Table for Fundamentals of Series: Part I: Basic Properties of Series and Products

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1 Binomial Identities

Remark 1.1 *Throughout these tables, we assume, unless specified, that* n, j, k, α *and* r *represent non-negative integers. Furthermore, we reserve* x, *and* y *for arbitrary real(complex) numbers.*

1.1 Basic Identities

Pascal's Formula

$$\binom{x}{k} = \binom{x+1}{k} - \binom{x}{k-1} \tag{1.1}$$

Committee/Chair Identity

$$(n+1)\binom{r}{n+1} = r\binom{r-1}{n} \tag{1.2}$$

Cancelation Identity

$$\binom{n}{r+a}\binom{r+a}{r} = \binom{n}{r}\binom{n-r}{a} \tag{1.3}$$

$$\binom{n}{r}\binom{n+r}{r} = \binom{n+r}{n-r}\binom{2r}{r} = \binom{n+r}{2r}\binom{2r}{r} = \frac{(n+r)!}{(r!)^2(n-r)!} \tag{1.4}$$

$$\binom{n}{r}\binom{2n}{n} = \binom{n+r}{r}\binom{2n}{n-r} \tag{1.5}$$

-1 *Transformation*

$$\binom{-x}{r} = (-1)^r\binom{x+r-1}{r} \tag{1.6}$$

$$\binom{n+r}{r} = \binom{n+r}{n} = (-1)^n\binom{-r-1}{n} \tag{1.7}$$

$$\frac{\binom{-x+\alpha-1}{n+r}\binom{x}{\alpha-r-n}}{\binom{x}{\alpha}} = (-1)^{n+r}\binom{\alpha}{n+r} \tag{1.8}$$

$\frac{-1}{2}$ *Transformation*

$$\boldsymbol{\binom{\frac{-1}{2}}{n} = (-1)^n\binom{2n}{n}\frac{1}{2^{2n}}} \tag{1.9}$$

$\frac{1}{2}$ *Transformation*

$$\boldsymbol{\binom{\frac{1}{2}}{n} = (-1)^{n+1}\binom{2n}{n}\frac{1}{2^{2n}(2n-1)}} \tag{1.10}$$

$$\binom{\alpha}{n+1}\binom{\alpha-n-1}{\alpha-j}\binom{n+1}{j-k} = \binom{\alpha}{j}\binom{j}{k}\binom{k}{j-n-1} \tag{1.11}$$

1.2 Binomial Identities From the Gamma Function

Identities from $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$

$$n!(-1-n)! = \frac{\pi}{\sin(n+1)\pi} \tag{1.12}$$

$$(-n-\frac{1}{2})!(n-\frac{1}{2})! = (-1)^n\pi \tag{1.13}$$

Identities from Duplication Formula: $\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$

$$\boldsymbol{(n-\frac{1}{2})! = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}} \tag{1.14}$$

$$\frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{n!\sqrt{\pi}}{2(n+\frac{1}{2})!} \tag{1.15}$$

$$\left(\frac{n}{2}\right)!\left(\frac{n-1}{2}\right)! = \frac{n!\sqrt{\pi}}{2^n} \tag{1.16}$$

$$\binom{n}{\frac{1}{2}} = \frac{2^{2n+1}}{\pi\binom{2n}{n}} \tag{1.17}$$

$$\binom{k}{\frac{1}{2}}\binom{n-k}{\frac{1}{2}} = \frac{2^{2n+2}}{\pi^2\binom{2k}{k}\binom{2n-2k}{n-k}} \tag{1.18}$$

$$\binom{n}{\frac{n}{2}} = \frac{2^{2n}(\frac{n-1}{2})!}{n!\pi} = \frac{2^{2n}}{\pi\binom{n}{\frac{n-1}{2}}} \tag{1.19}$$

1.3 Limit Formulas

$$\lim_{n\to\infty} \binom{2n}{n}^{\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n)!}{(n!)^2}} = 4 \tag{1.20}$$

$$\lim_{n\to\infty} \frac{n+1}{\sqrt[n]{n!}} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e \tag{1.21}$$

$$\lim_{n\to\infty} \frac{(\frac{n}{e})^n \sqrt{2\pi n}}{n!} = 1 \tag{1.22}$$

2 Series: The Basic Properties

2.1 Indices Properties

Remark 2.1 *In this chapter, we assume a is a nonnegative integer. We also let $[x]$ denote the floor of x, i.e. the greatest integer less than or equal to x.*

$$\sum_{k=a}^{n} f(k) = \sum_{k=0}^{n} f(k) - \sum_{k=0}^{a-1} f(k) \tag{2.1}$$

$$\sum_{j=k_0+1}^{k_m} f(j) = \sum_{i=0}^{m-1} \sum_{j=k_i+1}^{k_{i+1}} f(j), \qquad 1 \le m \le \infty \tag{2.2}$$

$$2 \sum_{k=[\frac{a+1}{2}]}^{[\frac{n}{2}]} f(2k) = \sum_{k=a}^{n} f(k) + \sum_{k=a}^{n} (-1)^k f(k), \qquad n \ge a+1 \tag{2.3}$$

$$2 \sum_{k=[\frac{a+2}{2}]}^{[\frac{n+1}{2}]} f(2k-1) = \sum_{k=a}^{n} f(k) - \sum_{k=a}^{n} (-1)^k f(k), \qquad n \ge a+1 \tag{2.4}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} x^{2k} = \frac{1}{2} \frac{(1+x)^n - (1-x)^n}{x}, \qquad n \ge 1 \tag{2.5}$$

2.1.1 Bifurcation Formulas

Bifurcation Formula

$$\boldsymbol{\sum_{k=a}^{n} f(k) = \sum_{k=[\frac{a+1}{2}]}^{[\frac{n}{2}]} f(2k) + \sum_{k=[\frac{a+2}{2}]}^{[\frac{n+1}{2}]} f(2k-1)}, \qquad n \ge a+1 \tag{2.6}$$

Let $i = \sqrt{-1}$ in the following two equations.

$$\sum_{k=0}^{n} i^{k^2} f(k) = \sum_{k=0}^{[\frac{n}{2}]} f(2k) + i \sum_{k=0}^{[\frac{n-1}{2}]} f(2k+1), \qquad n \ge 1 \tag{2.7}$$

$$\sum_{k=0}^{n} (-1)^{[\frac{k}{2}]} i^{k^2} f(k) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k f(2k) + i \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k f(2k+1), \qquad n \ge 1 \tag{2.8}$$

Generalized Bifurcation Formulas

$$\sum_{k=0}^{rn-1} f(k) = \sum_{j=0}^{r-1}\sum_{k=0}^{n-1} f(rk+j), \qquad r \geq 1 \tag{2.9}$$

$$\sum_{k=a}^{n} f(k) = \sum_{j=0}^{r-1} \sum_{k=[\frac{a+r-1-j}{r}]}^{[\frac{n-j}{r}]} f(rk+j), \qquad n-a+1 \geq r \tag{2.10}$$

$$\sum_{k=a}^{n} f(k) = \sum_{k=[\frac{a+2}{3}]}^{[\frac{n}{3}]} f(3k) + \sum_{k=[\frac{a+1}{3}]}^{[\frac{n-1}{3}]} f(3k+1) + \sum_{k=[\frac{a}{3}]}^{[\frac{n-2}{3}]} f(3k+2), \qquad n \geq a+2 \tag{2.11}$$

$$\sum_{k=a}^{n} f(k) = \sum_{k=[\frac{a+3}{4}]}^{[\frac{n}{4}]} f(4k) + \sum_{k=[\frac{a+2}{4}]}^{[\frac{n-1}{4}]} f(4k+1) + \sum_{k=[\frac{a+1}{4}]}^{[\frac{n-2}{4}]} f(4k+2) + \sum_{k=[\frac{a}{4}]}^{[\frac{n-3}{4}]} f(4k+3), \tag{2.12}$$

where $n \geq a+3$.

Alternating Bifurcation Formula

$$\sum_{k=a}^{n} (-1)^k f(k) = \sum_{k=[\frac{a+1}{2}]}^{[\frac{n}{2}]} f(2k) - \sum_{k=[\frac{a+2}{2}]}^{[\frac{n+1}{2}]} f(2k-1), \qquad n \geq a+1 \tag{2.13}$$

Generalized Alternating Bifurcation Formula

$$\sum_{k=a}^{n} (-1)^k f(k) = \sum_{j=0}^{r-1} \sum_{k=[\frac{a+r-1-j}{r}]}^{[\frac{n-j}{r}]} (-1)^{rk+j} f(rk+j), \qquad n \geq a+r-1 \tag{2.14}$$

$$\sum_{k=a}^{n} (-1)^k f(k) = \sum_{k=[\frac{a+2}{3}]}^{[\frac{n}{3}]} (-1)^k f(3k) - \sum_{k=[\frac{a+1}{3}]}^{[\frac{n-1}{3}]} (-1)^k f(3k+1) + \sum_{k=[\frac{a}{3}]}^{[\frac{n-2}{3}]} (-1)^k f(3k+2), \tag{2.15}$$

where $n \geq a+2$.

$$\sum_{k=a}^{n}(-1)^k f(k) = \sum_{k=[\frac{a+3}{4}]}^{[\frac{n}{4}]} f(4k) - \sum_{k=[\frac{a+2}{4}]}^{[\frac{n-1}{4}]} f(4k+1) + \sum_{k=[\frac{a+1}{4}]}^{[\frac{n-2}{4}]} f(4k+2) - \sum_{k=[\frac{a}{4}]}^{[\frac{n-3}{4}]} f(4k+3), \tag{2.16}$$

where $n \geq a+3$.

2.1.2 Basic Telescoping Identities

$$\sum_{k=1}^{n}(f(k) - f(k+r)) = \sum_{k=1}^{r}(f(k) - f(k+n)) \tag{2.17}$$

Let $\Delta_{k,r} f(k) = \frac{f(k+r)-f(k)}{r}$. *Then,*

$$\frac{1}{n}\sum_{k=1}^{n}\Delta_{k,r} f(k) = \frac{1}{r}\sum_{k=1}^{r}\Delta_{k,n} f(k) \tag{2.18}$$

2.1.3 Greatest Integer Function Identities

$$\sum_{k=1}^{n} f(k) = \frac{1}{2}\sum_{k=1}^{2n} f\left(\left[\frac{k+1}{2}\right]\right) \tag{2.19}$$

$$\sum_{k=1}^{n} f(k) = \frac{1}{2}\sum_{k=1}^{2n-1} f\left(\left[\frac{k}{2}\right]+1\right) \tag{2.20}$$

$$\sum_{k=1}^{n}(-1)^{k-1} f(k) = \frac{1}{2}\sum_{k=1}^{2n-1}(-1)^{[\frac{k}{2}]} f\left(\left[\frac{k}{2}\right]+1\right) \tag{2.21}$$

2.2 Expansions of $(1+i)^n$ and $(1-i)^n$

Remark 2.2 *In Section 2.2, we let* $i = \sqrt{-1}$.

Variation of Bifurcation Formula

$$\sum_{k=a}^{n} i^k f(k) = \sum_{k=[\frac{a+1}{2}]}^{[\frac{n}{2}]} (-1)^k f(2k) + i \sum_{k=[\frac{a+2}{2}]}^{[\frac{n+1}{2}]} (-1)^{k-1} f(2k-1) \tag{2.22}$$

Variation of Alternating Bifurcation Formula

$$\sum_{k=a}^{n} (-i)^k f(k) = \sum_{k=[\frac{a+1}{2}]}^{[\frac{n}{2}]} (-1)^k f(2k) - i \sum_{k=[\frac{a+2}{2}]}^{[\frac{n+1}{2}]} (-1)^{k-1} f(2k-1) \tag{2.23}$$

2.2.1 Expansion of $(1+i)^n$

$$(1+i)^n = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} - i \sum_{k=1}^{[\frac{n+1}{2}]} (-1)^k \binom{n}{2k-1} \tag{2.24}$$

2.2.2 Expansions Involving $(1-i)^n$

$$(1-i)^n = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} + i \sum_{k=1}^{[\frac{n+1}{2}]} (-1)^k \binom{n}{2k-1} \tag{2.25}$$

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} = \frac{(1+i)^n + (1-i)^n}{2} \tag{2.26}$$

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} = (\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right) \tag{2.27}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{4n}{2k} = (-1)^n 2^{2n} \tag{2.28}$$

$$\boldsymbol{\sum_{k=0}^{n} (-1)^k \binom{2n}{2k} = (-1)^{[\frac{n}{2}]} \left(1 + (-1)^n\right) 2^{n-1}} \tag{2.29}$$

$$\sum_{k=1}^{[\frac{n+1}{2}]}(-1)^{k-1}\binom{n}{2k-1}=\frac{(1+i)^n-(1-i)^n}{2i} \tag{2.30}$$

$$\sum_{k=1}^{[\frac{n+1}{2}]}(-1)^{k-1}\binom{n}{2k-1}=\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^{k-1}\binom{n}{2k+1}=(\sqrt{2})^n\sin\left(\frac{n\pi}{4}\right) \tag{2.31}$$

$$\sum_{k=0}^{2n-1}(-1)^k\binom{4n}{2k+1}=0 \tag{2.32}$$

$$\boldsymbol{\sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1}=(-1)^{[\frac{n}{2}]}\left(1-(-1)^n\right)2^{n-1}} \tag{2.33}$$

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}=(-1)^{[\frac{n}{2}]}2^n} \tag{2.34}$$

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k}=(-1)^{[\frac{n+1}{2}]}2^n} \tag{2.35}$$

2.2.3 Expansions of $\left(\cos\left(\frac{\pi}{3}\right)\underline{+}i\sin\left(\frac{\pi}{3}\right)\right)^n$

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}3^k=2^n\cos\left(\frac{n\pi}{3}\right) \tag{2.36}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}3^k=\frac{2^n\sqrt{3}}{3}\sin\left(\frac{n\pi}{3}\right) \tag{2.37}$$

2.2.4 Expansions of $\left(\cos\left(\frac{\pi}{6}\right)\underline{+}i\sin\left(\frac{\pi}{6}\right)\right)^n$

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}\frac{1}{3^k}=\left(\frac{2\sqrt{3}}{3}\right)^n\cos\left(\frac{n\pi}{6}\right) \tag{2.38}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}\frac{1}{3^k}=\sqrt{3}\left(\frac{2\sqrt{3}}{3}\right)^n\sin\left(\frac{n\pi}{6}\right) \tag{2.39}$$

2.3 Index Shift Formula with Applications

2.3.1 Index Shift Formula

Index Shift Formula: Version 1

$$\sum_{k=a}^{n} f(k) = \sum_{k=0}^{n-a} f(n-k), \qquad a \geq 0 \tag{2.40}$$

Index Shift Formula: Version 2

$$\sum_{k=a}^{n} f(k) = \sum_{k=a}^{n} f(a+n-k), \qquad a \geq 0 \tag{2.41}$$

2.3.2 Applications of Index Shift Formula

$$\sum_{k=0}^{n} \binom{2n}{k} = 2^{2n-1} + \frac{1}{2}\binom{2n}{n} \tag{2.42}$$

$$\sum_{k=0}^{n} \binom{2n}{k} = 2^{2n-1} + \binom{2n-1}{n}, \qquad n \geq 1 \tag{2.43}$$

$$\sum_{k=1}^{n} \binom{2n}{n-k} = \sum_{k=1}^{n} \binom{2n}{n+k} = 2^{2n-1} - \frac{1}{2}\binom{2n}{n} \tag{2.44}$$

$$\sum_{k=1}^{n} \binom{2n}{n-k} = \sum_{k=1}^{n} \binom{2n}{n+k} = 2^{2n-1} - \binom{2n-1}{n}, \qquad n \geq 1 \tag{2.45}$$

$$\sum_{k=0}^{n} \binom{2n-1}{k} = 2^{2n-2} + \binom{2n-1}{n}, \qquad n \geq 1 \tag{2.46}$$

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{2n}{2k} = 2^{2n-2} + \frac{1+(-1)^n}{2}\binom{2n-1}{n}, \tag{2.47}$$

otherwise, if $n = 0$*, the sum equals 1.*

$$\sum_{k=0}^{n} \binom{4n}{2k} = 2^{4n-2} + \binom{4n-1}{2n} \tag{2.48}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]} \binom{2n}{2k+1} = 2^{2n-2} + \frac{1-(-1)^n}{2}\binom{2n-1}{n}, \qquad n \geq 1 \tag{2.49}$$

$$\sum_{k=0}^{n-1} \binom{4n}{2k+1} = 2^{4n-2} = 4^{2n-1}, \qquad n \geq 1 \tag{2.50}$$

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{2n}{n-2k} = \sum_{k=0}^{[\frac{n}{2}]} \binom{2n}{n+2k} = 2^{2n-2} + \frac{1}{2}\binom{2n}{n} \tag{2.51}$$

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{2n}{n-2k} = \sum_{k=0}^{[\frac{n}{2}]} \binom{2n}{n+2k} = 2^{2n-2} + \binom{2n-1}{n} \tag{2.52}$$

$$\sum_{k=0}^{n} \binom{4n}{2n-2k} = \sum_{k=0}^{n} \binom{4n}{2n+2k} = 2^{4n-2} + \frac{1}{2}\binom{4n}{2n} \tag{2.53}$$

$$\sum_{k=0}^{n} \binom{4n}{2k} = \sum_{k=0}^{n} \binom{4n}{4n-2k} = 2^{4n-2} + \frac{1}{2}\binom{4n}{2n} \tag{2.54}$$

Variation of Index Shift Formula

$$\boldsymbol{\sum_{k=1}^{n} f(k) = \sum_{k=1}^{[\frac{n}{2}]}(f(k) + f(n-k+1)) + \frac{1-(-1)^n}{2} f\left(\left[\frac{n+1}{2}\right]\right)}, \qquad n \geq 2 \tag{2.55}$$

$$\sum_{k=0}^{n} f(k) = \sum_{k=1}^{[\frac{n-1}{2}]} (f(k) + f(n-k)) + \frac{1+(-1)^n}{2} f\left(\left[\frac{n}{2}\right]\right), \qquad n \geq 1 \tag{2.56}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{k} = 2^{n-1} - \frac{1+(-1)^n}{4}\binom{n}{[\frac{n}{2}]}, \qquad n \geq 1 \tag{2.57}$$

$$\sum_{k=0}^{n} \binom{2n+1}{k} = 2^{2n}, \qquad n \geq 0 \tag{2.58}$$

Application of Index Shift Formula with -1 Transformation

$$\boldsymbol{\sum_{k=n}^{r+n} \binom{n-1-k}{k} f(k) = (-1)^n \sum_{k=0}^{r} (-1)^k \binom{n+2k}{k} f(k+n)}, \qquad r, n \geq 0 \tag{2.59}$$

2.3.3 Iterated Index Shift Formula

$$\sum_{k=0}^{n}\sum_{j=0}^{k} f(k,j) = \sum_{j=0}^{n}\sum_{k=0}^{j} f(n-k,n-j) \tag{2.60}$$

2.4 Series Properties of Periodic Functions

In the following identity, suppose $f(x) = f(\pi - x)$.

$$\sum_{k=0}^{n-1} f\left(\frac{2k+1}{4n}\pi\right) = \sum_{k=0}^{n-1} f\left(\frac{4k+1}{4n}\pi\right), \qquad n \geq 1 \tag{2.61}$$

In the following identity, suppose $f(x) = -f(\pi - x)$.

$$\sum_{k=0}^{n-1}(-1)^k f\left(\frac{2k+1}{4n}\pi\right) = \sum_{k=0}^{n-1} f\left(\frac{4k+1}{4n}\pi\right), \qquad n \geq 1 \tag{2.62}$$

In the following identity suppose $f(x) = -f(2\pi - x)$

$$\sum_{k=0}^{2n-1}(-1)^k f\left(\frac{2k+1}{4n}\pi\right) = \sum_{k=0}^{2n-1} f\left(\frac{4k+1}{4n}\pi\right), \qquad n \geq 1 \tag{2.63}$$

3 Calculus Operations on Series

3.1 Four Basic Integral Formulas

Remark 3.1 *In Section 3.1, we assume a and b are nonnegative integers. We assume p is a positive integer. Furthermore, we assume that $a_b \leq a_{b+1} \leq \ldots \leq a_n \leq a_{n+1} \leq a_{n+2} \leq a_{n+3}$. Lastly, recall the $[x]$ is the greatest integer in x.*

3.1.1 First Integral Formula

$$\sum_{r=b}^{n} \varphi(r) \int_{a_r}^{a_{r+1}} f(x)\,dx = \varphi(b) \int_{a_b}^{a_{n+1}} f(x)\,dx + \sum_{r=b+1}^{n} (\varphi(r) - \varphi(r-1)) \int_{a_r}^{a_{n+1}} f(x)\,dx \tag{3.1}$$

Applications of First Integral Formula

$$\sum_{r=a}^{n} (\varphi(r) - \varphi(r-1))(n+1-r) = \sum_{r=a}^{n} \varphi(r) - (n-a+1)\varphi(a-1) \tag{3.2}$$

$$\sum_{r=a}^{n} ((-1)^r - (-1)^{r-1})(n-r+1) = \sum_{r=a}^{n} (-1)^r - (-1)^{a-1}(n-a+1) \tag{3.3}$$

$$\sum_{r=0}^{n} (-1)^r (n_r + 1) = \frac{(-1)^n + 2n + 3}{4} \tag{3.4}$$

$$\sum_{r=0}^{n} \left(\binom{n}{r} - \binom{n}{r-1} \right) (n-r+1) = \sum_{r=0}^{n} \binom{n}{r} \tag{3.5}$$

$$\sum_{r=0}^{n} \binom{n}{r} r = n2^{n-1} \tag{3.6}$$

$$\sum_{r=1}^{n} f(r) = nf(n) - \sum_{r=1}^{n-1} r\,(f(r+1) - f(r)) \tag{3.7}$$

$$\sum_{k=1}^{n} \frac{f(k)}{k} = \frac{nf(n+1)}{n+1} - \sum_{k=1}^{n} k \left(\frac{f(k+1)}{k+1} - \frac{f(k)}{k} \right) \tag{3.8}$$

$$\sum_{r=1}^{n} r! = nn! - \sum_{r=1}^{n-1} r^2 r! \tag{3.9}$$

$$\prod_{k=1}^{n} \left(\frac{f(k+1)}{f(k)} \right)^{k} = \frac{(f(n+1))^n}{\prod_{k=1}^{n} f(k)} \tag{3.10}$$

$$\sum_{k=1}^{n} \left(k^p - (k-1)^p\right) f(k) = n^p f(n+1) - \sum_{k=1}^{n} k^p \left(f(k+1) - f(k)\right) \tag{3.11}$$

$$\sum_{k=1}^{n} \left(k^p - (k-1)^p\right) f(k) = \sum_{j=1}^{p} (-1)^j \binom{p}{j} \sum_{k=1}^{n} k^{p-j} f(k) \tag{3.12}$$

$$\prod_{k=1}^{n} \left(\frac{f(k+1)}{f(k)} \right)^{k^p} = (f(n+1))^{n^p} \prod_{k=1}^{n} (f(k))^{(k-1)^p - k^p} \tag{3.13}$$

3.1.2 Second Integral Formula

$$\boldsymbol{\sum_{r=b}^{n} \int_{a_r}^{a_{r+1}} f(x)\, dx = \int_{a_b}^{a_{n+1}} f(x)\, dx} \tag{3.14}$$

Applications of Second Integral Formula

$$\sum_{r=0}^{n} \frac{2r+3}{(r+1)^2(r+2)^2} = 1 - \frac{1}{(n+2)^2} \tag{3.15}$$

$$\sum_{r=0}^{n} \frac{2r+5}{(r+2)^2(r+3)^2} = \frac{1}{4} - \frac{1}{(n+3)^2} \tag{3.16}$$

$$\sum_{r=0}^{n} \frac{2r+2a+1}{(r+a)^2(r+a+1)^2} = \frac{1}{a^2} - \frac{1}{(n+a+1)^2}, \qquad a \geq 1 \tag{3.17}$$

$$\sum_{r=0}^{n} \frac{3r^2+9r+7}{(r+1)^3(r+2)^3} = 1 - \frac{1}{(n+1)^3} \tag{3.18}$$

$$\sum_{r=1}^{\infty} \frac{3r^2+3r+1}{r^3(r+1)^3} = 1 \tag{3.19}$$

$$\sum_{r=0}^{n} (r^2+2r)(r!)^2 = (n+1)!-1 \tag{3.20}$$

$$\sum_{r=0}^{n} \left((r+1)^p - 1\right)(r!)^p = ((n+1)!)^p - 1 \tag{3.21}$$

$$\sum_{r=0}^{n} r r! = (n+1)! - 1 \tag{3.22}$$

$$\sum_{r=0}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!} \tag{3.23}$$

$$\sum_{r=0}^{n} \frac{r^2+2r}{(r+1)^2(r!)^2} = 1 - \frac{1}{((n+1)!)^2} \tag{3.24}$$

$$\sum_{r=0}^{n} \left(1 - \frac{1}{(r+1)^p}\right) \frac{1}{(r!)^p} = 1 - \frac{1}{((n+1)!)^p} \tag{3.25}$$

$$\sum_{r=0}^{n} \left(1 - \frac{1}{\sqrt{r+1}}\right) \frac{1}{\sqrt{r!}} = 1 - \frac{1}{(\sqrt{(n+1)!}} \tag{3.26}$$

$$\sum_{r=1}^{\infty} \frac{r-\sqrt{r}}{r\sqrt{(r-1)!}} = 1 \tag{3.27}$$

3.1.3 Third Integral Formula

$$\boldsymbol{\sum_{r=b}^{n}(-1)^r \int_{a_r}^{a_{r+2}} f(x)\,dx = (-1)^b \int_{a_b}^{a_{b+1}} f(x)\,dx + (-1)^n \int_{a_{n+1}}^{a_{n+2}} f(x)\,dx} \tag{3.28}$$

Applications of Third Integral Formula

$$\boldsymbol{\sum_{r=b}^{n}(-1)^r \left(F(r+2)-F(r)\right) = (-1)^b \left(F(b+1)-F(b)\right) + (-1)^n \left(F(n+2)-F(n+1)\right)} \tag{3.29}$$

$$\sum_{k=0}^{n}(-1)^k(k^2+3k+1)k! = (-1)^n(n+1)(n+1)! \tag{3.30}$$

$$\sum_{k=0}^{n}(-1)^k \frac{1}{(k+1)(k+3)} = \frac{1}{4} + \frac{(-1)^n}{2(n+2)(n+3)} \tag{3.31}$$

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{(k+1)(k+3)} = \frac{1}{4} \tag{3.32}$$

$$\sum_{k=0}^{n}(-1)^k \frac{k^2+3k+1}{(k+2)!} = (-1)^n \frac{n+1}{(n+2)!} \tag{3.33}$$

$$\sum_{k=0}^{\infty}(-1)^k \frac{k^2+3k+1}{(k+2)!} = 0 \tag{3.34}$$

3.1.4 Fourth Integral Formula

$$\boldsymbol{\sum_{r=0}^{n}(-1)^r \int_{a_r}^{a_{r+3}} f(x)\,dx = \sum_{r=2}^{n}(-1)^r \int_{a_r}^{a_{r+1}} f(x)\,dx + \int_{a_0}^{a_1} f(x)\,dx + (-1)^n \int_{a_{n+2}}^{a_{n+3}} f(x)\,dx} \tag{3.35}$$

Applications of Fourth Integral Formula

$$\sum_{r=0}^{n}(-1)^{r}\left(F(r+3)-F(r)\right)=\sum_{r=2}^{n}(-1)^{r}\left(F(r+1)-F(r)\right)+F(1)-F(0)$$
$$+(-1)^{n}\left(F(n+3)-F(n+2)\right) \tag{3.36}$$

$$\sum_{r=0}^{n}(-1)^{r}=\frac{1+(-1)^{n}}{2} \tag{3.37}$$

$$\sum_{r=0}^{n}(-1)^{r}r=\frac{(2n+1)(-1)^{n}-1}{4}=(-1)^{n}\left[\frac{n+1}{2}\right] \tag{3.38}$$

$$\sum_{r=0}^{n}(-1)^{r}r^{2}=(-1)^{n}\frac{n^{2}+n}{2} \tag{3.39}$$

3.2 Three Integration by Parts Formulas

3.2.1 First Integration by Parts Formula

$$\sum_{k=1}^{n}f(k)(\varphi(k)-\varphi(k-1))=f(n)\varphi(n)-f(1)\varphi(0)-\sum_{k=1}^{n-1}\varphi(k)(f(k+1)-f(k)) \tag{3.40}$$

Applications of First Integration by Parts Formula

$$\sum_{k=1}^{n}\frac{f(k)}{k(k+1)}=f(1)-\frac{f(n)}{n+1}-\sum_{k=1}^{n-1}\frac{f(k)-f(k+1)}{k+1} \tag{3.41}$$

$$\sum_{k=1}^{\infty}\frac{2k+1}{k^{2}(k+1)^{3}}=\sum_{k=1}^{\infty}\frac{1}{k^{3}(k+1)} \tag{3.42}$$

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)^{2}}=1-\sum_{k=1}^{\infty}\frac{1}{k^{2}(k+1)}=2-\frac{\pi^{2}}{6} \tag{3.43}$$

3.2.2 Second Integration by Parts Formula

Assume a is a nonnegative integer. Let $S_n = \sum_{i=0}^{n} a_i$.

$$\sum_{k=a+1}^{n} a_k b_k = \sum_{k=a+1}^{n} S_k \left(b_k - b_{k+1}\right) + S_n b_{n+1} - S_a b_{a+1} \tag{3.44}$$

3.2.3 Third Integration by Parts Formula

$$\int \prod_{i=0}^{n-1} u_i \, du_n = \prod_{i=1}^{n} u_i - \sum_{k=1}^{n-1} \int \left(\prod_{i=1}^{n} u_i \right) \frac{1}{u_k} \, du_k \tag{3.45}$$

3.3 Taylor's Theorem

Remark 3.2 *Let $f(x) = \sum_{i=0}^{n} a_i x^i$, where the a_i are independent of x. Let $f^{(k)}(x)$ denote the k^{th} derivative of $f(x)$. Let $f^{(k)}(y)$ be the k^{th} derivative with respect to x evaluated at y.*

Taylor's Theorem

$$f(x) = \sum_{k=0}^{n} \frac{(x-y)^k}{k!} f^{(k)}(y) \tag{3.46}$$

Two Variations of Taylor's Theorem

$$f(x+y) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(y) \tag{3.47}$$

$$f(x) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(0) \tag{3.48}$$

3.4 Taylor's Theorem for Real Valued Functions of Several Variables

Let φ be a real valued function of n variables, say $(x_1, x_2, ..., x_n)$. Let $\frac{\partial^{j_i}}{\partial x_i}$ be the partial derivative (with respect to the variable x_i) of $\varphi(x_1, ..., x_n)$ taken j_i times. Then,

$$\varphi(x_1 + h_1, x_2 + h_2, ..., x_n + h_n) =$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{\forall j, 0 \le j_i \le k, \\ \sum_{i=1}^{n} j_i = k}} \frac{k!}{j_1! j_2! ... j_n!} \left(h_1^{j_1} \frac{\partial^{j_1}}{\partial x_1} h_2^{j_2} \frac{\partial^{j_2}}{\partial x_2} ... h_n^{j_n} \frac{\partial^{j_n}}{\partial x_n} \right) \varphi(x_1, ..., x_n) \tag{3.49}$$

3.5 Leibnitz Formula: Generalized Product Rule for Differentiation

Remark 3.3 *In Section 3.5, we assume u and v be $r-$times differentiable functions of x, where $r \geq 0$ and integral.*

Leibnitz Formula

$$\frac{d^r}{dx^r}(uv) = \sum_{k=0}^{r} \binom{r}{k} \frac{d^{r-k}\, u}{dx^{r-k}} \frac{d^k\, v}{dx^k} \tag{3.50}$$

3.5.1 Applications of Leibnitz Formula

$$\sum_{k=0}^{n} \binom{k}{r} x^k = \frac{x^r}{(1-x)^{r+1}} - x^{n+1} \sum_{k=0}^{\infty} \binom{n+1+k}{r} x^k, \qquad |x| < 1 \tag{3.51}$$

$$\sum_{k=1}^{\infty} \binom{2n+k}{n} \frac{1}{2^k} = 2^{2n} \tag{3.52}$$

$$\sum_{k=0}^{n} \binom{k}{r} x^k = \frac{x^r}{(1-x)^{r+1}} + \binom{n}{r} x^n - \frac{n!}{r!} x^n \sum_{k=0}^{r} \binom{r}{k} \frac{k!}{(k-r+n)!} \frac{x^k}{(1-x)^{k+1}} \tag{3.53}$$

$$\sum_{k=0}^{2n} \binom{k}{n} x^k = \frac{x^n}{(1-x)^{n+1}} + \binom{2n}{n} x^{2n} - x^{2n} \sum_{k=0}^{n} \binom{2n}{k+n} \frac{x^k}{(1-x)^{k+1}} \tag{3.54}$$

$$\sum_{k=0}^{n} \binom{k+n}{n} x^k = \frac{1}{(1-x)^{n+1}} + \binom{2n}{n} x^n - \sum_{k=0}^{n} \binom{2n}{k+n} \frac{x^{k+n}}{(1-x)^{k+1}} \tag{3.55}$$

3.6 Three Versions of the Generalized Chain Rule

Remark 3.4 *In Section 3.6, we will let D_z represent differentiation with respect to z. Hence, $D_z^n f(x)$ is the n^{th} derivative of $f(x)$ with respect to z, i.e. $D_z^n f(x) = D_z D_z ... D_z f(x)$, where the product contains n factors. We will let D_x represent differentiation with respect to x. We also assume that x is a function of z, i.e. $x = x(z)$. Finally, we let, α, unless otherwise specified, denote a nonnegative integer.*

3.6.1 Version 1: Hoppe Form of Generalized Chain Rule

$$\boldsymbol{D_z^n f(x) = \sum_{\alpha=0}^{n} D_x^\alpha f(x) \frac{(-1)^\alpha}{\alpha!} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} x^{\alpha-j} D_z^n x^j} \tag{3.56}$$

Applications of Version 1

$$D_z^n f(x)|_{z=a+bx} = \frac{1}{b^n} D_x^n f(x), \qquad b \neq 0 \tag{3.57}$$

Remark 3.5 *In the following identity,* α *is* <u>*any*</u> *real number. Also, we assume* $u = u(x)$*.*

$$D_x^n u^\alpha = \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} u^{\alpha-j} D_x^n u^j \tag{3.58}$$

Remark 3.6 *In the following identity, let* $x = e^z$*. Then,* $D_z^n = (xD_x)^n$*.*

Gunnert's Formula

$$\boldsymbol{D_z^n f(x) = (xD_x)^n f(x) = \sum_{\alpha=0}^{n} D_x^\alpha f(x) e^{\alpha z} \frac{(-1)^\alpha}{\alpha!} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} j^n} \tag{3.59}$$

Derivatives of Reciprocal Functions

$$D_z^n \left(\frac{1}{x}\right) = \sum_{\alpha=0}^{n} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \frac{1}{x^{j+1}} D_z^n x^j \tag{3.60}$$

$$D_z^n \left(\frac{1}{x}\right) = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \frac{1}{x^{j+1}} D_z^n x^j \tag{3.61}$$

$$\boldsymbol{D_x^n \left(\frac{1}{f(x)}\right) = \sum_{\alpha=0}^{n} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \frac{1}{(f(x))^{j+1}} D_x^n (f(x))^j} \tag{3.62}$$

$$\boldsymbol{D_x^n \left(\frac{1}{f(x)}\right) = \sum_{j=0}^{\alpha} (-1)^j \binom{n+1}{j+1} \frac{1}{(f(x))^{j+1}} D_x^n (f(x))^j} \tag{3.63}$$

Remark 3.7 *In the following identity, assume* $f(x) = \sum_{i=0}^{\infty} a_i x^i$*, with* $f(0) = a_0 \neq 0$*. Note that all* a_i *are independent of* x*. Furthermore, assume for a nonnegative integer* β*,* $(f(x))^\beta = \sum_{j=0}^{\infty} b_j^{(\beta)} x^j$*. Once again, all* $b_j^{(\beta)}$ *are independent of* x*.*

$$\frac{1}{f(x)} = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j+1}{k+1} \frac{b_j^{(k)}}{a_0^{k+1}} \tag{3.64}$$

Remark 3.8 *In the following two identities, both u and x are functions of z.*

$$D_z^n \left(\frac{u}{x}\right) = \sum_{k=0}^{n} \binom{n}{k} D_z^{n-k} u \sum_{\alpha=0}^{k} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \frac{1}{x^{j+1}} D_z^k x^j \tag{3.65}$$

$$D_z^n \left(\frac{u}{x}\right) = \sum_{k=0}^{n} \binom{n}{k} D_z^{n-k} u \sum_{j=0}^{k} \binom{k+1}{j+1} \frac{(-1)^j}{x^{j+1}} D_z^k x^j \tag{3.66}$$

Remark 3.9 *In the following two identities, assume f is any $n-$times differentiable function. Also assume a is independent of x*

$$\frac{(a-x)^{n+1}}{n!} D_x^n \left(\frac{f(x)}{a-x}\right) = \sum_{k=0}^{n} \frac{(a-x)^k}{k!} D_x^k f(x) \tag{3.67}$$

$$\frac{(a-x)^{n+1}}{n!} D_x^n \left(\frac{f(a)-f(x)}{a-x}\right) = f(a) - \sum_{k=0}^{n} \frac{(a-x)^k}{k!} D_x^k f(x) \tag{3.68}$$

Remark 3.10 *In the following identity, due to G. H. Halphen, we assume f and ϕ are $n-$times differentiable functions. We also let $\phi^{(k)}\left(\frac{1}{x}\right)$ denote the k^{th} derivative of $\phi\left(\frac{1}{x}\right)$ with respect to $\frac{1}{x}$.*

$$D_x^n \left(f(x)\phi\left(\frac{1}{x}\right)\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{x^k} \phi^{(k)}\left(\frac{1}{x}\right) D_x^{n-k} \left(\frac{f(x)}{x^k}\right) \tag{3.69}$$

Remark 3.11 *The following identity is a generalization of the Version 1 due to R. Most. He assumes that both f and ϕ are $(n+m-)$times differentiable functions, where m is an arbitrary nonnegative integer.*

$$D_z^n f(x) = \sum_{k=0}^{n+m} \frac{(-1)^k}{k!} D_x^k \left(f(x)\phi(x)\right) \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{k-j} D_z^n \left(\frac{x^j}{\phi(x)}\right) \tag{3.70}$$

Remark 3.12 *In the following identity, due to R. Most, α and β are arbitrary real numbers.*

$$D_z^n x^\alpha = (\alpha+\beta)\binom{n+m-\alpha-\beta}{n+m} \sum_{j=0}^{n+m} (-1)^j \binom{n+m}{j} \frac{x^{\alpha+\beta-j}}{\alpha+\beta-j} D_z^n x^{j-\beta} \tag{3.71}$$

3.6.2 Version 2: Operator Form of Generalized Chain Rule

$$\boldsymbol{D_z^n f(x) = \sum_{j=0} A_j^n(z) D_x^j f(x),} \tag{3.72}$$

where $A_j^n(z)$ are independent of f and calculated by

$$\boldsymbol{A_j^n(z) = \frac{1}{j!} D_t^j \left(e^{-tx} D_z^n e^{tx}\right) |_{t=0}} \tag{3.73}$$

3.6.3 Version 3: Faa di Bruno's Formula for the Generalized Chain Rule

$$\boldsymbol{D_z^n f(x) = \sum_{k=1}^{n} D_x^k f(x) \frac{1}{k!} \sum \frac{n!}{j_1! j_2! \ldots j_\alpha!} \left(\frac{1}{k_1!} D_z^{k_1} x\right)^{j_1} \ldots \left(\frac{1}{k_\alpha!} D_z^{k_\alpha} x\right)^{j_\alpha}}, \tag{3.74}$$

where the inner sum is extended over all partitions such that $\sum_i^\alpha j_i = k$ and $\sum_{i=1}^\alpha j_i k_i = n$.

4 Iterative Series

Remark 4.1 *In this chapter, recall that $[x]$ is the greatest integer in x.*

4.1 First Example of an Iterative Series

$$\boldsymbol{\sum_{r=0}^{n}\sum_{k=0}^{[\frac{r}{2}]} A_{r,k} = \sum_{k=0}^{[\frac{n}{2}]}\sum_{2k}^{n} A_{r,k}} \tag{4.1}$$

4.1.1 Applications of the First Iterative Series

$$\sum_{r=0}^{n}\left[\frac{r}{2}\right] = n\left[\frac{n}{2}\right] - \left[\frac{n}{2}\right]^2 \tag{4.2}$$

$$\sum_{r=0}^{n}\left[\frac{r}{2}\right] f(r) = \left[\frac{n}{2}\right]\sum_{r=0}^{n} f(r) - \sum_{k=1}^{[\frac{n}{2}]}\sum_{r=0}^{2k-1} f(r) \tag{4.3}$$

$$\sum_{r=0}^{n}(-1)^r\left[\frac{n}{2}\right] = \frac{1+(-1)^n}{2}\left[\frac{n}{2}\right] \tag{4.4}$$

$$\sum_{r=0}^{n}\binom{n}{r}\left[\frac{r}{2}\right] = \left[\frac{n}{2}\right]2^n - \sum_{k=1}^{[\frac{n}{2}]}\sum_{r=0}^{2k-1}\binom{n}{r} \tag{4.5}$$

$$\sum_{r=0}^{n}\left[\frac{r}{2}\right]^2 = \left[\frac{n}{2}\right]\frac{1+3n[\frac{n}{2}]-4[\frac{n}{2}]^2}{3} \tag{4.6}$$

Remark 4.2 *In the following identity, assume $\{a_n\}_{n=0}^{\infty}$ is a sequence which obeys the Fibonacci recurrence, i.e. $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$*

$$\sum_{r=0}^{n}\sum_{k=0}^{[\frac{r}{2}}\binom{r-k}{k} = \sum_{r=0}^{n} a_r = a_{n+2} - 1 \tag{4.7}$$

$$\sum_{r=0}^{n} 2^r \cos\left(\frac{r\pi}{3}\right) = \frac{2^{n+1}\sqrt{3}}{3} \sin\left(\frac{(n+1)\pi}{3}\right) \tag{4.8}$$

Generalization of First Iterative Series

$$\boldsymbol{\sum_{k=a}^{n} \sum_{i=a}^{[\frac{k}{r}]} A_{i,k} = \sum_{i=a}^{[\frac{n}{r}]} \sum_{ri}^{n} A_{i,k},} \tag{4.9}$$

where r and a are integers such that $r \geq 1$ and $a \geq 0$.

4.2 Second Example of an Iterative Series

$$\boldsymbol{\sum_{k=1}^{n} \sum_{i=1}^{[\sqrt{k}]} A_{i,k} = \sum_{i=1}^{[\sqrt{n}]} \sum_{k=i^2}^{n} A_{i,k}} \tag{4.10}$$

4.2.1 Applications of the Second Iterative Series

$$\sum_{k=1}^{n} [\sqrt{k}] f(k) = \sum_{i=1}^{[\sqrt{n}]} \sum_{k=i^2}^{n} f(k) \tag{4.11}$$

$$\sum_{k=1}^{n} [\sqrt{k}] = [\sqrt{n}] \left(n + 1 - \frac{2[\sqrt{n}]^2 + 3[\sqrt{n}] + 1}{6}\right) \tag{4.12}$$

$$\sum_{k=1}^{n} \frac{[\sqrt{k}]}{2^k} = 2\sum_{i=1}^{[\sqrt{n}]} \frac{1}{2^{i^2}} - \frac{[\sqrt{n}]}{2^n} \tag{4.13}$$

$$\sum_{k=1}^{\infty} \frac{[\sqrt{k}]}{2^k} = \sum_{i=1}^{\infty} \frac{1}{2^{i^2-1}} \tag{4.14}$$

4.3 Third Example of an Iterative Series

$$\sum_{i=1}^{n} \sum_{k=1}^{2^i-1} A_{i,k} = \sum_{k=1}^{2^n-1} \sum_{i=1+[\log_2 k]}^{n} A_{i,k} \tag{4.15}$$

4.3.1 Applications of the Third Iterative Series

$$\sum_{k=1}^{2^n-1} [\log_2 k] f(k) = n \sum_{k=1}^{2^n-1} f(k) - \sum_{i=1}^{n} \sum_{k=1}^{2^i-1} f(k) \tag{4.16}$$

$$\sum_{k=1}^{2^n-1} [\log_2 k] = (n-2)2^n + 2 \tag{4.17}$$

$$\sum_{k=1}^{2^n-1} [\log_2 (2k)] = (n-1)2^n + 1 \tag{4.18}$$

$$\sum_{k=1}^{2^{n-1}} [\log_2 (2k-1)] = (n-2)2^{n-1} + 1 \tag{4.19}$$

$$\sum_{k=1}^{2^n-1} (-1)^k [\log_2 k] = 0 \tag{4.20}$$

$$\sum_{k=1}^{2^n} (-1)^k [\log_2 k] = n \tag{4.21}$$

$$\sum_{i=1}^{n} \sum_{k=1}^{2^i-1} f(i) = \sum_{k=1}^{2^n-1} \sum_{i=1+[\log_2 k]}^{n} f(i) \tag{4.22}$$

4.4 Standard Interchange Formula for Iterative Series

Remark 4.3 *In Section 4.4, we assume a is a nonnegative integer with $a \leq n$.*

$$\sum_{k=a}^{n} \sum_{i=a}^{k} A_{i,k} = \sum_{i=a}^{n} \sum_{k=i}^{n} A_{i,k} \tag{4.23}$$

4.4.1 Variations of Standard Interchange Formula

$$\boldsymbol{\sum_{k=a}^{n}\sum_{i=a}^{k} A_{i,k} = \sum_{k=0}^{n-a}\sum_{i=a}^{n-k} A_{i,n-k}} \tag{4.24}$$

$$\sum_{k=a}^{n}\sum_{i=a}^{k} A_{i,k} = \sum_{i=a}^{n}\sum_{k=0}^{n-i} A_{i,k+i} \tag{4.25}$$

$$\sum_{k=a}^{n}\sum_{i=a}^{k} A_{i,k} = \sum_{i=0}^{n-a}\sum_{k=n-i}^{n} A_{n-i,k} \tag{4.26}$$

$$\sum_{k=a}^{n}\sum_{i=a}^{k} A_{i,k} = \sum_{i=0}^{n-a}\sum_{k=0}^{i} A_{n-i,k+n-i} \tag{4.27}$$

$$\sum_{k=a}^{n}\sum_{i=a}^{k} A_{i,k} = \sum_{k=0}^{n-a}\sum_{i=k}^{n-a} A_{n-i,k+n-i} \tag{4.28}$$

$$\sum_{k=a}^{n}\sum_{i=a}^{k} A_{i,k} = \sum_{k=a}^{n}\sum_{i=k-a}^{n-a} A_{n-i,k-a+n-i} \tag{4.29}$$

$$\sum_{k=a}^{n}\sum_{i=a}^{k} A_{i,k} = \sum_{k=a}^{n}\sum_{i=k}^{n} A_{n-i+a,k+n-i} \tag{4.30}$$

4.4.2 Applications of Standard Interchange Formula

$$\sum_{k=0}^{n}\sum_{i=0}^{2n-k} A_{i,k} = \sum_{i=0}^{2n}\sum_{k=0}^{i} A_{i-k,k} - \sum_{i=n+1}^{2n}\sum_{k=n+1}^{i} A_{i-k,k} \tag{4.31}$$

$$\boldsymbol{\sum_{k=a}^{n}\sum_{j=a}^{k} f(j) = \sum_{j=a}^{n}\sum_{k=j}^{n} f(j)} \tag{4.32}$$

$$\sum_{j=a}^{n} jf(j) = (n+1)\sum_{j=a}^{n} f(j) - \sum_{k=a}^{n}\sum_{j=a}^{k} f(j) \tag{4.33}$$

$$\sum_{k=1}^{n} k\left[\frac{x+2^{k-1}}{2^k}\right] = \sum_{k=1}^{n}\left[\frac{x}{2^k}\right] - (n+1)\left[\frac{x}{2^n}\right] + [x] \tag{4.34}$$

$$\sum_{k=0}^{n} k\binom{k}{m} = \frac{mn+n+m}{m+2}\binom{n+1}{m+1} \tag{4.35}$$

where m is a nonnegative integer

4.5 Fourth Example of an Iterative Series

Remark 4.4 *In Section 4.5, we assume a and r are integers such that $0 \geq a \geq n$ and $r \geq 1$.*

$$\boldsymbol{\sum_{k=a}^{n}\sum_{i=a}^{rk} A_{i,k} = \sum_{i=a}^{rn}\sum_{k=[\frac{i+r-1}{r}]}^{n} A_{i,k}} \tag{4.36}$$

4.5.1 Applications of the Fourth Iterative Series

$$\sum_{k=1}^{n}\sum_{i=1}^{2k} A_{i,k} = \sum_{i=1}^{2n}\sum_{k=[\frac{i+1}{2}]}^{n} A_{i,k} \tag{4.37}$$

$$\sum_{k=1}^{n}\sum_{i=1}^{3k} A_{i,k} = \sum_{i=1}^{3n}\sum_{k=[\frac{i+2}{3}]}^{n} A_{i,k} \tag{4.38}$$

$$\sum_{i=1}^{rn}\left[\frac{i-1}{r}\right] = \frac{rn(n-1)}{2} \tag{4.39}$$

$$\sum_{k=0}^{n}\sum_{i=k}^{2k} A_{i,k} = \sum_{i=0}^{2n}\sum_{k=[\frac{i+1}{2}]}^{i-[\frac{i}{n+1}](i-n)} A_{i,k} \tag{4.40}$$

$$\sum_{k=0}^{\infty}\sum_{i=k}^{2k} A_{i,k} = \sum_{i=0}^{\infty}\sum_{k=[\frac{i+1}{2}]}^{i} A_{i,k} \tag{4.41}$$

$$\sum_{k=0}^{\infty}\sum_{i=k}^{3k} A_{i,k} = \sum_{i=0}^{\infty}\sum_{k=[\frac{i+2}{3}]}^{i} A_{i,k} \tag{4.42}$$

$$\sum_{k=1}^{\infty}\sum_{i=k}^{2k} A_{i,k} = \sum_{i=1}^{\infty}\sum_{k=0}^{[\frac{i}{2}]} A_{i,i-k} \tag{4.43}$$

4.6 Fifth Example of an Iterative Series

Remark 4.5 *In Section 4.6, we assume p is a positive integer.*

$$\boldsymbol{\sum_{k=1}^{n}\sum_{i=1}^{k^p} A_{i,k} = \sum_{i=1}^{n^p}\sum_{k=1+[\sqrt[p]{i-1}]}^{n} A_{i,k}} \tag{4.44}$$

4.6.1 Applications of the Fifth Iterative Series

$$\sum_{k=1}^{n}\sum_{i=1}^{k^2} A_{i,k} = \sum_{i=1}^{n^2}\sum_{k=1+[\sqrt{i-1}]}^{n} A_{i,k} \tag{4.45}$$

$$\sum_{k=1}^{n}\sum_{i=1}^{k^3} A_{i,k} = \sum_{i=1}^{n^3}\sum_{k=1+[\sqrt[3]{i-1}]}^{n} A_{i,k} \tag{4.46}$$

$$\sum_{k=2}^{\infty}\sum_{i=2k}^{k^2} A_{i,k} = \sum_{i=2}^{\infty}\sum_{k=1+[\sqrt{i-1}]}^{[\frac{i}{2}]} A_{i,k} \tag{4.47}$$

4.7 Two Special Iterative Series

Remark 4.6 *In the following identity, we define* $!^{-1}$ *as the inverse function of* $!$. *That is,* $x! = n$ *if and only if* $!^{-1}n = x$. *Furthermore, we assum* $!^{-1}1 = 1$.

$$\sum_{k=2}^{n}\sum_{i=2}^{k!} A_{i,k} = \sum_{i=2}^{n!}\sum_{k=1+[!^{-1}(i-1)]}^{n} A_{i,k} \tag{4.48}$$

Remark 4.7 *In the following identity, we assume* $g(x)$ *is a function such that* $x^x = z$ *if and only if* $x = g(z)$.

$$\sum_{k=2}^{n}\sum_{i=2}^{k^2} A_{i,k} = \sum_{i=2}^{n^n}\sum_{k=1+[g(i-1)]}^{n} A_{i,k} \tag{4.49}$$

4.8 Iterations of the Hockey Stick Identity

Let

$$\sum_{(r)_{k_1}} f(k_1) \equiv \sum_{k_r=0}^{n}\sum_{k_{r-1}=0}^{k_r}\sum_{k_{r-2}=0}^{k_{r-1}} \dots \sum_{k_1=0}^{k_2} f(k_1)$$

be the $r-$*fold iterated sum of* $f(k_1)$.

Iterated Hockey Stick Identity

$$\sum_{(r)_j}^{n}\binom{j}{k} = \binom{n+r}{k+r}, \qquad r \geq 1 \qquad n,k \geq 0 \tag{4.50}$$

4.9 Iterated Sums with Deleted Terms

$$\sum_{j=0}^{n}\sum_{i=0}^{n} A_{i,j} - \sum_{i=0}^{n} A_{i,i} = \sum_{j=0}^{n}\sum_{\substack{i=0\\ i\neq j}}^{n} A_{i,j} = \sum_{i=0}^{n}\sum_{\substack{j=0\\ j\neq i}}^{n} A_{i,j} \tag{4.51}$$

4.9.1 Applications of Deleted Terms Identity

Remark 4.8 *In the following identities, let $A_{i,j} = u_i v_j$.*

$$\sum_{j=0}^{n} v_j \sum_{\substack{i=0 \\ i\neq j}}^{n} u^i = \sum_{j=0}^{n}\sum_{i=0}^{n} u_i v_j - \sum_{i=0}^{n} u_i v_j \tag{4.52}$$

$$\sum_{j=0}^{n}\sum_{\substack{i=0 \\ i\neq j}}^{n} u^i = n\sum_{i=0}^{n} u_i \tag{4.53}$$

$$\sum_{k_r=0}^{n}\ \sum_{\substack{k_{r-1}=0 \\ k_r\neq k_r}}^{n} \dots \sum_{\substack{k_1=0 \\ k_1\neq k_2}}^{n} u_{k_1} = n^{r-1}\sum_{j=0}^{n} u_j, \tag{4.54}$$

where the left hand side is an $r-$fold iterated sum for fixed positive integers r and n.

$$\prod_{j=0}^{n}\prod_{\substack{i=0 \\ i\neq j}}^{n} u_i = \left(\prod_{i=0}^{n} u_i\right)^{n} \tag{4.55}$$

$$\prod_{k_r=0}^{n}\ \prod_{\substack{k_{r-1}=0 \\ k_r\neq k_r}}^{n} \dots \prod_{\substack{k_1=0 \\ k_1\neq k_2}}^{n} u_{k_1} = \left(\prod_{j=0}^{n} u_j\right)^{n^{r-1}} \tag{4.56}$$

where the left hand side is an $r-$ iterated product, for r and n fixed positive integers.

$$\sum_{j=0}^{n}\sum_{\substack{i=0 \\ i\neq j}}^{n}(j+1)u_i = \frac{n^2+3n}{2}\sum_{i=0}^{n} u_i - \sum_{i=0}^{n} iu_i \tag{4.57}$$

$$\sum_{j=0}^{n}\ \sum_{\substack{i=0 \\ i\neq j,n-j}} u_i = (n-1)\sum_{i=0}^{n} u_i + \frac{(-1)^n+1}{2}u_{[\frac{n}{2}]} \tag{4.58}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n} A_{i,j} - \sum_{i=0}^{n} A_{i,n-i} = \sum_{j=0}\ \sum_{\substack{i=0 \\ i\neq n-j}}^{n} A_{i,j} \tag{4.59}$$

5 Three Convolution Formulas for Finite Series

5.1 First Convolution Formula

$$\sum_{j=1}^{n}\sum_{i=1}^{n} f(i)\varphi(j) = \left(\sum_{i=1}^{n} f(i)\right)\left(\sum_{j=1}^{n}\varphi(j)\right)$$
$$= \sum_{k=1}^{n} f(k)\varphi(k) + \sum_{j=1}^{n-1}\sum_{i=1}^{n-j}\left(f(i)\varphi(i+j) + f(i+j)\varphi(i)\right),\ n \geq 2 \tag{5.1}$$

5.1.1 Applications of First Convolution Formula

$$\left(\sum_{i=0}^{n} A_i\right)\left(\sum_{j=0}^{n} B_j\right) = \sum_{k=0}^{n} A_k B_k + \sum_{r=1}^{n}\sum_{k=0}^{n-r}\left(A_k B_{k+r} + A_{k+r} B_k\right),\ n \geq 1 \tag{5.2}$$

$$\left(\sum_{i=1}^{n} f(i)\right)^2 = \sum_{k=1}^{n}\left(f(k)\right)^2 + 2\sum_{r=1}^{n-1}\sum_{k=1}^{n-r} f(k)f(k+r),\ n \geq 2 \tag{5.3}$$

$$\left(\sum_{k=1}^{n}\frac{1}{k!}\right)^2 = \sum_{k=1}^{n}\frac{1}{(k!)^2} + 2\sum_{r=1}^{n-1}\sum_{k=1}^{n-r}\frac{1}{k!(k+r)!},\ n \geq 2 \tag{5.4}$$

$$\sum_{r=1}^{\infty}\sum_{k=r+1}^{\infty}\frac{1}{k^2(k-r)^2} = \sum_{k=2}^{\infty}\sum_{r=1}^{k-1}\frac{1}{k^2(k-r)^2} = \frac{\pi^4}{120} \tag{5.5}$$

$$\left(\sum_{k=1}^{n} f(k)\right)\left(\sum_{k=1}^{n}\frac{1}{f(k)}\right) = n + \sum_{r=1}^{n-1}\sum_{k=r+1}^{n}\frac{(f(k-r))^2 + (f(k))^2}{f(k-r)f(k)},\ n \geq 2 \tag{5.6}$$

$$\sum_{j=1}^{n}\frac{1}{j} = \frac{1}{n-1}\sum_{r=1}^{n-1}\sum_{k=1}^{n-r}\left(\frac{1}{k+r} + \frac{1}{k}\right),\ n \geq 2 \tag{5.7}$$

5.2 Cauchy Convolution Formula

Remark 5.1 *In this Section 5.2, we let $[x]$ denote the greatest integer in x.*

$$\left(\sum_{k=0}^{n} f(k)\right)^2 = \sum_{k=0}^{2n}\ \sum_{i=[\frac{k}{n+1}](k-n)}^{k-[\frac{k+1}{n+1}](k-n)} f(i)f(k-i) \tag{5.8}$$

5.2.1 Applications of Cauchy Convolution Formula

Vandermonde Convolution

$$\boldsymbol{\sum_{i=0}^{k}\binom{n}{i}\binom{n}{k-i}=\binom{2n}{k}} \tag{5.9}$$

$$\sum_{i=0}^{k}\binom{r}{i}\binom{q}{k-i}=\binom{r+q}{k}, \tag{5.10}$$

where k, r, and q are nonnegative integers.

$$\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n} \tag{5.11}$$

$$\left(\sum_{i=0}^{n}\binom{n}{i}x^i f(i)\right)\left(\sum_{j=0}^{n}\binom{n}{j}x^j\varphi(j)\right)=\sum_{k=0}^{2n}x^k\sum_{i=0}^{k}\binom{n}{i}\binom{n}{k-i}f(i)\varphi(k-i) \tag{5.12}$$

$$\boldsymbol{\left(\sum_{k=0}^{\infty}a_k x^k\right)^2=\sum_{k=0}^{\infty}\sum_{i=0}^{k}a_i a_{k-i}x^k} \tag{5.13}$$

$$e^{2x}=\sum_{k=0}^{\infty}\frac{(2x)^k}{k!}=\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{i=0}^{k}\binom{k}{i} \tag{5.14}$$

$$(\cosh x)^2=\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}\sum_{j=0}^{k}\binom{2k}{2j}=\frac{1}{2}\cosh(2x)+\frac{1}{2} \tag{5.15}$$

$$\left(\sum_{k=0}^{\infty}\frac{x^k}{(k!)^2}\right)^2=\sum_{k=0}^{\infty}\frac{(2k)!}{(k!)^4}x^k \tag{5.16}$$

Companion Binomial Theorem: Let n be a positive integer

$$\boldsymbol{\frac{1}{(1-x)^n}=\sum_{k=0}^{infty}\binom{k+n-1}{k}x^k=\left(\sum_{i=0}^{\infty}x^i\right)^n,\qquad |x|<1} \tag{5.17}$$

5.2.2 r^{th} Power of an Infinite Series

Let r be a positive integer. Assume $f(x) = \sum_{i=0}^{\infty} a_{1,i}x^i$. Then,

$$\boldsymbol{(f(x))^r = \sum_{k=0}^{\infty} a_{r,k}x^k,} \tag{5.18}$$

where

$$a_{r,k} = \sum_{i=0}^{k} a_{r-1,i}a_{1,k-i}$$

5.3 Third Formula Convolution Formula

Remark 5.2 *In Section 5.3, we let $[x]$ denote the greatest integer in x.*

$$\boldsymbol{\sum_{i=0}^{n} a_i a_{n-i} = 2\sum_{i=0}^{[\frac{n-1}{2}]} a_i a_{n-i} + \frac{1+(-1)^n}{2} a_{[\frac{n}{2}]}^2, \qquad n \geq 1} \tag{5.19}$$

Variation of Third Convolution Formula

$$\boldsymbol{\sum_{i=1}^{n} a_i a_{n-i+1} = 2\sum_{i=1}^{[\frac{n}{2}]} a_i a_{n-i+1} + \frac{1-(-1)^n}{2} a_{[\frac{n+1}{2}]}^2, \qquad n \geq 2} \tag{5.20}$$

5.3.1 Applications of Third Convolution Formula

$$\sum_{i=1}^{2n} a_i a_{2n-i+1} = 2\sum_{i=1}^{n} a_i a_{2n-i+1}, \qquad n \geq 1 \tag{5.21}$$

$$\sum_{i=1}^{2n+1} a_i a_{2n-i+2} = 2\sum_{i=1}^{n} a_i a_{2n-i+2} + a_{n+1}^2, \qquad n \geq 1 \tag{5.22}$$

$$\sum_{k=1}^{n-1} a_k a_{2n-1-k} = \frac{a_{2n-1}}{2} \tag{5.23}$$

$$\sum_{i=0}^{[\frac{n-1}{2}]} \binom{n}{i}^2 = \frac{1}{2}\binom{2n}{n} - \frac{1+(-1)^n}{4}\binom{n}{[\frac{n}{2}]}^2, \qquad n \geq 1 \tag{5.24}$$

$$\sum_{i=0}^{n-1} \binom{2n}{i}^2 = \frac{1}{2}\binom{4n}{2n} - \frac{1}{2}\binom{2n}{n}^2, \qquad n \geq 1 \tag{5.25}$$

$$\sum_{i=0}^{n} \binom{2n+1}{i}^2 = \frac{1}{2}\binom{4n+2}{2n+1}, \qquad n \geq 1 \tag{5.26}$$

6 Finite Products: Elementary Properties

Remark 6.1 *In this chapter, we assume, unless otherwise specified, that a and p are nonnegative integers.*

6.1 Basic Properties

6.1.1 Communativity Property

$$\prod_{k=a}^{n} \boldsymbol{f(k)\varphi(k)} = \prod_{k=a}^{n} \boldsymbol{f(k)} \prod_{k=a}^{n} \boldsymbol{\varphi(k)} \tag{6.1}$$

Applications of Communativity Property

$$\prod_{k=a}^{n} (f(k))^p = \left(\prod_{k=a}^{n} f(k)\right)^p, \qquad p \geq 1 \tag{6.2}$$

$$\prod_{k=0}^{n} f(2k)f(2k+1) = \prod_{k=0}^{2n+1} f(k), \qquad n \geq 0 \tag{6.3}$$

$$\prod_{k=0}^{n} \frac{f(2k)}{f(2k+1)} = \prod_{k=0}^{2n+1} (f(k))^{(-1)^k} \tag{6.4}$$

$$\prod_{k=0}^{\infty} \frac{f(2k)}{f(2k+1)} = \prod_{k=0}^{\infty} (f(k))^{(-1)^k} \tag{6.5}$$

6.1.2 Exponent Property

$$\prod_{k=a}^{n} \boldsymbol{\alpha}^{f(k)} = \boldsymbol{\alpha}^{\sum_{k=a}^{n} f(k)} \tag{6.6}$$

Applications of Exponent Property

$$\prod_{k=a}^{n} \alpha = \alpha^{n-a+1} \tag{6.7}$$

$$\prod_{k=a}^{n} \alpha^{k} = \alpha^{\frac{n_n^2 - a^2 + a}{2}} \tag{6.8}$$

$$\prod_{k=0}^{n} x^{(-1)^k \binom{n}{k}} = 1, \qquad n \neq 0, \qquad x \neq 0 \tag{6.9}$$

otherwise, the previous product equals x, when $n = 0$.

$$\prod_{k=0}^{n} x^{\binom{n}{k}} = x^{2^n} \tag{6.10}$$

6.1.3 Logarithm of Product Property

$$\log_b \prod_{k=a}^{n} (f(k))^p = p \sum_{k=a}^{n} \log_b f(k) \tag{6.11}$$

6.1.4 Product as an Exponential Function

$$\prod_{k=a}^{n} f(k) = e^{\sum_{k=a}^{n} \ln f(k)} \tag{6.12}$$

$$\prod_{k=a}^{n} (1 + f(k)) = e^{\sum_{k=a}^{n} f(k)} e^{\sum_{k=a}^{n} \sum_{j=2}^{\infty} (-1)^{j-1} \frac{(f(k))^j}{j}}, \ |f(x)| < 1, \ a \leq x \leq n \tag{6.13}$$

6.1.5 Factorial as a Finite Product

$$\prod_{k=1}^{n} k = n! \tag{6.14}$$

Remark 6.2 *In the following identity, we assume b is a nonnegative integer. If the reader wants to let b be an arbitrary complex number, then he or she must use the convention $\Gamma(b) = (b-1)!$.*

$$\prod_{k=a}^{n} (k+b) = \frac{(n+b)!}{(b+a-1)!} \tag{6.15}$$

Remark 6.3 *In the following identity, we assume b is a positive integer greater than n. Otherwise, the reader must use the fact that $\Gamma(b) = (b-1)!$ whenever b is a complex number which is not a negative integer.*

$$\prod_{k=a}^{n} (b-k) = \frac{(b-a)!}{(b-n-1)!} \tag{6.16}$$

Remark 6.4 *In the next eight identities, x is any complex number for which the corresponding factorial expression will be defined via the Gamma function (see Remark 6.2).*

$$\prod_{j=0}^{n}(2j+x) = 2^{n+1}\frac{\left(n+\frac{x}{2}\right)!}{\left(\frac{x}{2}-1\right)!}, \qquad n \geq 0 \tag{6.17}$$

$$\prod_{j=0}^{n}(2j-x) = 2^{n+1}\frac{\left(n-\frac{x}{2}\right)!}{\left(-\frac{x}{2}-1\right)!}, \qquad n \geq 0 \tag{6.18}$$

$$\prod_{j=0}^{n}(4j^2-x^2) = 2^{2n+2}\frac{\left(n+\frac{x}{2}\right)!\left(n-\frac{x}{2}\right)!}{\left(\frac{x}{2}-1\right)!\left(-\frac{x}{2}-1\right)!}, \qquad n \geq 0 \tag{6.19}$$

$$\prod_{j=0}^{n}(2j+1+x) = 2^{n+1}\frac{\left(n+\frac{x+1}{2}\right)!}{\left(\frac{x+1}{2}-1\right)!}, \qquad n \geq 0 \tag{6.20}$$

$$\prod_{j=0}^{n}(2j+1-x) = 2^{n+1}\frac{\left(n+\frac{-x+1}{2}\right)!}{\left(\frac{-x+1}{2}-1\right)!}, \qquad n \geq 0 \tag{6.21}$$

$$\prod_{j=0}^{n}\left((2j+1)^2-x^2\right)=2^{2n+2}\binom{n+\frac{x+1}{2}}{n+1}\binom{n-\frac{x-1}{2}}{n+1}((n+1)!)^2, \qquad n\geq 0 \tag{6.22}$$

$$\prod_{j=0}^{n}(x^2-j^2)=\frac{x(x+n)!}{(x-n-1)!}, \qquad n\geq 0 \tag{6.23}$$

$$\prod_{j=0}^{n}(j^2-x^2)=(-1)^{n+1}\frac{x(x+n)!}{(x-n-1)!}, \qquad n\geq 0 \tag{6.24}$$

Remark 6.5 *In the following identity, we assume b is a positive integer. The resulting factorial expressions are evaluated by use of the Gamma function (see Remark 6.2).*

$$\prod_{k=1}^{n}(1+bk)=\frac{b^n\left(n+\frac{1}{b}\right)!}{\left(\frac{1}{b}\right)!} \tag{6.25}$$

$$\boldsymbol{\prod_{k=1}^{n}(2k+1)=\frac{(2n+1)!}{2^n n!}, \qquad n\geq 1} \tag{6.26}$$

$$\boldsymbol{\prod_{k=1}^{n}2k=2^n n!, \qquad n\geq 1} \tag{6.27}$$

$$\prod_{k=1}^{n}(1+k)^p=((n+1)!)^p \tag{6.28}$$

6.1.6 Binomial Coefficient as Finite Product

$$\boldsymbol{\prod_{k=1}^{n}\left(1+\frac{a}{k}\right)=\prod_{k=1}^{a}\left(1+\frac{n}{k}\right)=\binom{n+a}{a}, \qquad n\geq 1, \qquad a\geq 1} \tag{6.29}$$

$$\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)^p=n^p, \qquad n\geq 1 \tag{6.30}$$

$$\prod_{k=1}^{n}\left(1+\frac{n}{k}\right)=\frac{(2n)!}{(n!)^2}, \qquad n\geq 1 \tag{6.31}$$

$$\prod_{k=1}^{n-1}\left(1-\frac{n}{k}\right)=(-1)^{n+1}, \qquad n\geq 2 \tag{6.32}$$

Remark 6.6 *In the following three identities, we assume a is a positive integer. The corresponding factorials are evaluated via the Gamma function (see Remark 6.2).*

$$\prod_{k=1}^{n}\left(1\pm\frac{1}{ak}\right)=\binom{n\pm\frac{1}{a}}{n},\qquad n\geq 1 \tag{6.33}$$

$$\prod_{k=1}^{n}\left(1-\frac{1}{a^2k^2}\right)=\frac{\left(n+\frac{1}{a}\right)!\left(n-\frac{1}{a}\right)!}{\left(\frac{1}{a}\right)!\left(\frac{-1}{a}\right)!(n!)^2},\qquad n\geq 1 \tag{6.34}$$

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{a^2k^2}\right)=\frac{1}{\left(\frac{1}{a}\right)!\left(\frac{-1}{a}\right)!}=\frac{\sin\left(\frac{\pi}{a}\right)}{\frac{\pi}{a}} \tag{6.35}$$

$$\prod_{j=1}^{k-1}\left(1-\frac{n^2}{j^2}\right)=\frac{(-1)^{k-1}}{2^{2k}}\frac{k^2}{n^2}\binom{2k}{}\sum_{j=0}^{k}\binom{2n}{2j}\binom{n-j}{k-j} \tag{6.36}$$

$$\prod_{j=1}^{n-1}\left(1-\frac{n^2}{j^2}\right)=\frac{(-1)^{n-1}}{2}\binom{2n}{n},\qquad n\geq 1 \tag{6.37}$$

6.1.7 Index Shift Formula

$$\boldsymbol{\prod_{k=a}^{n}f(k)=\prod_{k=a}^{n}f(n-k+a)} \tag{6.38}$$

Applications of Index Shift Formula

$$\prod_{k=1}^{n}k^2=(n!)^2,\qquad n\geq 1 \tag{6.39}$$

$$\prod_{k=1}^{n-1}(kn-k^2)=((n-1)!)^2,\qquad n\geq 2 \tag{6.40}$$

$$\prod_{k=1}^{n}\left(\frac{n+1}{k}-1\right)=1,\qquad n\geq 1 \tag{6.41}$$

$$\prod_{k=0}^{n} k! = \prod_{k=0}^{n-1}(k+1)^{n-k} = \prod_{k=0}^{n-1}(n-k)^{k+1} = \prod_{k=0}^{n}(n-k)! \tag{6.42}$$

$$\prod_{k=1}^{n} \frac{1}{k!} = \prod_{k=0}^{n-1} n^{-(k+1)} \prod_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^{-(k+1)}, \qquad n \geq 1 \tag{6.43}$$

6.1.8 Two Cancellation Properties

$$\boldsymbol{\prod_{k=a}^{n} \frac{f(k+1)}{f(k)} = \frac{f(n+1)}{f(a)}} \tag{6.44}$$

$$\boldsymbol{\prod_{k=a}^{n} \frac{f(k-1)}{f(k)} = \frac{f(a-1)}{f(n)}, \qquad a \geq 1} \tag{6.45}$$

Applications of the Cancellation Properties

$$\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right)^{p} = (n+1)^{p}, \qquad n \geq 1 \tag{6.46}$$

$$\prod_{r=1}^{n} \prod_{k=1}^{r} \left(1 + \frac{1}{k}\right)^{p} = \prod_{r=1}^{n}(r+1)^{p} = ((n+1)!)^{p}, \qquad n \geq 1 \tag{6.47}$$

$$\prod_{k=a}^{n} \left(1 + \frac{1}{k}\right)^{p} = \left(\frac{n+1}{a}\right), \qquad a \geq 1 \tag{6.48}$$

$$\prod_{k=a}^{n} \left(1 - \frac{1}{k}\right)^{p} = \left(\frac{a-1}{n}\right)^{p}, \qquad a \geq 2 \tag{6.49}$$

$$\prod_{k=a}^{n} \left(1 - \frac{1}{k^2}\right)^{p} = \left(\frac{a-1}{a}\left(1 + \frac{1}{n}\right)\right)^{p}, \qquad a \geq 2 \tag{6.50}$$

$$\prod_{k=a}^{\infty} \left(1 - \frac{1}{k^2}\right)^{p} = \left(\frac{a-1}{a}\right)^{p}, \qquad a \geq 2 \tag{6.51}$$

$$\prod_{k=2}^{n} \frac{k+1}{k-1} = \frac{n^2+n}{2} = \sum_{k=1}^{n} k, \qquad n \geq 2 \tag{6.52}$$

$$\sum_{k=2}^{n} \ln(k+1) - \sum_{k=2}^{n} \ln(k-2) = \ln \sum_{k=1}^{n} k = \ln\left(\frac{n^2+n}{2}\right), \qquad n \geq 2 \tag{6.53}$$

$$\prod_{k=1}^{n} \frac{(1+k)^{2p}}{k^p} = (n+1)^{2p}(n!)^p, \qquad n \geq 1 \tag{6.54}$$

$$\prod_{k=1}^{n} \left(\frac{r+k}{k}\right)^k = \frac{1}{(n!)^r} \prod_{j=1}^{r} (n+j)^{n-r+j}, \qquad r, n \geq 1 \tag{6.55}$$

Remark 6.7 *In the following identity, we assume r is a positive integer such that $r \geq n+1$, for fixed integer $n \geq 1$. If the reader prefers to let r represent an arbitrary complex number, the factorials must be evaluated by the Gamma function (See Remark 6.2).*

$$\prod_{k=1}^{n} \frac{r-k}{k+1} = \frac{\binom{r}{n+1}}{\binom{r}{1}} = \frac{(r-1)!}{(n+1)!(r-n-1)!} \tag{6.56}$$

Remark 6.8 *In the following identity, we assume $r = 0$ or $r = 1$. If the reader prefers to let r be any complex number which is not a positive integer greater than or equal to 2, he or she should ignore the binomial coefficient representation and evaluate the factorial by the Gamma function (See Remark 6.2).*

$$\prod_{k=1}^{n} \frac{k+1}{k+1-r} = \frac{\binom{n+1}{r}}{\binom{1}{r}} = \frac{(n+1)!(1-r)!}{(n+1-r)!}, \qquad n \geq 1 \tag{6.57}$$

6.1.9 Three Product Identities From Identity (3.7)

$$\prod_{k=a}^{n} \left(1 + \frac{1}{k}\right)^{k} = \frac{(a-1)!(n+1)^{n}}{a^{a-1} n!}, \qquad a \neq 1 \tag{6.58}$$

$$\prod_{k=1}^{n} k^{2k-1} = (n+1)^{n^2} \prod_{k=1}^{n} \left(\frac{k}{k+1}\right)^{k^2}, \qquad n \geq 1 \tag{6.59}$$

$$\prod_{k=1}^{n} k^{2k-1} \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right)^{k^2} = (n+1)^{n^2}, \qquad n \geq 1 \tag{6.60}$$

6.1.10 Iterative Product Formula

$$\boldsymbol{\prod_{i=0}^{n} \prod_{k=0}^{i} f(k) = \prod_{k=0}^{n} (f(k))^{n-k+1} = \prod_{k=0}^{n} (f(n-k))^{k+1}} \tag{6.61}$$

Applications of Iterative Product Formula

$$\prod_{k=1}^{n} k^{k} k! = (n!)^{n+1}, \qquad n \geq 1 \tag{6.62}$$

$$\prod_{k=1}^{n} k^{k-1} k! = (n!)^{n}, \qquad n \geq 1 \tag{6.63}$$

6.2 Trigometric Products

Remark 6.9 *In Section 6.2, we assume the reader is familiar with the Weirstrass Factorization Theorem. The reader may find this important theorem on the Wikkipedia website.*

Product for $\sin(\theta)$

$$\sin(\theta) = \theta \prod_{k=1}^{\infty} \left(1 - \frac{\theta^2}{k^2\pi^2}\right) \tag{6.64}$$

Product for $\cos(\theta)$

$$\cos(\theta) = \prod_{k=1}^{\infty} \left(1 - \frac{4\theta^2}{(2k-1)^2\pi^2}\right) \tag{6.65}$$

7 Intermediate Level Calculations Involving Products

Remark 7.1 *In the following chapter, we assume, unless otherwise specified, that r and n are positive integers.*

7.1 Defining $n!$ as a Product Limit

$$\lim_{r\to\infty} \frac{r^n n!}{(r+1)(r+2)...(n+r)} = n! \tag{7.1}$$

$$\lim_{r\to\infty} \frac{r^n r!}{(n+1)(n+2)...(n+r)} = n! \tag{7.2}$$

$$\lim_{r\to\infty} r^n \prod_{k=1}^{r} \frac{1}{\left(1+\frac{n}{k}\right)} = n! \tag{7.3}$$

$$\lim_{r\to\infty} r^{-n} \prod_{k=1}^{r} \left(1+\frac{n}{k}\right) = \frac{1}{n!} \tag{7.4}$$

7.2 Products From a Recursive Sequence

Remark 7.2 *In Section 7.2, we assume a such that $a \geq 2$. We assume n is a nonnegative integer. We define the sequence $(u_{0,n})_{n=0}^{\infty}$ by the recursive definition $(an-1)u_{0,n} = u_{0,n+1}$ with $u_{0,1} = 1$.*

$$u_{0,n+1} = u_{0,n-r+1} \prod_{k=0}^{r-1} (an - ka - 1) \tag{7.5}$$

$$u_{0,n+1} = u_{0,0} \prod_{k=0}^{n} (an - ka - 1) \tag{7.6}$$

Remark 7.3 *In the following four identities, we let $a = 2$. Also, any noninteger factorial is evaluted by the Gamma function, i.e. $\Gamma(x+1) = x$, for all complex numbers x, except negative integers.*

$$u_{0,n} = \frac{2^{n-1}}{\sqrt{\pi}} \Gamma\left(\frac{2n-1}{2}\right) \tag{7.7}$$

$$\prod_{k=0}^{n} (2n - 2k - 1) = -\frac{2^n}{\sqrt{\pi}} \left(n - \frac{1}{2}\right)! \tag{7.8}$$

$$\prod_{k=0}^{r-1} (2n - 2k - 1) = 2^r \frac{\left(n - \frac{1}{2}\right)!}{\left(n - r - \frac{1}{2}\right)!} \tag{7.9}$$

$$\prod_{k=0}^{n} (2k+1) = \frac{2^{n+1}\sqrt{\pi}}{(-1)^{n+1}\left(-n - \frac{3}{2}\right)!} \tag{7.10}$$

7.3 Applications of Binomial Coefficient as Product Formula

$$\boldsymbol{\binom{n+\frac{2r+1}{2}}{n} = \frac{1}{2^{2n}}\binom{2n}{2}\prod_{k=0}^{r}\frac{2n+2k+1}{2k+1}}, \tag{7.11}$$

where n is a positive integer and r is a nonnegative integer.

$$\binom{n+\frac{1}{2}}{n} = \binom{2n}{n}\frac{2n+1}{2^{2n}} \tag{7.12}$$

$$\binom{n+\frac{3}{2}}{n} = \binom{2n}{n}\frac{(2n+1)(2n+3)}{3*2^{2n}} \tag{7.13}$$

$$\binom{n+\frac{5}{2}}{n} = \binom{2n}{n}\frac{(2n+1)(2n+3)(2n+5)}{3*5*2^{2n}} \tag{7.14}$$

$$\binom{n+\frac{7}{2}}{n} = \binom{2n}{n}\frac{(2n+1)(2n+3)(2n+5)(2n+7)}{3*5*7*2^{2n}} \tag{7.15}$$

Remark 7.4 *In the following three identities, any non integer factorials are evaluated via the Gamma function (see Remark (7.3))*

$$\boldsymbol{\binom{n+\frac{\alpha}{2}}{n} = \frac{(2n+\alpha)!\left(\frac{\alpha-1}{2}\right)!}{2^{2n}\alpha!n!\left(n+\frac{\alpha-1}{2}\right)!}}, \tag{7.16}$$

where n is a positive integer and α is a nonnegative integer.

$$\boldsymbol{\binom{n+k}{n} = \frac{(2n+2k)!\left(\frac{2k-1}{2}\right)!}{2^{2n}(2k)!n!\left(n+\frac{2k-1}{2}\right)!}}, \tag{7.17}$$

where n is a positive integer and k is <u>any</u> real number.

$$\binom{2n+2k}{2n}\binom{2n}{n} = 2^{2n}\binom{n+k}{n}\binom{n+\frac{2k-1}{2}}{n}, \tag{7.18}$$

where n is a <u>nonnegative</u> integer and k is any real number.

Remark 7.5 *In the following three identities x is an arbitrary complex number, h is any nonzero complex number and n is a positive integer.*

$$\prod_{k=0}^{n-1}(x+kh) = h^n n!\binom{\frac{x}{h}+n-1}{n} \tag{7.19}$$

$$\prod_{k=0}^{n-1}(x-kh) = h^n n!\binom{\frac{x}{h}}{n} \tag{7.20}$$

$$\lim_{h\to 0} h^n n!\binom{\frac{x}{h}}{n} = x^n \tag{7.21}$$

7.4 Induction on Three Product Expansions

Remark 7.6 *In Section 7.4, we let $[x]$ denote the greatest integer in x.*

7.4.1 First Product Expansion

$$\prod_{k=2}^{n} k^{2k-1} = \prod_{k=1}^{n^2-1}(1+[\sqrt{k}]), \qquad n \geq 2 \tag{7.22}$$

$$([\sqrt{n}]+1)^{n-[\sqrt{n}]^2+1}\prod_{k=2}^{[\sqrt{n}]} k^{2k-1} = \prod_{k=1}^{n}(1+[\sqrt{k}]), \qquad n \geq 2 \tag{7.23}$$

7.4.2 Second Product Expansion

$$\prod_{k=1}^{n}\left(1+\frac{(-1)^{k-1}}{k}\right) = 1+\frac{1-(-1)^n}{2n}, \qquad n \geq 1 \tag{7.24}$$

$$\prod_{k=2}^{n}\left(1-\frac{(-1)^{k}}{k}\right) = \frac{1}{2}+\frac{1-(-1)^n}{4n}, \qquad n \geq 2 \tag{7.25}$$

$$\prod_{k=1}^{\infty}\left(1+\frac{(-1)^{k-1}}{k}\right) = 1 \tag{7.26}$$

7.4.3 Third Product Expansion

$$\prod_{k=2}^{n}\left(1+\frac{(-1)^k}{k}\right)=1+\frac{1+(-1)^n}{2n}, \qquad n\geq 2 \tag{7.27}$$

7.5 Three Product Functions

7.5.1 First Product Function

$$\prod_{i=1}^{n}(1+x^i)(1-x^{2i-1}=\prod_{j=n+1}^{2n}(1-x^j)=\prod_{j=1}^{n}(1-x^{j+n}) \tag{7.28}$$

7.5.2 Second Product Function

$$\prod_{i=1}^{n}(1-x^{2i})(1-x^{2i-1})=\prod_{j=1}^{2n}(1-x^j) \tag{7.29}$$

7.5.3 Third Product Function

$$\prod_{i=1}^{n}(1+x^{2i})(1+x^{2i-1})=\prod_{j=1}^{n}(1+x^j) \tag{7.30}$$

7.6 The Product Functions $\prod_{k=0}^{n} x^{a^k}$ and $\prod_{k=0}^{n}(1+x^{a^k})$

Remark 7.7 *In Section 7.6, we assume a is any nonzero real number, except 1. Also, we may assume that x is any nonzero complex number for which the products and resulting functions are defined.*

$$\prod_{k=0}^{n} x^{a^k}=x^{\frac{a^{n+1}-1}{a-1}} \tag{7.31}$$

$$\prod_{k=0}^{n}\left(1+\frac{1}{x^{2^k}}\right)=\frac{x}{x-1}\left(1-\frac{1}{x^{2^{n+1}}}\right) \tag{7.32}$$

$$\prod_{k=0}^{n}\left(1+x^{2^k}\right)=\frac{x^{2^{n+1}}-1}{x-1} \tag{7.33}$$

$$\prod_{k=0}^{n-1}\left(1+x^{2^k}\right)=\frac{1-x^{2^n}}{1-x} \tag{7.34}$$

$$\sum_{k=1}^{n} x^{2^k}\left(x^{2^k}-1\right)=x^2-x^{2^{n+1}}, \qquad n\geq 1 \tag{7.35}$$

$$\sum_{k=1}^{n}\frac{1}{x^{2^k}}\left(1-\frac{1}{x^{2^k}}\right)=\frac{1}{x^2}-\frac{1}{x^{2^{n+1}}}, \qquad n\geq 1 \tag{7.36}$$

$$\sum_{k=1}^{\infty}\frac{1}{x^{2^k}}\left(1-\frac{1}{x^{2^k}}\right)=\frac{1}{x^2}, \qquad |x|\geq 1 \tag{7.37}$$

Remark 7.8 *In the following four identities, we let $x=2$.*

$$\prod_{k=0}^{n} 2^{2^k}=2^{2^{n+1}-1} \tag{7.38}$$

$$\prod_{k=0}^{n}\left(1+\frac{1}{2^{2^k}}\right)=2\left(1-\frac{1}{2^{2^{n+1}}}\right) \tag{7.39}$$

$$\prod_{k=0}^{n}\left(1+2^{2^k}\right)=2^{2^{n+1}}-1 \tag{7.40}$$

$$\prod_{k=0}^{\infty}\left(1+\frac{1}{2^{2^k}}\right)=2 \tag{7.41}$$

7.6.1 Product Identities Involving Geometric Series

$$\boldsymbol{\prod_{k=1}^{n}\sum_{i=0}^{r-1} x^{ir^{k-1}}=\sum_{j=0}^{r^n-1} x^j=\frac{1-x^{r^n}}{1-x}, \qquad n,r\geq 1,\ x\neq 1} \tag{7.42}$$

$$\prod_{k=1}^{n}\left(1+x^{3^{k-1}}+x^{2*3k-1}\right)=\frac{1-x^{3^n}}{1-x}, \qquad n\geq 1,\ x\neq 1 \tag{7.43}$$

8 Relationships Between Finite Series and Finite Products

8.1 Series as a Product

$$\sum_{k=1}^{n} f(k) = f(1)\prod_{k=2}^{n}\left(1+\frac{f(k)}{\sum_{i=1}^{k-1} f(i)}\right), \qquad n \geq 2 \tag{8.1}$$

8.1.1 Applications of Series as Product Formula

$$\sum_{k=1}^{n}\frac{1}{2^{k-1}} = \prod_{k=2}^{n}\left(1+\frac{1}{2^k-2}\right) = 2-\frac{1}{2^{n-1}} \tag{8.2}$$

$$\prod_{k=2}^{\infty}\left(1+\frac{1}{2^k-2}\right) = 2 \tag{8.3}$$

$$2\sum_{k=1}^{n}\frac{1}{k(k+1)} = \prod_{k=2}^{n}\left(1+\frac{1}{k^2-1}\right) = \frac{2n}{n+1} \tag{8.4}$$

$$\prod_{k=2}^{\infty}\left(1+\frac{1}{k^2-1}\right) = 2 \tag{8.5}$$

$$\sum_{k=1}^{n}\frac{1}{k(k+2)} = \frac{1}{3}\prod_{k=2}^{n}\left(1+\frac{4(k+1)}{(k-1)(k+2)(3k+2)}\right) = \frac{1}{4}\frac{n(3n+5)}{(n+1)(n+2)} \tag{8.6}$$

$$\prod_{k=2}^{\infty}\left(1+\frac{4(k+1)}{(k-1)(k+2)(3k+2)}\right) = \frac{9}{4} \tag{8.7}$$

$$2\sum_{k=1}^{n}\frac{k}{(k+1)!} = \prod_{k=2}^{n}\left(1+\frac{k}{(k+1)!-k-1}\right) = 2-\frac{2}{(n+1)!} \tag{8.8}$$

$$\prod_{k=2}^{\infty}\left(1+\frac{k}{(k+1)!-k-1}\right) = 2 \tag{8.9}$$

$$\sum_{k=1}^{n} \frac{k(k+2)}{(k+1)^2(k!)^2} = \frac{3}{4} \prod_{k=2}^{n} \left(1 + \frac{k(k+2)}{(k+1)^2((k!)^2-1)}\right) = 1 - \frac{1}{((n+1)!)^2} \tag{8.10}$$

$$\prod_{k=2}^{\infty} \left(1 + \frac{k(k+2)}{(k+1)^2((k!)^2-1)}\right) = \frac{4}{3} \tag{8.11}$$

8.2 Product as a Series

$$\prod_{k=1}^{n} (1+f(k)) = 1 + f(1) + \sum_{k=2}^{n} f(k) \prod_{i=1}^{k-1} (1+f(i)) \tag{8.12}$$

8.2.1 Applications of Product as Series Formula

Remark 8.1 *In the following five identities, we assume r and n are positive integers.*

$$\sum_{k=1}^{n} \binom{k+r-1}{r} \frac{1}{k} = \frac{1}{r}\binom{n+r}{r} - \frac{1}{r} \tag{8.13}$$

$$\lim_{r\to\infty} \frac{1}{r^{n-1}} \sum_{k=1}^{n} \binom{k+r-1}{r} \frac{1}{k} = \frac{1}{n!} \tag{8.14}$$

$$\sum_{k=1}^{n} \binom{k+r}{k} \frac{1}{k+r} = \frac{\binom{n+r}{r} - 1}{r} \tag{8.15}$$

$$\lim_{r\to\infty} \frac{1}{r^{n-1}} \sum_{k=1}^{n} \binom{k+r}{r} \frac{1}{k+r} = \frac{1}{n!} \tag{8.16}$$

$$\lim_{r\to 0} \frac{\binom{n+r}{r} - 1}{r} = \sum_{k=1}^{n} \frac{1}{k} \tag{8.17}$$

Remark 8.2 *In the following two identities, we assume $0^0 = 1$.*

$$\sum_{k=0}^{n} \frac{(1+k)^k - k^k}{k!} = \frac{(n+1)^n}{n!} \tag{8.18}$$

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{(1+k)^k - k^k}{k!}\right)^{\frac{1}{n}} = e \tag{8.19}$$

8.3 Schlomilch Series to Product Identity

$$\sum_{i=0}^{n} u_i = \frac{u_0}{1} \prod_{k=0}^{n-1} \frac{\sum_{i=0}^{k+1} u_i}{\sum_{i=0}^{k} u_i} \tag{8.20}$$

8.4 Schlomilch Product to Series Identity

$$\prod_{j=0}^{n} v_j = v_0 + v_0(v_1 - 1) + v_0 v_1 (v_2 - 1) + ... + v_0 v_1 v_2 ... v_{n-1}(v_n - 1) \tag{8.21}$$