Linear Independence, span, basis, dimension

The **span** of a set of vectors is the subspace consisting of all linear combinations of the vectors in the set.

Given a subspace we say a set S of vectors **spans the subspace** if the span of the set S is the subspace.

A **basis** of a subspace is a set of vectors that spans the subspace and is linearly independent.

If you have a basis of a subspace and you add any vector $\bar{v}$ to it, the resulting set is no longer linearly independent.

(Reason: $\bar{v}$ is in the span of the basis, hence adding $\bar{v}$ to the basis renders it linearly dependent)

If you have a basis of a subspace, and you remove any vector from that set, the resulting set does not span the subspace.

(Reason: The vector you removed cannot be in the span of the remaining vectors because the original basis is linearly independent)

The subspace consisting of solutions of Ax=0 is called the **nullspace** of the matrix A. It has a basis consisting of the “fundamental solutions” of Ax=0 that we know how to calculate.

The span of a given set of vectors is a subspace. When we put these vectors in a matrix, that subspace is called the **column space** of the matrix: to find a basis of the span, put the vectors in a matrix A. The columns of A that wind up with leading entries in Gaussian elimination form a basis of that subspace.

The **dimension** of a subspace U is the number of vectors in a basis of U. (There are many choices for a basis, but the number of vectors is always the same.) There are many possible choices of a basis for any vector space; different bases can have different useful features.

Example: Find a basis for the space spanned by the vectors $\begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \\ 2 \end{bmatrix},$

$\begin{bmatrix} 0 \\ 6 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$

Solution: This method finds a basis from among the given vectors. We put them in a matrix and reduce the matrix: $\begin{bmatrix} -1 & 2 & 0 & 1 & 1 \\ 2 & 2 & 6 & 4 & -1 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & 2 & 4 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The leading entries appear in columns 1,2,5 so the vectors in those columns in the original matrix are our basis. The other columns can be generated from these, and, by themselves, columns 1,2,5 are linearly independent. So here is our basis:

$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ The dimesion of this subspace is 3. (Note that we need only go far enough to determine where the leading entries will be - row echelon form is sufficient for that)

Why is this result true: From the row reduced echelon form we see that column 3 is a linear combination of cols 1 and 2 and the same is true of column 4. Columns 1,2,5 are linearly independent and their span includes columns 2 and 3 and hence their span is the same as the five original columns.

Row spaces: Note that the span of the rows of a matrix is also a subspace, called the **row space** of the matrix, in this case a subspace of R^n (if the matrix is $m \times n$). Now every nonzero row in the row echelon (or reduced row echelon) form of the matrix is in the row space (they were obtained by a sequence of linear combinations from the original rows) and also every row of the original matrix is in the span of the nonzero rows of the row echelon (or reduced row echelon) form, because we can go backwards and recover the original rows. Hence the row space is also the span of the nonzero rows of the reduced matrix. On the other hand, the rows of the echelon (or reduced row echelon) form are linearly independent - this is easy to see for the reduced row echelon form, and not hard even for the row echelon form. So: **The nonzero rows of the row echelon form** (or reduced row echelon form) **of a matrix are a basis for the row space**. This is a useful basis. Although we have not chosen this basis from among the original rows (we could do that if we turned the rows into columns and used the method above), this basis is useful in that it is easy to compute the linear combination needed to generate any vector in the subspace from the basis, or easy to check whether any given vector is in the subspace.

In the example above:

$A = \begin{bmatrix} -1 & 2 & 0 & 1 & 1 \\ 2 & 2 & 6 & 4 & -1 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & 2 & 4 & 3 & 2 \end{bmatrix}$. If we consider the span of the rows, a basis for this subspace are the nonzero rows of the reduced form on the right:

$$\begin{bmatrix} -1 & 2 & 0 & 1 & 1 \\ 2 & 2 & 6 & 4 & -1 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & 2 & 4 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

. The (nonzero) rows of the reduced matrix on the right are our basis. There are three such vectors so the dimension of the row space of this matrix is 3.

This gives rise to the following famous and important observation about subspaces associated with matrices:

dimension of column space=**dimension of row space**=**rank of the matrix**

because all of these quantities are equal to the number of leading entries in the (reduced) row echelon form of the matrix.