Lubrication approximation for thin viscous films: asymptotic behavior of nonnegative solutions

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Abstract
We use standard regularized equations and adapted entropy functionals to prove exponential asymptotic decay in the $H^1$ norm for nonnegative weak solutions of fourth-order nonlinear degenerate parabolic equations of lubrication approximation for thin viscous film type. The weak solutions considered arise as limits of solutions for the regularized problems. Relaxed problems, with second-order nonlinear terms of porous media type are also successfully treated by the same means. The problems investigated here are one-dimensional in space, with power-law nonlinearities. Our approach is direct and natural, as it is adapted to deal with the more complex nonlinear terms occurring in the regularized, approximating problems.

Keywords. Asymptotic decay, Entropy dissipation, Fourth-order nonlinear parabolic equation, Exponential decay.

AMS subject classification. 35B40, 35K25, 35K45, 35K55, 35K65.

1 Introduction
To our knowledge, the sharpest asymptotic decay result for thin viscous film type equations is due to Carlen and Ulusoy [11], and it involves an initial power-law decay followed by exponential decay as the solutions uniformly approach equilibrium. This result is also a first in that it gives decay in the $H^1$ norm, an obvious improvement to $L^1$ or even $L^\infty$ results [2], [9], [10], [15]. The analysis in [11] is performed for classical solutions only and using entropy functionals first introduced by Laugesen [13]. As we shall discuss later, it may turn out to be extremely difficult to use these for weak solutions. Our goal in this paper is to quantify the rate of convergence in the $H^1$ norm by more direct methods that do extend to control weak solutions as well. We shall prove asymptotic decay in the $H^1$ norm for the classical solution of the regularized problem by using the Dirichlet energy as a Lyapunov functional. Our main achievement is that we extend the result to the weak solutions of the original problem obtained by the regularization procedure described in the sequel. We also show that perturbations of porous media type [8] lead to better rates of decay; we would like to thank the referee for raising this last issue. The $H^1$ decay of the solution to its mean value is exponential in time, and it holds for the whole range of the subdiffusive exponent ($n$ from the equation below). Within a certain range of this exponent an
argument similar to the one in [11] may be used, as we remark in the sequel. To explain, one can initially obtain a power-law decay after which, as soon as the solution gets uniformly close enough to its mean value, exponential decay sets in. “Close enough” in this context relates to the waiting time necessary to achieve sufficient proximity to the mean value in order to match the rate of decay for the linearized problem [11]. For reasons we shall explain later, however, we choose to follow a line of argument which will lead us directly to exponential decay.

Let $I := (-1, 1)$ and consider the IVP with no-flux boundary conditions:

\begin{align}
  u_t &= -(u^n u_{xxx})_x \quad \text{in } Q := I \times (0, \infty), \\
  u_x &= 0 \quad \text{on } \partial I \times (0, \infty), \\
  u_{xxx} &= 0 \quad \text{on } \partial I \times (0, \infty), \\
  u(\cdot, 0) &= u_0 \quad \text{in } I.
\end{align}

(1.1)

The existence of nonnegative weak solutions for this problem was proved by Bernis and Friedman [4] for $n \geq 1$, and Bertozzi and Pugh [9] for all $n > 0$. These solutions satisfy (1.1) in some sense (reviewed in the next section). Whereas positive weak solutions (which are, in fact, classical) are known to be unique [4], uniqueness of weak solutions is an open question. Thus, it is important that we specify from the start that our results apply only to the nonnegative weak solutions arising as limits from the standard regularized problems (see Section 3).

Different values of $n$ correspond to various physical interpretations: silicon oxidation in semiconductors ($n = 3$) [19], thin flows in Hele-Shaw cells ($n = 1$) [12], etc. Problems of this type also appear in modelling the motion of viscous droplets that are spread over a solid surface ($n = 3$) [5]. We have found the survey paper [3] of great help in offering at least a general idea about the applications of such problems; it also contains a discussion of the state of the art in the field.

A brief and concise qualitative comparison between (1.1) and second-order degenerate parabolic problems of porous media type can be found in [9]. In essence, the common features exhibited are the parabolicity, the divergence structure and the existence of nonnegative solutions corresponding to nonnegative initial data. The main difference is the lack of a maximum principle for (1.1). As a consequence, (1.1) is analyzed via dissipation of certain nonlinear entropies, and no universal result on positivity preservation of the initial data is available (it is not known whether initial positivity is preserved for $n < 3.5$ see, e.g., [6]; for $n \geq 4$ it was proved by Bernis and Friedman in their seminal paper [4], then Bertozzi & al. [7] improved it to $n \geq 3.5$). The maximum principle does not allow for the development of zeros in the second-order case.

If the first equation is formally differentiated with respect to $x$ and then integrated by parts against $u_x$, by taking into account the boundary conditions we obtain

\[ \frac{1}{2} \frac{d}{dt} \int_I u_x^2(x, t) dx = - \int_I u^n(x, t) u_{xxx}(x, t) dx. \]

(1.2)

Thus, the energy

\[ J[u] := \frac{1}{2} \int_I u_x^2(x, t) dx \]

(1.3)

is a Lyapunov functional for nonnegative smooth solutions of (1.1). If we formally integrate (1.1) over $I$ for any fixed $t > 0$, we observe that the total mass of $u(\cdot, t)$ is conserved in time. As we shall see in the next section, another useful quantity is the mass of $u^{2-n}(\cdot, t)$, which is dissipated in time for suitable values of $n$ [4].

Remark: Here we would like to mention that, for $n = 1$, there is a stronger connection between the evolution described by (1.1) and the functional $J$. Recent developments in the optimal mass
transportation theory are concerned with the study of gradient flows with respect to certain optimal cost metrics [1], [21]. Formally it is easy to see that (1.1) is the gradient flow of $J$ with respect to the Wasserstein metric on a special manifold [1], [21]. The analysis in [17] can ostensibly be adapted to (1.1). Then, there are standard results concerning the asymptotic decay of solutions for gradient flows driven by convex functionals. However, one should be cautious at this point, as $J$ is not convex with respect to the geometry of the Wasserstein space. Here we allude to the well-developed theory of displacement convexity [16].

The structure of this paper is the following: Section 2 introduces some energy-energy dissipation inequalities for positive classical solutions of (1.1), and the useful Lemma 1 as the key ingredient in the proof of Theorem 2. The next section discusses the regularized problem and recalls existence results for nonnegative weak solutions corresponding to nonnegative initial data. Most importantly, we prove convergence of the energy corresponding to the regularized problem to the energy corresponding to the original problem. Entropy dissipation is essential to strong convergence. In Section 4 we prove exponential decay in energy (and, implicitly, in the $H^1$ norm) by analyzing two cases differentiated by the range of the subdiffusive exponent $n$. It turns out that the case $1 \leq n \leq 2$ is harder, as we were not able to extend to this case the entropy based inequalities from the other case. Nevertheless, we use the entropy bound to obtain the global bound (crucial to our analysis) on the $L^2$ norm of the second derivative. Section 5 quantifies the effect of “porous media” type perturbations on the decay rates obtained for the original problem (1.1). We deal with a more general setting than the one already existing in the literature and we show how the analysis performed in the previous sections works here as well. In fact, the second-order perturbation allows for better decay rates. Finally, the last section discusses related open problems.

## 2 Energy-energy dissipation inequality

Previous research (e.g. [11]) has only used the dissipation expressed in (1.2) to control the rate of decay of $J[u]$ when $\|u - \langle u \rangle_I\|_\infty$ is small (here $\langle u \rangle_I$ denotes the average of $u$ over $I$); then it is trivial to get a lower bound for the dissipation in terms of $J[u]$. As we now show, it is possible to obtain such a bound even without assuming anything about the size of $\|u - \langle u \rangle_I\|_\infty$.

**Proposition 1.** Assume that $0 < n < \infty$ and $u$ is a positive classical solution of (1.1) corresponding to an initial probability density $u_0 \in H^1(I)$. The following are true:

(i) If $0 < n < 2$, then there exists a constant $C > 0$ depending only on $J[u_0]$ and $n$ such that

$$
\int_I u^n(x, t) u^2_{xx}(x, t) dx \geq C J^2[u(\cdot, t)], \quad \text{for all } t > 0.
$$

(ii) For $n = 2$, (2.1) holds with $C = 9/8$ (and thus, independent of $u_0$);

(iii) If $n > 2$ and $\int_I u_0^{2-n} dx < +\infty$, then there exists $C > 0$ depending only on $\int_I u_0^{2-n} dx$ such that (2.1) holds.

First, let us prove the following lemma, which is the key ingredient in the proofs of Proposition 1 and Theorem 2.

**Lemma 1.** For any measurable $w : I \to [0, \infty)$ and any nonnegative $v \in H^3(I)$ with $v_x(\pm 1) = 0$, one has

$$
\int_I \frac{v^2(x)}{w(x)} dx \int_I w(x) v_{xx}^2(x) dx \geq \frac{9}{16} \left( \int_I v_x^2(x) dx \right)^2.
$$

(2.2)
Proof: The Schwarz inequality gives
\[
\left( \int_I \frac{v^2}{w} \, dx \right)^{1/2} \left( \int_I w u_{xx}^2 \, dx \right)^{1/2} \geq - \int_{x_0}^x v u_{xxx} \, dy
\]
for all \( x_0, x \in I \). If we denote the left hand side by \( A \in [0, \infty] \), we obtain
\[
A \geq -v(x)v_{xx}(x) + v(x_0)v_{xx}(x_0) + \frac{1}{2} v_x^2(x) - \frac{1}{2} v_x^2(x_0).
\]
Integrating in \( x \) over \( I \) gives, after making the assumption \( v_x(x_0) = 0 \) (we know that we have at least two such points),
\[
2A \geq 2v(x_0)v_{xx}(x_0) + 3 \int_I v_x^2 \, dx.
\]
Of course, if either \( v_{xx}(-1) \) or \( v_{xx}(1) \) is nonnegative, then we are done by choosing \( x_0 \) to be that endpoint. If, however, both \( v_{xx}(-1) \) and \( v_{xx}(1) \) are negative, we conclude the proof by observing that there exists at least one \( x_0 \in I \) for which \( v_x(x_0) = 0 \) and \( v_{xx}(x_0) \geq 0 \).

We may now proceed with the proof of the proposition.

Proof of Proposition 1: Note that (1.2) implies
\[
t \to \|u_x(\cdot, t)\|_{L^2(I)} \text{ is nonincreasing .}
\]
Therefore, by using the fact that \( u(\cdot, t) \) is a probability density for all \( t \geq 0 \),
\[
\|u(\cdot, t)\|_\infty \leq \frac{1}{2} + 2^{1/2}\|u'_0\|_{L^2(I)} =: M_0.
\]
We shall apply Lemma 1 with \( w = u(\cdot, t)^n \) and \( v \equiv u(\cdot, t) \) for some \( t > 0 \). Thus,
\[
\int_I u^{2-n}(x,t) \, dx \int_I u^n(x,t) u_{xx}^2(x,t) \, dx \geq \frac{9}{16} \left( \int_I u_x^2(x,t) \, dx \right)^2.
\]
Thus, if \( 0 < n \leq 2 \), (2.1) follows with \( C = 9/\{8M_0^{2-n}\} \) and \( (i), (ii) \) are proved. To prove \( (iii) \) note that \( n \geq 2 \) implies
\[
t \to \int_I u^{2-n}(x,t) \, dx \text{ is nonincreasing,}
\]
which is already known [4]. Indeed, due to the boundary conditions,
\[
\frac{d}{dt} \int_I u^{2-n}(x,t) \, dx = (2-n) \int_I u^{1-n}(x,t) u_t(x,t) \, dx = (n-2) \int_I u^{1-n}(x,t) [u^n(x,t) u_{xxx}(x,t)]_x \, dx = (n-2)(n-1) \int_I u^{-n}(x,t) u^n(x,t) u_x(x,t) u_{xxx}(x,t) \, dx = -(n-2)(n-1) \int_I u_{xx}^2(x,t) \, dx \leq 0.
\]
In view of this and Lemma 1 (again with \( w = u^n \) and \( v \equiv u \) we infer (2.1) with \( C = 9/\{4 \int_I u_0^{2-n} \, dx\} \). Note that \( (ii) \) also comes out of this. \( \square \)
Remark: It is interesting that, in the case $0 < n \leq 2$ we can obtain a third-power-law decay as in [11] by a different argument, one which does not use Lemma 1 but uses conservation of mass instead. We give it next because roughly the same idea shall be used in Subsection 4.2. Here we simplify the notation by neglecting the dependencies of $u$. Note that, by Schwarz’s inequality, at any $t > 0$ we have

$$
\int_I u^n u^2_{xxxx} dx \int_I u^{2-n} u^2_x dx \geq \left( \int_I uu_{xxxx} dx \right)^2.
$$

The boundary conditions yield

$$
\int_I uu_{xxxx} dx = - \int_I uu^2_{xx} dx - \int_I u^2_x u_{xx} dx = - \int_I uu^2_x dx.
$$

Thus,

$$
\int_I u^n u^2_{xxxx} dx \int_I u^{2-n} u^2_x dx \geq \left( \int_I uu^2_x dx \right)^2.
$$

(2.3)

Again by Schwarz, we may write

$$
\int_I uu^2_x dx \int_I u dx \geq \left( \int_I uu_x dx \right)^2.
$$

Since $u(\cdot, t)$ is a probability density for all $t > 0$ (one simply integrates the first equation in (1.1) over $I$ and takes into account the boundary conditions to observe that the initial mass is conserved for all times), we may use the boundary conditions once again to deduce

$$
\int_I uu^2_x dx = \int_I u dx \int_I uu^2_x dx \geq \left( \int_I u^2_x dx \right)^2.
$$

Combining this with (2.3) we obtain

$$
\int_I u^n u^2_{xxxx} dx \int_I u^{2-n} u^2_x dx \geq \left( \int_I u^2_x dx \right)^4
$$

(2.4)

which, since $n < 2$, leads to

$$
\int_I u^n u^2_{xxxx} dx \geq 8M_0^{n-2} J^3[u].
$$

Thus, we obtain the same power-law decay as in [11].

Obviously, Proposition 1 implies

$$
J[u(\cdot, t)] \leq (J[u_0]^{-1} + Ct)^{-1}, \ t > 0
$$

(2.5)

which is no better than the initial decay in [11]. Then, as in [11], one can show that the exponential decay sets in for sufficiently large $t > 0$. Indeed, as soon as $J[u]$ is driven sufficiently small by (2.5), $u$ becomes uniformly bounded from below away from zero (even in the case $0 < n \leq 2$) by, say, $\alpha > 0$.

Thus,

$$
\int_I u^n u^2_{xxxx} dx \geq \alpha^n \int_I u^2_{xxxx} dx.
$$

We then use the boundary conditions to deduce

$$
\int_I u^2_{xxxx} dx \int_I u^2_x dx \geq \left( \int_I u^2_x dx \right)^2 \geq \left( \beta \int_I u^2_x dx \right)^2,
$$

(2.6)
where the last inequality is the Poincaré inequality with constant $\beta > 0$. Combining the last inequality with the one from the previous display, we get
\[
\int_I u^n u_{xx}^2 \, dx \geq \alpha^n \beta^2 \int_I u_x^2 \, dx
\]
which, in view of (1.2), leads to exponential decay.

**Remark:** Eventually, we shall not obtain the same initial rate of decay as in (2.5) for the weak solution that arises as the limit of solutions for the regularized problems (see next section). The reason is that, in the case of the regularized problem, we have been unable to obtain an inequality similar to (2.5) for all values of $n$. Therefore, as it has already been pointed out, we shall prove exponential decay directly for all $n \in (0, \infty)$. The role of Proposition 1 is to show, at least for classical solutions, that (1.3) and (1.2) provide sufficient information for obtaining rates of asymptotic decay to the mean value.

### 3 Weak solution, regularization and entropy

The theory of nonnegative weak solutions for (1.1) was introduced by Bernis and Friedman in [4] for $1 \leq n < \infty$. There are a number of definitions for nonnegative weak solutions of (1.1) with $u_0 \in H^1(I)$, $u_0 \geq 0$. The reference [9] covers the whole range $0 < n < \infty$ in that respect. What is important to us is that all these weak solutions arise as limits of classical solutions of the regularized problem first introduced by Bernis and Friedman in the original paper [4]. Let $f(s) = s^n$ and, for all $\varepsilon > 0$, define
\[
f_\varepsilon(s) := \frac{s^4 f(s)}{\varepsilon f(s) + s^4}, \quad s > 0.
\]
Of course, if we replace the first equation and the initial data in (1.1) by
\[
u_t + [f_\varepsilon(u)u_{xx}]_x = 0, \quad u(\cdot, 0) = u_0 + \delta(\varepsilon),
\]
we still have a degenerate problem. However, for $0 < n < 4$, $f_\varepsilon$ behaves like $s^4/\varepsilon$ for small $s > 0$. Thus, just as in the case $n = 4$, the authors of [4] proved that (3.2) with no-flux boundary conditions admits a positive classical solution $u_\varepsilon$ for any $\varepsilon > 0$. With appropriate choices of the constant $\delta(\varepsilon) > 0$, the same paper contains proofs of convergence as $\varepsilon \downarrow 0$ to a weak solution of (1.1) for a certain range of $n$. Bertozzi and Pugh [9] extended these convergence results to all $0 < n < \infty$. We will see in the next section that the sense in which $u_\varepsilon$ converges to $u$ always allows to infer $u_\varepsilon(\cdot, t) \rightharpoonup u(\cdot, t)$ strongly in $H^1(I)$ for a.e. $t > 0$.

While weak solutions of (1.1) generally do not possess enough regularity for the conclusion of Proposition 1 to hold, we plan to prove similar inequalities for the regularized problems, with enough control to pass to the limit as $\varepsilon \downarrow 0$. However, the analysis of (3.2) turns out to be somewhat complicated. To explain, the integration by parts that went very smoothly in the proof of Proposition 1 will be less elegant for (3.2) and will not provide a straight-forward answer, mainly due to the more sophisticated expression for $f_\varepsilon$. That is, we were told by the authors of [11], the reason why such an approach seems impractical if one tries to use the same regularized equations to prove asymptotic decay via the Laugesen-type functionals used in [11]. Such functionals are also Lyapunov functionals for our evolution but they have a more complicated expression than the energy $J$, the square of the derivative being integrated against a negative power (in a certain range) of the density itself [13].

Let us now take a closer look at the regularized problem (3.2) with no-flux boundary conditions. Standard parabolic Schauder estimates yield existence of a smooth, classical solution up to a finite
The entropy first considered by Bernis and Friedman in [4] is defined by

\[ C \]

The uniform bound on the one obtains \( \int \)

\[ \int_{0}^{t} f_{e}(u_{\varepsilon}(x, \tau)) u^{2}_{\varepsilon, xx}(x, \tau) d\tau = \int_{I} u^{2}_{0, x}(x) dx. \] (3.3)

The entropy first considered by Bernis and Friedman in [4] is defined by

\[ H_{\varepsilon}[u(\cdot, t)] := \int_{I} G_{\varepsilon}(u(x, t)) dx, \] (3.4)

where \( G_{\varepsilon}'(s) = 1/f_{e}(s) \) and the constants of integration are chosen so that \( H_{\varepsilon} \geq 0 \). To suit our purpose, we integrate twice from 0 to \( s \) to obtain

\[ G_{\varepsilon}(s) = \begin{cases} \frac{\varepsilon}{6s^{2}} + \frac{s^{2-n}}{(n-1)(n-2)} & \text{if } n \in (0, 1) \cup (1, 2) \cup (2, \infty), \\ \frac{\varepsilon}{6s^{2}} + s \log s - s & \text{if } n = 1, \\ \frac{\varepsilon}{6s^{2}} - \log s & \text{if } n = 2. \end{cases} \] (3.5)

Again for \( t > 0 \) in the smooth regime,

\[ H_{\varepsilon}[u_{\varepsilon}(\cdot, t)] + \int_{0}^{t} \int_{I} u^{2}_{\varepsilon, xx}(x, \tau) d\tau = H_{\varepsilon}[u_{0, \varepsilon}]. \] (3.6)

The uniform bound on the \( C^{0,1/2}\)-Hölder norm of \( u_{\varepsilon}(\cdot, t) \) given by (3.3) [4], along with the positive uniform lower bound on \( u_{\varepsilon} \) given by (3.6) [9], shows that \( u_{\varepsilon} \) is uniformly parabolic in the initial time interval of smoothness. Therefore, it may be extended indefinitely in time as a smooth classical solution. With the right choice of \( \delta(\varepsilon) \), one can then pass to the limit as \( \varepsilon \downarrow 0 \) and obtain a weak solution for (1.1) (see [4], [9]).

For fixed \( T > 0 \) define

\[ Q_{T} := I \times (0, T), \quad P_{T}(u) := \{ (x, t) \in Q_{T} \mid u(x, t) > 0 \}. \]

All test functions used in the sequel are in

\[ \mathcal{F} := C^{\infty}_{c}((0, T); C^{\infty}(I)). \]

There are, as already mentioned, multiple definitions of weak solutions, depending on the regularity one can show that the limit of \( \{ u_{\varepsilon} \}_{\varepsilon>0} \) inherits. Different ranges for \( n \) dictate this regularity. The strongest notion of weak solution involves two integrations by parts

\[ \iint_{Q_{T}} u_{\varepsilon} d\tau dt = \iint_{Q_{T}} \left[ u^{n} u_{x} \zeta_{xx} + nu^{n-1} u_{x} u_{xx} \zeta_{x} \right] d\tau dt, \quad \zeta \in \mathcal{F}. \]

A weaker formulation transfers one more derivative on \( \zeta \)

\[ \iint_{Q_{T}} u_{\varepsilon} \zeta d\tau dt = - \iint_{Q_{T}} \left[ u^{n} u_{x} \zeta_{xx} + \frac{3m}{2} u^{n-1} u_{x}^{2} \zeta_{xx} + \frac{n(n-1)}{2} u^{n-2} u_{x}^{3} \zeta_{x} \right] d\tau dt, \]

whereas the weakest version reads

\[ \iint_{Q_{T}} u_{\varepsilon} \zeta d\tau dt = - \iint_{P_{T}(u)} u^{n} u_{x} \zeta_{x} dx dt. \] (3.7)

An all encompassing result in [9] (adapted to the no-flux boundary conditions) states:
Theorem 1. (Bertozzi-Pugh [9]) Consider $0 < n < \infty$ and an initial probability density $u_0 \in H^1(I)$. Let $u_\varepsilon$ be the unique smooth solution of (3.2) with no-flux boundary conditions and

$$\delta(\varepsilon) = \varepsilon^\theta \text{ for some } 0 < \theta < 2/5.$$ 

Then there exists a subsequence (not relabelled) \{\{u_\varepsilon\}_{P_\varepsilon}\} which converges pointwise uniformly and weakly in $L^2((0,T); H^2(I))$ and $L^\infty((0,T); H^1(I))$ to some $u \in L^2((0,T); H^2(I)) \cap L^\infty((0,T); H^1(I))$.

Moreover, $u$ is a solution for (1.1) in the sense of (3.7).

Since our goal is to prove, via the regularized solutions, the decay in the $H^1$ norm to the constant equilibrium solution $\bar{u} \equiv 1/2$, we first seek to prove convergence of $\|u_\varepsilon\|_{H^1(I)}$ to $\|u\|_{H^1(I)}$. Theorem 1 gives the uniform convergence of $u_\varepsilon$ to $u$. As for the derivatives, we have the following proposition:

Proposition 2. Let $u_0 \in H^1(I)$ be a probability density. Assume that $u_0$ satisfies

$$\int_I |\log u_0| dx < +\infty \text{ if } n = 2$$

or

$$\int_I u_0^{2-n} dx < +\infty \text{ if } n > 2.$$ 

Then there exists a subsequence of $\{u_\varepsilon\}_{P_\varepsilon}$ such that

$$u_\varepsilon,\alpha(\cdot, t) \to u_\alpha(\cdot, t) \text{ strongly in } L^2(I), \text{ for a.e. } t > 0.$$ 

Remark: Note that we have no restrictions on $u_0$ for $0 < n < 2$.

Most of the entropy properties from the proof below are already in the literature. However, since this result is central to our approach we present full details here.

Proof of Proposition 2: As in [4], [9], we would like to make use of a deep result due to Aubin-Lions [14]. The starting point is (3.6). We will show that $H_\varepsilon[u_\varepsilon(\cdot, t)]$ is bounded from below and $H_\varepsilon[u_0,\varepsilon]$ is bounded from above independently of $\varepsilon$ and $t$. These imply, by (3.6),

$$\int_0^\infty \int_I u_\varepsilon^2,xx(x,t) dx dt < C$$

for some $C > 0$ independent of $\varepsilon$. From (3.3) we also deduce that

$$\int_0^\infty \int_I f_\varepsilon(u_\varepsilon(x,t)) u_\varepsilon^2,xxx(x,t) dx dt$$

is uniformly bounded.

Since $0 \leq f_\varepsilon(s) \leq f(s)$, we get $f_\varepsilon(u_\varepsilon) \leq M^n$, where $M$ is the upper bound for $u_\varepsilon$ for all $0 < \varepsilon < 1$. Indeed, it is trivial to show that such a uniform upper bound exists. One just uses the conservation of mass

$$\int_I u_\varepsilon(x,t) dx = 1 + 2\varepsilon^\theta, \text{ for all } t > 0$$

to infer

$$u_\varepsilon(x,t) - \frac{1}{2} - \varepsilon^\theta \leq \sqrt{2}\|u_\varepsilon,\alpha(\cdot, t)\|_{L^2(I)} \leq \sqrt{2}\|u_0\|_{L^2(I)}, \text{ } x \in I, \text{ } t > 0.$$ 

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Thus, the uniform $L^2$ bound on $f^{1/2}_\varepsilon (u_\varepsilon) u_{\varepsilon,xxx}$ implies a uniform $L^2$ bound on $f_\varepsilon (u_\varepsilon) u_{\varepsilon,xxx}$, i.e. $\partial u_\varepsilon / \partial t$ is uniformly bounded in $L^2((0, \infty); H^{-1}(I))$. On the other hand, due to (3.11), a subsequence \{ $u_\varepsilon$ \}_{\varepsilon \downarrow 0} converges weakly in $L^2((0, \infty); H^1(I))$ to $u$. The Aubin-Lions lemma guarantees the existence of a subsequence that converges strongly in $L^2((0, \infty); H^1(I))$ to $u$. Therefore, all that is left to prove are the uniform lower and upper bounds on $H_\varepsilon [u_\varepsilon (\cdot, t)]$ and $H_\varepsilon [u_{0, \varepsilon}]$, respectively. We have

$$H_\varepsilon [u] = \begin{cases} \int_I \left[ \frac{\varepsilon}{6u^2} + \frac{u^{2-n}}{(n-1)(n-2)} \right] dx & \text{if } n \in (0, 1) \cup (1, 2) \cup (2, \infty), \\ \int_I \left( \frac{\varepsilon}{6u^2} + u \log u - u \right) dx & \text{if } n = 1, \\ \int_I \left( \frac{\varepsilon}{6u^2} - \log u \right) dx & \text{if } n = 2. \end{cases}$$

Since $0 \leq u_\varepsilon \leq M < \infty$ uniformly in $\varepsilon$, it is easy to see that $u_\varepsilon^{2-n}/[(n-1)(n-2)] \geq 0$ if $n < 1$ or $n > 2$. If $1 < n < 2$, we have $u_\varepsilon^{2-n}/[(n-1)(n-2)] \geq 0 \geq M^2-n/[n-1)(n-2)]$. For the cases $n = 1, 2$, note that function $z \mapsto z \log z$ is bounded and $z \mapsto \log z$ is bounded from above on $[0, M]$. Also, we take into account (3.12) for $n = 1$. We have thus shown that $H_\varepsilon [u_\varepsilon]$ is bounded from below independently of $0 < \varepsilon < 1$.

As for the upper bound on $H_\varepsilon [u_{0, \varepsilon}]$, note that $\varepsilon u_{0, \varepsilon}^2 \leq \varepsilon^{1-2\theta} < 1$ for $0 < \varepsilon < 1$. Due to part of the observations above and, whenever appropriate, the conditions (3.8) and (3.9), we obtain the desired uniform bounds. \hfill \Box

Remark: In fact, we do not need the full strength of (3.10). Therefore, we may avoid using the Aubin-Lions compactness result by using the fact, stated by Theorem 1, that \{ $u_\varepsilon$ \}_{\varepsilon \downarrow 0} converges to $u$ pointwise uniformly and weakly in $L^\infty((0, T); H^1(I))$. Indeed, since $u_\varepsilon$ are smooth and uniformly bounded, we first deduce that $u$ is continuous (noted in [9]) and bounded on $[-1, 1] \times (0, T)$. Now, since $u \in L^\infty(0, T; H^1(I))$, if we fix $t_0 \in (0, T)$, then we can find $t_n \rightarrow t_0$ a time sequence converging to $t_0$ such that $u(\cdot, t_n) \in H^1(I)$ and $\| u(\cdot, t_n) \|_{H^1(I)}$ are uniformly bounded with respect to $n$. If $\xi \in C_c^\infty (I)$, we can pass to the limit up to a subsequence in

$$\int_I u(x, t_n) \xi(x) dx = - \int_I u_\varepsilon (x, t_n) \xi(x) dx$$

to obtain that $u(\cdot, t_0) \in H^1(I)$. Indeed, the left hand side converges trivially and the right hand side converges up to a subsequence due to the fact that $\{ u_\varepsilon (\cdot, t_n) \}_{\varepsilon}$ is bounded in $L^2(I)$. Also, $u(\cdot, t_0)$ is bounded, thus in $L^2(I)$. Therefore, the claim is proved and, since $t_0$ was arbitrary, we conclude $u(\cdot, t) \in H^1(I)$ for all $t \in (0, T)$. Now, the uniform pointwise convergence of $u_\varepsilon$ to $u$ also implies

$$\int_I u_\varepsilon (x, t) \xi(x) dx \rightarrow \int_I u(x, t) \xi(x) dx$$

for all $t \in (0, T)$ and any $\xi \in C_c^\infty (I)$ which is equivalent to

$$u_{\varepsilon, \varepsilon} (\cdot, t) \rightarrow u_\varepsilon (\cdot, t) \in L^2(I) \text{ as } \varepsilon \downarrow 0.$$
4 Asymptotic exponential decay

4.1 Case 0 < \( n < 1 \) or \( n > 2 \)

This case turns out to be easier than the other. Looking back at the expression of the entropy \( H_\varepsilon \) for \( n \in (0, 1) \cup (2, \infty) \), i.e.
\[
H_\varepsilon[u] = \int_I \left[ \frac{\varepsilon}{6u^2} + \frac{u^{2-n}}{(n-1)(n-2)} \right] dx,
\]
we observe that both terms are positive in this case. Thus, it is easy to see that
\[
H_\varepsilon[u] \geq \begin{cases} 
\frac{1}{6} \int_I \left( \frac{\varepsilon}{u^2} + u^{2-n} \right) dx & \text{if } n \in (0, 1) \cup (2, 4], \\
\frac{1}{(n-1)(n-2)} \int_I \left( \frac{\varepsilon}{u^2} + u^{2-n} \right) dx & \text{if } n > 4.
\end{cases}
\]

Thus, according to (3.6),
\[
H_\varepsilon[u_0, \varepsilon] \geq \begin{cases} 
\frac{1}{6} \int_I \left( \frac{\varepsilon}{u_0^2} + u_0^{2-n} \right) dx & \text{if } n \in (0, 1) \cup (2, 4], \\
\frac{1}{(n-1)(n-2)} \int_I \left( \frac{\varepsilon}{u_0^2} + u_0^{2-n} \right) dx & \text{if } n > 4.
\end{cases}
\]

But, as we already discussed in the proof of Proposition 2,
\[
J[u_0, \varepsilon] = \int_I (\frac{\varepsilon}{u_0^2} + u_0^{2-n}) dx,
\]
which gives us a uniform upper bound on \( H_\varepsilon[u_\varepsilon] \) as \( \varepsilon \downarrow 0 \) provided that (3.9) holds.

We are now ready to prove:

**Theorem 2.** Assume \( n \in (0, 1) \cup (2, \infty) \) and \( u_0 \in H^1(I) \) is a probability density satisfying (3.9). There exists a constant \( C > 0 \) depending only on \( J_\varepsilon u_0^{2-n} dx \) and \( n \) such that the solution \( u \) of (1.1) given by Theorem 1 satisfies
\[
J[u(\cdot, t)] \leq J[u_0] \exp\left\{ -Ct \right\} \text{ for all } t > 0.
\]

**Proof:** Let \( t > 0 \) be given. Note that, since \( u_\varepsilon \) is a classical solution for (3.2), we infer, just as in deducing (1.2), that
\[
\frac{d}{dt} J[u_\varepsilon(\cdot, t)] = -\int_I f_\varepsilon(u_\varepsilon(t)) u_{\varepsilon,xxx}(x,t) dx.
\]

Next we shall apply Lemma 1 with \( v \equiv u_\varepsilon(\cdot, t) \) and \( P \equiv f_\varepsilon \). Note that
\[
\frac{u_0^2}{f_\varepsilon(u_\varepsilon(\cdot, t))} = \frac{\varepsilon}{u_0^2(\cdot, t)} + u_\varepsilon^{2-n}(\cdot, t).
\]

Thus, according to (4.1) and (2.2), one has
\[
c_n H_\varepsilon[u_{\varepsilon,0}] \int_I f_\varepsilon(u_\varepsilon(t)) u_{\varepsilon,xxx}(x,t) dx \geq \frac{9}{16} \left( \int_I u_{\varepsilon,x}(x,t) dx \right)^2 = \frac{9}{4} J^2[u_\varepsilon(\cdot, t)],
\]

(4.4)
where \( c_n = 6 \) if \( n \in (0, 1) \cup (2, 4) \) and \( c_n = (n - 1)(n - 2) \) if \( n > 4 \). Now, (4.3) implies
\[
\frac{d}{dt} J[u_\epsilon(\cdot, t)] \leq -\frac{9}{4cnH_\epsilon[u_\epsilon,0]} J^2[u_\epsilon(\cdot, t)]. \tag{4.5}
\]
Next, let \( \lambda > 0 \) to be fixed later. Note that (4.4) implies
\[
\int_I f_\epsilon(u_\epsilon(x,t))u^2_{\epsilon,xxx}(x,t)dx \geq \frac{9\lambda}{4cnH_\epsilon[u_\epsilon,0]} J[u_\epsilon(\cdot, t)] \text{ if } J_\epsilon[u_\epsilon(\cdot, t)] \geq \lambda. \tag{4.6}
\]
Let \( v \in H^3(I) \) with \( \int_I vdx = r > 0 \) and \( v_\epsilon(\pm 1) = 0 \). If \( J[v] < \lambda \), then the Poincaré inequality yields
\[
\inf v > \frac{r}{2} - 2\sqrt{\lambda} > 0 \text{ if } \lambda < \frac{r^2}{16}.
\]
Thus, if \( 0 \leq J[v] < \lambda < r^2/16 \), one has
\[
\int_I f_\epsilon(v)v^2_{xxx}dx \geq f_\epsilon(r/2 - 2\sqrt{\lambda}) \int_I v^2_{xxx}dx
\geq \frac{\pi^4}{16} f_\epsilon(r/2 - 2\sqrt{\lambda}) \int_I v^2_xdx
= \frac{\pi^4}{8} f_\epsilon(r/2 - 2\sqrt{\lambda}) J[v]. \tag{4.7}
\]
Indeed, we used the fact that \( f_\epsilon \) is increasing on \([0, \infty)\) for any \( \varepsilon > 0 \), and the inequalities (the second one with optimal Poincaré constant)
\[
\frac{\int_I v^2_{xxx}dx}{\int_I v^2_xdx} \geq \left( \int_I v^2_{xx}dx \right)^2
\geq \frac{\pi^2}{4} \left( \int_I v^2_xdx \right)^2,
\]
due to \( v \in H^3(I) \) and \( v_\epsilon(\pm 1) = 0 \). If we apply (4.7) to \( v \equiv u_\epsilon \) and use (4.6), we infer
\[
\int_I f_\epsilon(u_\epsilon(x,t))u^2_{\epsilon,xxx}(x,t)dx \geq \sup_{\lambda \in [0, r_\epsilon^2/32]} \min \left\{ \frac{9\lambda}{8cnH_\epsilon[u_\epsilon,0]}, \frac{\pi^4}{8} f_\epsilon(r_\epsilon/2 - 2\sqrt{\lambda}) \right\} J[u_\epsilon(\cdot, t)], \tag{4.8}
\]
where \( r_\epsilon := 1 + 2\varepsilon^6 \). As \( \lambda \to r_\epsilon/2 - 2\sqrt{\lambda} \) is decreasing on \([0, r_\epsilon^2/16] \), we conclude that \( \lambda \to f_\epsilon(r_\epsilon/2 - 2\sqrt{\lambda}) \) is decreasing on \([0, r_\epsilon^2/16] \) being zero at \( r_\epsilon^2/16 \). It follows that, if for any \( \varepsilon > 0 \) we pick an arbitrary constant \( A_\varepsilon > 0 \), the quantity
\[
C_\varepsilon := \sup_{0 \leq \lambda \leq r^2/16} \min \{ A_\varepsilon \lambda, \pi^4 f_\epsilon(r_\epsilon/2 - 2\sqrt{\lambda}) \}
\]
is attained at the unique solution \( \lambda_{0,\varepsilon} \in [0, r_\varepsilon^2/16] \) of \( 8A_\varepsilon \lambda = \pi^4 f_\epsilon(r_\epsilon/2 - 2\sqrt{\lambda}) \). We now choose \( A_\varepsilon := 9/\{8cnH_\epsilon[u_\epsilon,0]\} \) to deduce that (4.8) implies
\[
\int_I f_\epsilon(u_\epsilon(x,t))u^2_{\epsilon,xxx}(x,t)dx \geq C_\varepsilon J[u_\epsilon(\cdot, t)]. \tag{4.9}
\]
Thus, by (4.3),
\[
J[u_\epsilon(\cdot, t)] \leq J[u_{0,\varepsilon}] \exp \{- C_\varepsilon t\} \text{ for all } t > 0.
\]
But $J[u_0, \varepsilon] = J[u_0]$ and
\[
H_\varepsilon[u_{\varepsilon, 0}] \to \frac{1}{(n-1)(n-2)} \int_I u_0^{2-n} \, dx \text{ as } \varepsilon \downarrow 0,
\]
which means
\[
A_\varepsilon \to \frac{9(n-1)(n-2)}{4c_n \int_I u_0^{2-n} \, dx} =: A \text{ as } \varepsilon \downarrow 0.
\]
Also, $f_\varepsilon(s) \to \pi s$ and $r_\varepsilon \to 1$ as $\varepsilon \downarrow 0$. Therefore, if we denote by $\lambda_0$ the unique solution in $(0, 1/16)$ of the equation
\[
8 A \lambda = \frac{\pi}{4} \left( \frac{1}{2} - 2\sqrt{\lambda} \right)^n,
\]
we obtain (4.2) by letting $\varepsilon \downarrow 0$ and using (3.14). The decay constant is
\[
C = \lim_{\varepsilon \downarrow 0} C_\varepsilon = A \lambda_0 = \frac{\pi^4}{8} \left( \frac{1}{2} - 2\sqrt{\lambda_0} \right)^n.
\]

Remark: Note that we could have continued from (4.5) by a different argument. Indeed, (4.5) gives
\[
J[u_{\varepsilon}(\cdot, t)] \leq J[u_{0, \varepsilon}][1 + A_\varepsilon J[u_{0, \varepsilon}]t]^{-1}.
\]
We pass to the limit to infer
\[
J[u(\cdot, t)] \leq J[u_0][1 + A J[u_0]t]^{-1}.
\]
Next, as explained in the introduction (also, see [11]), one may easily infer that, once $J[u(\cdot, t)]$ becomes small enough, exponential decay sets in (see the last section for details).

4.2 Case $0 < n \leq 2$

Let us begin with the following algebraic lemma.

**Lemma 2.** For any $2 \leq m < 4$, let $p$ be fixed in the interval $[5/2 - 4/m, 3/2)$. Then, for any $\varepsilon > 0$ one has
\[
\varepsilon^2 (\varepsilon + s^m)^{-5/2} < C \varepsilon^\omega s^{\beta - 2} \text{ for all } s \geq 0,
\]
where $C = 2^{p-5/2}$, and
\[
\omega = 3/4 - p/2 > 0 \text{ and } 1 > \beta = m(2p - 5)/4 + 2 \geq 0.
\]

**Proof:** We write
\[
\varepsilon^2 (\varepsilon + s^m)^{-5/2} = \left( \frac{\varepsilon}{\varepsilon + s^m} \right)^{p-5/2} \varepsilon^{2-p(\varepsilon + s^m)^{p-5/2}}
\leq \varepsilon^2 \varepsilon^{2-p(\varepsilon + s^m)^{p-5/2}} \leq 2^{p-5/2} \varepsilon^{3/4 - p/2} s^{m(2p-5)/4}.
\]
Note that we have used $\varepsilon \leq \varepsilon + s^m$ and $p > 0$ for the first inequality. The second inequality follows from $\varepsilon + s^m \geq 2\varepsilon^{1/2}s^{m/2}$ and the fact that $p < 5/2$. Also, due to $m \in [2, 4)$ and $p \in [5/2 - 4/m, 3/2)$, (4.12) becomes obvious. \qed
The next lemma appears, essentially, in [15]. We only make the observation that the restriction $\beta \in (0,1)$ may be eliminated. The proof is exactly as in [15].

**Lemma 3.** For any $\beta \in \mathbb{R}$ and any strictly positive $v \in H^2(I)$ with $v_x(\pm 1) = 0$, one has

$$
\int_I v^\beta v_x^2 \, dx \geq \frac{1}{9} (1 - \beta)^2 \int_I v^{\beta-2} v_x^4 \, dx.
$$

(4.13)

**Proof:** Since

$$
\int_I (\lambda v^{\beta/2} v_x + v^{\beta/2-1} v_x^2)^2 \, dx \geq 0
$$

for all $\lambda \in \mathbb{R}$, we conclude that the discriminant of the quadratic polynomial in $\lambda$ obtained by expanding the square is nonpositive. Taking into account the boundary conditions on the derivative, we conclude by observing that

$$
\int_I v^{\beta-1} v_x v_x^2 \, dx = \frac{1}{3} \int_I v^{\beta-1} [v_x]^3 \, dx = \frac{1 - \beta}{3} \int_I v^{\beta-2} v_x^4 \, dx.
$$

□

**Remark:** First, note that $\beta = 1$ trivially upholds the inequality. Secondly, note that the solutions $u_x$ at fixed time levels satisfy (4.13) since they satisfy the boundary conditions and are strictly positive [9] on $I$.

**Lemma 4.** For any $v \in H^2(I)$ of positive mass that satisfies $v_x(\pm 1) = 0$, one has

$$
\int_I v^2 v_x^2 \, dx \geq \frac{\pi^2}{16(\pi + 1)^2} \left( \int_I v \, dx \right)^2 \int_I v_x^2 \, dx.
$$

(4.14)

**Proof:** Let

$$
\int_I v \, dx =: r > 0.
$$

The Schwarz inequality gives, after one integration by parts,

$$
2 \int_I v^2 v_x^2 \, dx \geq \left( \int_I v_x^2 \, dx \right)^2.
$$

Thus, for any $0 < \lambda < r^2/8$,

$$
\int_I v^2 v_x^2 \, dx \geq \frac{\lambda}{2} \int_I v_x^2 \, dx \quad \text{whenever} \quad \int_I v_x^2 \, dx \geq \lambda.
$$

(4.15)

Next, observe that

$$
\sup \{|v - r/2|^2\} \leq 2 \int_I v_x^2 \, dx.
$$

It follows

$$
\inf v \geq r/2 - \sqrt{2\lambda} > 0 \quad \text{provided that} \quad \int_I v_x^2 \, dx \leq \lambda.
$$

Consequently, with the help of the standard Poincaré inequality for $v_x$, we obtain

$$
\int_I v^2 v_x^2 \, dx \geq \frac{\pi^2}{4} \left( \frac{r}{2} - \sqrt{2\lambda} \right)^2 \int_I v_x^2 \, dx \quad \text{whenever} \quad \int_I v_x^2 \, dx < \lambda.
$$

(4.16)

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Combining (4.15) and (4.16), we deduce
\[ \int_I v^2 v_{x}^2 \, dx \geq \frac{1}{2} \min \left\{ \lambda, \frac{\pi^2}{2} \left( \frac{r}{2} - \sqrt{2\lambda} \right)^2 \right\} \int_I v^2 \, dx \text{ for all } \lambda \in (0, r^2/8). \] (4.17)

It is easy to see that
\[ \sup_{0 < \lambda < r^2/8} \min \left\{ \lambda, \frac{\pi^2}{2} \left( \frac{r}{2} - \sqrt{2\lambda} \right)^2 \right\} = \frac{\pi^2 r^2}{8(\pi + 1)^2}, \]
which, taking (4.17) into account, concludes the proof. \(\square\)

Before we state the next proposition, let us make available some useful identities. First, denote
\[ g_{\varepsilon}(s) := f_{\varepsilon}^{1/2}(s) = s^2(\varepsilon + s^m)^{-1/2}, \text{ where } m := 4 - n. \]
The first derivative, i.e.
\[ g'_{\varepsilon}(s) = [2\varepsilon s + (2 - m/2)s^{m+1}](\varepsilon + s^m)^{-3/2}, \]
gives the second
\[ g''_{\varepsilon}(s) = [2\varepsilon^2 + \varepsilon(8 - 3m - m^2)s^{m}/2 + (1 - m/2)(2 - m/2)s^{2m}](\varepsilon + s^m)^{-5/2}. \] (4.18)

We now have all the necessary tools to prove the main result.

**Theorem 3.** Assume \(0 < n \leq 2\) and \(u_0 \in H^1(I)\) is a probability density. There exists a constant \(C > 0\) depending only on \(J[u_0]\) and \(n\) such that the solution \(u\) of (1.1) given by Theorem 1 satisfies
\[ J[u(\cdot, t)] \leq J[u_0] \exp\{-Ct\} \text{ for all } t > 0. \] (4.19)

**Proof:** For \(t \geq 0\) denote
\[ J_{\varepsilon}(t) := \frac{1}{2} \int_I u_{\varepsilon,x}^2(x,t) \, dx, \quad I_{\varepsilon}(t) := \int_I f_{\varepsilon}(u_{\varepsilon}(x,t))u_{\varepsilon,x,x,x}(x,t) \, dx. \]
The Schwarz inequality implies
\[ (2I_{\varepsilon}J_{\varepsilon})^{1/2} \geq -\int_I f_{\varepsilon}^{1/2}(u_{\varepsilon})u_{\varepsilon,x}u_{\varepsilon,x,x} \, dx = -\int_I g_{\varepsilon}(u_{\varepsilon})u_{\varepsilon,x}u_{\varepsilon,x,x} \, dx, \]
where, for notational simplification, we dropped the time dependence. On the other hand, integration by parts taking into account the boundary conditions leads to
\[ \int_I g_{\varepsilon}(u_{\varepsilon})u_{\varepsilon,x,x}u_{\varepsilon,x} \, dx = -\int_I g_{\varepsilon}(u_{\varepsilon})u_{\varepsilon,x,x}^2 \, dx - \int_I g_{\varepsilon}'(u_{\varepsilon})u_{\varepsilon,x}u_{\varepsilon,x,x} \, dx = -\int_I g_{\varepsilon}(u_{\varepsilon})u_{\varepsilon,x,x}^2 \, dx + \frac{1}{3} \int_I g_{\varepsilon}''(u_{\varepsilon})u_{\varepsilon,x,x}^2 \, dx. \] (4.20)
Thus, (4.18) yields
\[ (2I_{\varepsilon}J_{\varepsilon})^{1/2} \geq \int_I g_{\varepsilon}(u_{\varepsilon})u_{\varepsilon,x,x}^2 \, dx + \frac{\varepsilon}{6} (m^2 + 3m - 8) \int_I u_{\varepsilon}^m(\varepsilon + u_{\varepsilon}^m)^{-5/2}u_{\varepsilon,x}^4 \, dx \]
\[ - \frac{1}{3} (1 - m/2)(2 - m/2) \int_I u_{\varepsilon}^{2m}(\varepsilon + u_{\varepsilon}^m)^{-5/2}u_{\varepsilon,x}^4 \, dx - \frac{2}{3} \varepsilon \int_I (\varepsilon + u_{\varepsilon}^m)^{-5/2}u_{\varepsilon,x}^4 \, dx. \]
Since $2 \leq m < 4$, we deduce that the two terms in the middle of the right hand side are nonnegative. Thus, 
\begin{equation}
(2I_\varepsilon J_\varepsilon)^{1/2} \geq \int_I g_\varepsilon(u_\varepsilon)u_{\varepsilon,x,x}^2 \, dx - \frac{2}{3} \varepsilon^2 \int_I (\varepsilon + u_\varepsilon^m)^{-5/2} u_{\varepsilon,x}^4 \, dx. \tag{4.21}
\end{equation}

Let 
\[ M_\varepsilon := \frac{1}{2} + \varepsilon^0 + \sqrt{2}\|u_0\|_{L^2(I)}. \]

According to (3.13), $M_\varepsilon$ is an upper bound for $u_\varepsilon(\cdot, t)$ uniform with respect to $t$. Thus, 
\[ u_{\varepsilon}^2(\cdot, t)(\varepsilon + u_\varepsilon^m(\cdot, t))^{-1/2} \geq (\varepsilon + M_\varepsilon^m)^{-1/2} u_{\varepsilon}^2(\cdot, t) \text{ for all } t > 0. \]

For the second term in the right hand side of (4.21) we apply (4.11) from Lemma 2 with $p = 5/2 - 4/m$ to obtain 
\[ \varepsilon^2 \int_I (\varepsilon + u_\varepsilon^m)^{-5/2} u_{\varepsilon,x,x}^2 \, dx < 2^{-4/m} \varepsilon^\omega \int_I u_{\varepsilon,x}^{-2} u_{\varepsilon,x}^4 \, dx, \]
where, according to (4.12), $\omega = (4 - m)/(2m) = n/[2(4 - n)] > 0$. These observations, together with (4.21), lead to 
\begin{equation}
(2I_\varepsilon J_\varepsilon)^{1/2} \geq (\varepsilon + M_\varepsilon^m)^{-1/2} \int_I u_{\varepsilon,x,x}^2 \, dx - \frac{2^{1-4/m}}{3} \varepsilon^\omega \int_I u_{\varepsilon,x}^{-2} u_{\varepsilon,x}^4 \, dx. \tag{4.22}
\end{equation}

Denote 
\[ c := 2^{1-4/m}/27 \text{ and } K_\varepsilon := \pi^2(\varepsilon + M_\varepsilon^m)^{-1/2}(1 + 2\varepsilon^0)^2/[16(\pi + 1)^2] \]
to infer, using (4.14), (4.13) and (4.22), 
\[ (2I_\varepsilon J_\varepsilon)^{1/2} \geq 2K_\varepsilon J_\varepsilon - c\varepsilon^\omega \int_I u_{\varepsilon,x,x}^2 \, dx. \]

It follows that 
\[ J_\varepsilon^{1/2} \leq \frac{1}{4K_\varepsilon} \left[ 2^{1/2}I_\varepsilon^{1/2} + \left( 2K_\varepsilon + 8c\varepsilon^\omega K_\varepsilon \int_I u_{\varepsilon,x,x}^2 \, dx \right)^{1/2} \right], \]
which implies 
\begin{equation}
2K_\varepsilon^2 J_\varepsilon \leq I_\varepsilon + 2c\varepsilon^\omega K_\varepsilon \int_I u_{\varepsilon,x,x}^2 \, dx. \tag{4.23}
\end{equation}

The combination of this inequality and (4.3) gives 
\begin{equation}
\frac{d}{dt} J_\varepsilon(t) \leq -2K_\varepsilon^2 J_\varepsilon(t) + 2c\varepsilon^\omega K_\varepsilon \int_I u_{\varepsilon,x,x}^2 \, dx. \tag{4.24}
\end{equation}

A classical version of Gronwall’s lemma applies to yield 
\begin{equation}
J_\varepsilon(t) \leq J_\varepsilon(0) \exp\left\{ -2K_\varepsilon^2 t \right\} + 2c\varepsilon^\omega K_\varepsilon \int_0^t \exp\left[ -2K_\varepsilon^2 (t - \tau) \right] \int_I u_{\varepsilon,x,x}(x, \tau) \, dx \, d\tau
\leq J_\varepsilon(0) \exp\left\{ -2K_\varepsilon^2 t \right\} + 2c\varepsilon^\omega K_\varepsilon \int_0^t \int_I u_{\varepsilon,x,x}(x, \tau) \, dx \, d\tau.
\end{equation}

In view of $\omega > 0$, (3.11), and the fact that 
\[ \lim_{\varepsilon \to 0} K_\varepsilon = \lim_{\varepsilon \to 0} \pi^2(\varepsilon + (1/2 + \varepsilon^0 + 2^{1/2}\|u_0\|_{L^2(I)})^m)^{-1/2}(1 + 2\varepsilon^0)^2/[16(\pi + 1)^2] = \pi^2(1/2 + 2^{1/2}\|u_0\|_{L^2(I)})^{-2+n/2}/[16(\pi + 1)^2] =: K, \]
we may pass to the limit as $\varepsilon \downarrow 0$ to deduce, using (3.14), 
\begin{equation}
J[u(\cdot, t)] \leq J[u_0] \exp\left\{ -2K_\varepsilon^2 t \right\} \text{ for all } t > 0. \tag{4.25}
\end{equation}

Therefore, (4.19) holds with $C = 2K_\varepsilon^2$. \qed
Remark: As soon as $J[u]$ is driven sufficiently small, the solution becomes strictly positive and, thus, classical in finite time. Upper bounds for the time at which the solution becomes classical and stays that way may be explicitly computed in terms of the constants given by Theorem 2 and Theorem 3.

Remark: Obviously, our analysis extends to the case of periodic boundary conditions.

Remark: Note that (4.20) suggests that things would be much simpler if $g_\varepsilon$ were concave. Thus, it is natural to ask whether one could use different regularizing functions $f_\varepsilon$ such that $g_\varepsilon = f_\varepsilon^{1/2}$ is concave. Nevertheless, it is quite easy to see that this is impossible. Indeed, even though the limiting function $g_\varepsilon(s) = s^{n/2}$ is concave for $n \in (0, 2]$, note that the main feature of $f_\varepsilon$ as a regularizing function [4] is that $f_\varepsilon(s)/s^4 \to 1/\varepsilon$ as $s \downarrow 0$. However, if $g_\varepsilon$ were concave, then $f_\varepsilon(s)/s^4$ would diverge as $s \downarrow 0$.

5 A relaxed problem

Let us now consider, for some $\sigma > 0$ and $0 < \nu < \infty$, the problem

$$
\begin{aligned}
    u_t &= -(u^n u_{xxx})_x + \sigma (u^\nu)_{xx} \quad \text{in } Q := I \times (0, \infty), \\
    u_x &= 0 \quad \text{on } \partial I \times (0, \infty), \\
    u_{xxx} &= 0 \quad \text{on } \partial I \times (0, \infty), \\
    u(\cdot, 0) &= u_0 \quad \text{in } I.
\end{aligned}
$$

This problem with $\sigma = 1$ and $1 < \nu < 2$ is considered by Bertozzi and Pugh in [8], the addition of the second-order term being justified by its behavior as a cut-off of van der Waals interactions. If $\sigma = 0$, then we are, obviously, back to (1.1).

If we remove the fourth-order term in the equation above, then we end up with the classical porous media problem for which there exists a well developed theory; we have uniqueness of weak solutions, a maximum/minimum (comparison) principle, finite-speed support propagation etc. [18], [20]. In fact, if the initial data is bounded from below away from zero on $I$, then the weak solution becomes a classical one for all positive times and does not drop below the initial lower bound. If $\nu \in (1/2, 2]$, then one can show (it follows from our arguments below by removing the higher-order terms) that it asymptotically decays to its mean value in the $H^1$ norm. Thus, we do not expect (5.1) to asymptotically behave “worse” than (1.1).

If we consider a positive classical solution for (5.1), then

$$
\frac{1}{2} \frac{d}{dt} \int_I u^2_t \, dx = - \int_I u^n u_{xxx}^2 \, dx - \nu \sigma \int_I u^{\nu-1} u_{xx}^2 \, dx + \frac{1}{3} \nu(\nu - 1)(\nu - 2) \sigma \int_I u^{\nu-3} u_x^4 \, dx. \quad (5.2)
$$

Clearly, $J$ is a Lyapunov functional for nonnegative smooth solutions of (5.1) provided that $1 \leq \nu \leq 2$. In fact, this remains true even if we extend the range of $\nu$ to $[0.5, 2]$ (we thank the referee for this observation). Indeed, if $\nu < 2$, then (5.2) may be rewritten as

$$
\frac{1}{2} \frac{d}{dt} \int_I u^2_t \, dx = - \int_I u^n u_{xxx}^2 \, dx - \nu \sigma \frac{2\nu - 1}{2 - \nu} \int_I u^{\nu-1} u_{xx}^2 \, dx - \nu \sigma \frac{3(1 - \nu)}{2 - \nu} \left[ \int_I u^{\nu-1} u_{xx}^2 \, dx - \frac{(2 - \nu)^2}{9} \int_I u^{\nu-3} u_x^4 \, dx \right]. \quad (5.3)
$$

We can apply Lemma 3 with $v \equiv u(\cdot, t)$ and $\beta = \nu - 1$ to conclude

$$
\int_I u^{\nu-1} u_{xx}^2 \, dx \geq \frac{(2 - \nu)^2}{9} \int_I u^{\nu-3} u_x^4 \, dx. \quad (5.4)
$$
Thus, in view of (5.3), the right hand side of (5.2) is still negative even if \( \nu \in [0.5, 1] \).

Before estimating the new terms appearing in the right hand side of (5.2) as a consequence of the perturbation, let us check whether the assertions in Proposition 1 are still valid. Clearly, (i) and (ii) are unaffected. As to (iii), we observe that (by repeating the integration performed in the proof of Proposition 1 for the fourth order term)

\[
\frac{d}{dt} \int J\left[u^{\nu-1}u_{xx}^2\right] \, dx = (2 - n) \int u^{1-n}(x,t)u_t(x,t) \, dx \\
= -(n-2)(n-1) \int u_{xxt}^2(x,t) \, dx + (2-n)\sigma \int u^{1-n}(x,t)(u^\nu(x,t))_{xx} \, dx \\
= -(n-2)(n-1) \left[ \int u_{xxt}^2(x,t) \, dx + \sigma \int u^{\nu-n}(x,t)u_x(x,t)(u^\nu(x,t))_{x} \, dx \right].
\]

The last integral in the display above yields

\[
\int u^{\nu-n}(x,t)u_x(x,t)(u^\nu(x,t))_{x} \, dx = \nu \int u^{\nu-n-1}(x,t)u_x^2(x,t) - u^{\nu-n}(x,t)u_{xx}(x,t) \, dx \\
= \nu \int u^{\nu-n-1}(x,t)u_x^2(x,t) \, dx
\]

after integrating by parts twice and taking into account that \( u_x \) is zero on the boundary. It follows that the mass of \( u^{2-n} \) is still dissipating for \( n > 2 \), which proves that (iii) from Proposition 1 remains valid.

If \( \nu \leq 3 \), then we can use the uniform upper bound \( M_0 \) on \( u \) along with the Schwarz inequality to bound the right hand side of (5.4) from below and obtain

\[
\int u^{\nu-1}u_{xx}^2 \, dx \geq \frac{2(2-\nu)^2M_0^{\nu-3}}{9} J^2[u].
\]

Thus, if \( 1 \leq \nu \leq 2 \), then (5.2) directly implies

\[
\frac{d}{dt} J[u(\cdot,t)] \leq -(C + C_{\nu,\sigma})J^2[u(\cdot,t)],
\]

where \( C > 0 \) is the constant defined in Proposition 1 and

\[
C_{\nu,\sigma} := \frac{2\sigma}{9} \nu(2-\nu)(2\nu-1)M_0^{\nu-3}.
\]

In the \( 1/2 \leq \nu \leq 1 \) case we drop the third term in the right hand side of (5.3) and use (5.5) again to obtain (5.6) with precisely the same constant \( C_{\nu,\sigma} \). Therefore, if \( 1/2 \leq \nu \leq 2 \), then positive classical solutions of (5.1) satisfy (2.5) with an improved constant, depending on \( \nu \) and \( \sigma \). This encourages us to determine whether the analysis from the previous sections applies to (5.1) as well.

Let us begin by stating an extension of a fundamental result from [8].

**Proposition 3.** Assume \( n \in (0, \infty) \) and \( \nu \in [1/2, 2] \). Then there exists a unique positive classical solution \( u_\varepsilon \) for

\[
u u_t + [f_\varepsilon(u)u_{xxx}]_x = \nu u^\nu, \quad u(\cdot,0) = u_0 + \delta(\varepsilon)
\]

with \( u_\varepsilon = 0 = u_{xxx} \) at \( \{-1,1\} \times (0, \infty) \), where \( f_\varepsilon \) is the regularizing function introduced in (3.1) and \( \delta(\varepsilon) > 0 \).

Note that the original Theorem 3.1. in [8] has the restriction \( 1 < \nu < 2 \). Based on Proposition 3, the existence results from [8] may also be extended and summarized in the following statement:
Theorem 4. (Bertozzi-Pugh [8] if \(1 < \nu < 2\)) Consider \(0 < n < \infty\), \(1/2 \leq \nu \leq 2\) and a probability density \(u_0 \in H^1(I)\). Let \(u_\varepsilon\) be the unique smooth solution of (5.7) with no-flux boundary conditions and 
\[
\delta(\varepsilon) = \varepsilon^d
\]
for some \(0 < \theta < 2/5\).

Then there exists a subsequence (not relabelled) \(\{u_\varepsilon\}_{\varepsilon > 0}\) which converges pointwise uniformly and weakly in \(L^2((0,T); H^2(I))\) and \(L^\infty((0,T); H^1(I))\) to some
\[
u = \text{some constant}
\]

Moreover, \(u\) is a solution for (5.1) in the sense
\[
\int_{Q_T} u_\varepsilon \zeta dx dt = - \int_{\Gamma_T(u)} u_\varepsilon u_{xxx} \zeta dx dt + \sigma \int_{Q_T} (u_\nu)_x \zeta dx dt.
\]  
(5.8)

Remark: Whereas this result is proved in [8] for the case \(\sigma = 1\) and \(1 < \nu < 2\), a careful inspection of the proof shows that it is, in fact, valid for all \(\sigma > 0\), and, as long as Proposition 3 holds, for all \(1/2 \leq \nu \leq 2\).

Let us now state the main result of this section.

Theorem 5. Assume \(n \in (0,\infty)\), \(1/2 \leq \nu \leq 2\), and \(u_0 \in H^1(I)\) is a probability density. There exists a constant \(C > 0\) depending only on \(n, \nu, \sigma, J[u_0]\), and \(\int u_0^{-n} dx\) in the case \(n \geq 2\), such that the solution \(u\) of (5.1) given by Theorem 4 satisfies
\[
J[u(\cdot, t)] \leq J[u_0] \exp\{-Ct\} \quad \text{for all } t > 0.
\]  
(5.9)

The proofs of Proposition 3 and Theorem 5 are based on the following identity, obtained just as in deducing (5.2):
\[
\frac{d}{dt} J[u_\varepsilon] = - \int_I f_\varepsilon(u_\varepsilon) u_{\varepsilon,xxx} dx - \nu \sigma \int_I u_\varepsilon^{\nu - 1} u_{\varepsilon,xx}^2 dx + \frac{1}{3} \nu (\nu - 1)(\nu - 2) \sigma \int_I u_\varepsilon^{\nu - 3} u_{\varepsilon,xx}^4 dx
\]  
(5.10)
\[
= - \int_I f_\varepsilon(u_\varepsilon) u_{\varepsilon,xxx} dx - \nu \sigma \left[ \frac{2\nu - 1}{2} - \nu \right] \int_I u_\varepsilon^{\nu - 1} u_{\varepsilon,xx}^2 dx
\]
\[
- \nu \sigma \frac{3(1 - \nu)}{2 - \nu} \left[ \int_I u_\varepsilon^{\nu - 1} u_{\varepsilon,xx}^4 dx - \frac{(2 - \nu)^2}{9} \int_I u_\varepsilon^{\nu - 3} u_{\varepsilon,xx}^4 dx \right].
\]  
(5.11)

(Obviously, the last equation holds provided that \(\nu \neq 2\).) Note that, exactly as in (5.2) or (5.3), the extra-terms appearing (relative to (4.3)) are both nonpositive if \(1 \leq \nu \leq 2\) or they combine into something nonpositive if \(1/2 \leq \nu \leq 1\). Thus, it is easy to check that (3.3) and (3.6) still hold for the entropy (3.4) with “\(\leq\)” replacing the equality sign. Therefore, we expect that everything will go through smoothly and we will be able to obtain even better decay rates. We shall next show how the analysis from the previous section extends to (5.7) and, ultimately, to (5.1).

Proof of Theorem 5: Let us begin by using the upper bound \(M_\varepsilon\) on \(u_\varepsilon\) and (4.14) to infer
\[
\int_I u_\varepsilon^{\nu - 1} u_{\varepsilon,xx}^2 dx \geq M_\varepsilon^{\nu - 3} \int_I u_\varepsilon^2 u_{\varepsilon,xx}^2 dx
\]
\[
\geq \frac{\pi^2}{16(\pi + 1)^2} M_\varepsilon^{\nu - 3} (1 + 2\varepsilon^d)^2 J[u_\varepsilon].
\]  
(5.12)

As for the third term from the right hand side of (5.10), there is no hope to get a lower bound for
\[
\int_I u_\varepsilon^{\nu - 3} u_{\varepsilon,xx}^4 dx \text{ in terms of } J[u_\varepsilon],
\]
can only get one in terms of \(J^2[u_\varepsilon]\) (see the remark below). However,
if $1 \leq \nu \leq 2$, then the term becomes nonpositive so that we can safely throw it away. Thus, we have

$$\frac{d}{dt} J[u_\varepsilon] \leq - \int I f(\varepsilon x) u_{\varepsilon,xxx}^2 (x, t) dx - \frac{\pi^2 \nu^\varepsilon}{16(\pi + 1)^2} M_{\varepsilon}^{-\delta} (1 + 2\varepsilon^\delta)^2 J[u_\varepsilon].$$  \hfill (5.13)

This inequality may now be combined with (4.9) in the case $n \in (0, 1) \cup (2, \infty)$ or (4.23) along with (4.24) if $n \in (0, 2)$. Therefore, for all $n \in (0, \infty)$ we obtain an improvement in the rate of exponential decay by an additive factor of

$$\lim_{\varepsilon \to 0} \frac{\pi^2 \nu^\varepsilon}{16(\pi + 1)^2} M_{\varepsilon}^{-\delta} (1 + 2\varepsilon^\delta)^2 = \frac{\pi^2 \nu^\varepsilon}{16(\pi + 1)^2} (1/2 + 2^{1/2} ||u_0||_{L^2(I)})^{\nu-3}$$

provided that $1 \leq \nu \leq 2$. If $1/2 \leq \nu \leq 1$, then we use (5.11) and throw away the quantity inside the brackets (which is nonnegative according to Lemma 3). Thus, in this case the additive factor mentioned above gets multiplied by $(2\nu - 1)/(2 - \nu)$.

\[\Box\]

For a complete proof of Proposition 3 we refer the reader to [9], [8]. Here we only motivate the extension of (1, 2) to $[1/2, 2]$ for $\nu$.

**Proof of Proposition 3:** The point is that, for some short time $t > 0$ (for which classical parabolic Schauder estimates give existence of smooth solutions for (5.7)) one can integrate (5.10) on $[0, t]$ to obtain

$$J[u_\varepsilon(\cdot, t)] + \int_0^t \int_I f(\varepsilon x) u_{\varepsilon,xxx}^2 dxdt + \nu \sigma \int_0^t \int_I |u_{\varepsilon}^{\nu-1} u_{\varepsilon,xxx}^2 dxdt + \frac{1}{3} \nu(\nu - 1)(2 - \nu) \sigma \int_0^t \int_I |u_{\varepsilon}^{\nu-4} u_{\varepsilon,xxx}^4 dxdt = J[u_0].$$

Next we argue as in (5.11) to infer that the sum of the last two terms in the left hand side is nonnegative for $\nu \in [0.5, 2]$ (a larger interval than the obvious [1, 2]). Thus, we obtain the *a priori* bound on $J[u_\varepsilon(\cdot, t)]$ (uniformly in $\varepsilon$ and $t$) necessary to conclude the argument as in [8]. For the important steps we refer back to the paragraph containing (3.3) and (3.4) in Section 3.

**Remark:** In [8] the authors throw away the first two terms from the right hand side of (5.10), then they easily bound the third one from above in terms on $-J^2[u_\varepsilon]$ (the restriction $1 < \nu < 2$ becoming crucial). In this way they are led to a power-law decay in the $H^1$-norm which is used initially for all $n \in (0, \infty)$. However, this approach completely loses sight of the term $\int_I f(\varepsilon x) u_{\varepsilon,xxx}^2 dx$ and thus is far from being optimal (the contribution of the fourth-order term is neglected). Since in our case $\sigma$ is any positive number, only if we keep this term and let $\sigma \downarrow 0$ may we recover the estimates for the original problem (1.1).

Also, note that by using the power-law decay in the $H^1$-norm, the authors of [8] are only able to obtain exponential decay in the $L^\infty$-norm.

**Open problems**

If $n \in [1, 2]$, then the power-law decay from the case of smooth, positive solutions of (5.1) (see (5.6)) is not available for (5.7). The reason is that within this range of $n$ we have been unable to bound $\int_I u_{\varepsilon,xxx}^2 f(\varepsilon x) dx$ from above. We came across the same difficulty in Section 4. There we needed to study the case $n \in [1, 2]$ separately in order to obtain exponential decay directly (as seen, the arguments there worked for $n \in (0, 2]$). That is why, as explained in the Introduction, we have decided to prove directly.
exponential decay for all \( n \in (0, \infty) \). It remains an interesting question whether, in the cases where we can prove initial power-law decay for the regularized problem, the power-law-then-exponential decay approach from [11] is better (as to optimal decay rates) compared to ours.

Another question, related to the previous one, concerns the concept of optimality in the case of algebraic-then-exponential decay approach. Indeed, let us go back to the remark surrounding (4.10) and let \( 0 < \delta < 1/2 \) be fixed, and \( T_{\delta} > 0 \), \( m_{\delta} \) be given by

\[
T_{\delta} := \frac{4J[u_0] - \delta^2}{AJ[u_0]\delta^2}, \quad m_{\delta} := \frac{1}{2} - \delta.
\]

Then, since

\[
\|u(\cdot, t) - 1/2\|_\infty \leq 2\sqrt{J[u(\cdot, t)]},
\]

we infer by (4.10)

\[
\frac{1}{2} - \delta \leq \inf_{x \in I} u(x, t), \text{ if } t \geq T_{\delta}.
\]

Since for \( t \geq T_{\delta} \) the solution \( u \) becomes positive, thus classical, we can employ (1.2) to infer

\[
\frac{d}{dt} J[u(\cdot, t)] \leq -m_{\delta} \frac{\pi^4}{8} J[u(\cdot, t)] \text{ for } t \geq T_{\delta},
\]

which leads to

\[
J[u(\cdot, t)] \leq J[u(\cdot, T_{\delta})] \exp \left\{ m_{\delta} \frac{\pi^4}{8} (T_{\delta} - t) \right\} \text{ for } t \geq T_{\delta}.
\]

To estimate \( J[u(\cdot, T_{\delta})] \) we use (4.10), then by the definitions of \( T_{\delta} \) and \( m_{\delta} \) we conclude

\[
J[u(\cdot, t)] \leq \frac{\delta^2}{4} \exp \left\{ \frac{\pi^4}{8} \left( \frac{1}{2} - \delta \right)^n \left( \frac{4J[u_0] - \delta^2}{AJ[u_0]\delta^2} - t \right) \right\} \text{ for } t \geq \frac{4J[u_0] - \delta^2}{AJ[u_0]\delta^2}.
\] (5.14)

The exponential rate of decay is thus, as argued in [11], essentially optimal (\( \delta \)-optimal) for large time for any given \( 0 < \delta < 1/2 \).

Let us now look at (5.14) from a different perspective. If we fix a large enough time horizon \( T_0 > 0 \) (such that the solution \( u \) is positive and classical on \([T_0, \infty)\), and such that \( J[u_0]/(1 + AJ[u_0]T_0) < 1/16 \), we may ask the following question: is there a \( \delta > 0 \) such that \( 2\sqrt{J[u_0]}/(1 + AJ[u_0]T_0) \leq \delta \leq 1/2 \) and for which the inequality (5.14) becomes optimal at \( t = T_0 \)? In other words, we are looking at the problem of minimizing

\[
\varphi(\delta) := \frac{\delta^2}{4} \exp \left\{ \frac{\pi^4}{8} \left( \frac{1}{2} - \delta \right)^n \left( \frac{4J[u_0] - \delta^2}{AJ[u_0]\delta^2} - T_0 \right) \right\} \text{ in the interval } [2\sqrt{J[u_0]}/(1 + AJ[u_0]T_0), 1/2].
\]

If there exists such a minimizer, say, \( \delta_0 \), then the strategy to obtain an optimal (by the algebraic-then-exponential decay technique) upper bound for \( J[u(\cdot, T_0)] \) is to use (4.10) up to \( t := T_{\delta_0} \) (defined just as \( T_3 \) above for this particular \( \delta_0 \)), then use exponential decay as shown above. Of course, the optimal bound is then given by the minimum value \( \varphi(\delta_0) \). However, it may happen that for some values of \( n, T_0 \), and some initial densities \( u_0 \in H^1(I) \) the minimum is not attained in the open interval. Then it is attained at the left endpoint \( \delta_0 = 2\sqrt{J[u_0]}/(1 + AJ[u_0]T_0) \) (since \( \varphi \) achieves a greater value at 1/2 than at the left endpoint). This simply means that the optimal bound for \( J[u(\cdot, T_0)] \) is obtained directly by the algebraic decay given by (4.10).

To summarize, the question is this: if given a time \( T_0 > 0 \) sufficiently large, is there a way to estimate (in terms of \( n, u_0 \) and \( T_0 \)) the value of \( \delta > 0 \) for which (5.14) becomes optimal (again, by the algebraic-then-exponential decay technique) at \( t = T_0 \)?
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