1)!C_k^{n+r+1}|} = \frac{1}{e} \tag{8.6}$$

9 Appendix C: Number Theoretic Definitions of Stirling Numbers

9.1 Kramp-Ettinghausen Definitions of Stirling Numbers

$$S_1(n-1,k) = \sum \frac{n(n-1)(n-2)...(n-\gamma+1)}{\alpha_1!\alpha_2!\alpha_3!...2^{\alpha_1}3^{\alpha_2}4^{\alpha_3}}, \tag{9.1}$$

where the summation is over all possible integers $\alpha_\beta \geq 1$ such that $\alpha_1 + 2\alpha_2 + 3\alpha_3 + ... = k$, and $\gamma = \alpha_1 + \alpha_2 + \alpha_3 + ... + k$.

$$S_2(n,k) = \sum \frac{(n+k)(n+k-1)...(n+k-\gamma+1)}{\alpha_1!\alpha_2!\alpha_3!...(2!)^{\alpha_1}(3!)^{\alpha_2}(4!)^{\alpha_3}}, \tag{9.2}$$

where the summation is over all possible integers $\alpha_\beta \geq 1$ such that $\alpha_1 + 2\alpha_2 + 3\alpha_3 + ... = k$, and $\gamma = \alpha_1 + \alpha_2 + \alpha_3 + ... + k$.

9.2 Iterated Definitions of $S_1(n,k)$ and $S_2(n,k)$

$$S_1(n,k) = \sum_{j_k=1}^{n-k+1} j_k \sum_{j_{k-1}=j_k+1}^{n-k+2} j_{k-1} \sum_{j_{k-2}=j_{k-1}+1}^{n-k+3} j_{k-2} \cdots \sum_{j_2=j_3+1}^{n-1} j_2 \sum_{j_1=j_2+1}^{n} j_1 \tag{9.3}$$

$$S_1(n,k) = \prod_{i=1}^{k} \sum_{j_i=j_{i+1}+1}^{n-i+1} j_i, \qquad \textit{with } j_{k+1} \equiv 0 \tag{9.4}$$

$$S_2(n,k) = \sum_{j_k=1}^{n} j_k \sum_{j_{k-1}=j_k}^{n} j_{k-1} \sum_{j_{k-2}=j_{k-1}}^{n} j_{k-2} \cdots \sum_{j_2=j_3}^{n} j_2 \sum_{j_1=j_2}^{n} j_1 \tag{9.5}$$

$$S_2(n,k) = \prod_{i=1}^{k} \sum_{j_i=j_{i+1}}^{n} j_i, \qquad \textit{with } j_{k+1} \equiv 1 \tag{9.6}$$

9.2.1 Restatements of Equations (9.3) and (9.5)

$$S_1(n,k) = \sum_{1 \leq r_1 < r_2 < r_3 < ... < r_k \leq n} r_1 r_2 r_3 ... r_k, \quad \textit{where } r_i \textit{ is a positive integer} \tag{9.7}$$

$$S_2(n,k) = \sum_{1 \leq r_1 \leq r_2 \leq r_3 \leq ... \leq r_k \leq n} r_1 r_2 r_3 ... r_k, \quad \textit{where } r_i \textit{ is a positive integer} \tag{9.8}$$

9.2.2 Application of Equation (9.5)

$$S_1(n, n-k) = n! \prod_{i=1}^{k} \sum_{j_i=i}^{j_{i+1}-1} \frac{1}{j_i}, \qquad \textit{with } j_{k+1} \equiv n+1 \tag{9.9}$$

9.3 Properties of $S_1(n,k)$ and $S_2(n,k)$

$$\boldsymbol{S_1(n-1,k) = (-1)^k n! C^n_{n-k}} \tag{9.10}$$

$$\boldsymbol{S_2(n,k) = \frac{1}{n!} B^{n+k}_{n,n}} \tag{9.11}$$

$$S_1(n,k) = S_1(n-1,k) + nS_1(n-1,k-1), \qquad n \geq 1,\ k \geq 1 \tag{9.12}$$

$$S_2(n,k) = S_2(n-1,k) + nS_2(n,k-1), \qquad n \geq 1,\ k \geq 1 \tag{9.13}$$

$$S_1(n-1,k) = \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} S_2(j,k) \tag{9.14}$$

$$\boldsymbol{S_2(n,k) = S_1(-n-1,k)} \tag{9.15}$$

$$\boldsymbol{S_1(n,k) = S_2(-n-1,k)} \tag{9.16}$$

9.3.1 Generalization of $S_2(n,k)$

Remark 9.1 *In the following identity, we assume z is an arbitrary real or complex number.*

$$S_2(z,k) = \sum_{j=0}^{k} \binom{k+z}{k+j} \binom{k-z}{k-j} S_2(j,k) \tag{9.17}$$

9.3.2 Basis Expansions Involving $S_1(n,k)$ and $S_2(n,k)$

Remark 9.2 *In the following two formulas, we assume x is a real or complex number.*

$$\prod_{k=0}^{n-1} (1+kx) = \sum_{k=0}^{n} S_1(n-1,k) x^k, \qquad n \geq 1 \tag{9.18}$$

$$\frac{1}{\prod_{k=1}^{n}(1-kx)} = \sum_{k=0}^{\infty} S_2(n,k) x^k, \qquad n \geq 1,\ |x| < \frac{1}{n} \tag{9.19}$$

9.3.3 Hagen Recurrences

Remark 9.3 *The following identities are found in Hagen's Synopsis der hoeheren Mathematik, Berlin 1891, Volume I, Page 60.*

$$S_1(n-1,k) = \frac{1}{k}\sum_{j=0}^{k-1}\binom{n-j}{k+1-j}S_1(n-1,j), \qquad n \geq 1 \qquad k \geq 1 \tag{9.20}$$

$$nC_{n-k}^{n} = (-1)^k \sum_{j=0}^{k}(-1)^j \binom{n-j}{k+1-j}C_{n-j}^{n} \tag{9.21}$$

$$S_2(n,k) = \frac{1}{k}\sum_{j=0}^{k-1}\binom{-n-j}{k+1-j}S_2(n,j), \qquad k \geq 1 \tag{9.22}$$

$$-nB_{n,n}^{n+k} = \sum_{j=0}^{k}\binom{-n-j}{k+1-j}B_{n,n}^{n+j} \tag{9.23}$$