1. Prove by definition (in $\epsilon - \delta$ language) that $f(x) = \sqrt{1 + x^2}$ is uniformly continuous in $(0, 1)$. Is $f(x)$ uniformly continuous in $(1, \infty)$? Prove your conclusion.

2. Let $f_n(x) = \frac{nx}{n + x}$.
   (a) Find $f(x) = \lim_{n \to \infty} f_n(x)$;
   (b) Does $f_n$ converge to $f$ uniformly $(0,1)$? Prove your conclusion.
   (c) Does $f_n$ converge to $f$ uniformly $(1,\infty)$? Prove your conclusion.

3. If $\lim_{n \to \infty} s_n \geq 0$, which of the following statements are true? Explain your conclusion.
   (a) $\exists N > 0$ such that $\forall n > N, s_n \geq 0$.
   (b) $\forall N > 0, \exists n > N$ such that $s_n \geq 0$.
   (c) $\exists \epsilon > 0$ such that $\forall N > 0, \exists n > N$ with $s_n \geq 0$.
   (d) $\lim_{n \to \infty} s_n^2 \leq (\lim_{n \to \infty} s_n)^2$.

4. Let $g(x)$ be defined as follows
   \[ g(x) = \int_{-\infty}^{\infty} \frac{\cos(xy^3)}{1+y^2}dy. \]
   (a) Prove $g(x)$ is continuous in $(-\infty, \infty)$.
   (b) Is $g(x)$ uniformly continuous in $(-\infty, \infty)$? absolutely continuous in $(-\infty, \infty)$? Prove your conclusion.

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying $\sum_{n=1}^{\infty} |a_n| < \infty$. Define $f$ by $f(x) = \sum_{n=1}^{\infty} a_n x^n$.
   (a) Show that $f$ is a function of bounded variation on $x \in [-1,1]$.
   (b) If $a_1 \neq 0$ and $a_n = 0$ for $n \geq 2$, is $f$ absolutely continuous on $x \in \mathbb{R}$. Explain your answer.
   (c) If $a_1 \neq 0$, $a_2 \neq 0$, and $a_n = 0$ for $n \geq 3$, is $f$ absolutely continuous on $x \in \mathbb{R}$. Explain your answer.
6. For $1 \leq p < \infty$ define
\[ \|f\|_p = \left( \int_E |f|^p \right)^{1/p} \]
and denote $L^p(E)$ to be the set of functions $f$ for which $\int_E |f|^p < \infty$.
Either prove the statement or show a counter example.

(a) If $E = [0,1]$, then there is a constant $c > 0$ for which
\[ \|f\|_1 \leq c \|f\|_2 \text{ for all } f \in L^2(E). \]

(b) Suppose $E = [1, \infty)$. If $f$ is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$.

7. Suppose that $f(x)$ is a uniformly continuous and Lebesgue integrable function on $\mathbb{R}$.
Show that
\[ \lim_{|x| \to \infty} f(x) = 0. \]

8. Let $\{u_n\}_{n=1}^\infty$ be a sequence of Lebesgue measurable functions on $[0,1]$ and assume
$\lim_{n \to \infty} u_n(x) = 0$ a.e. on $[0,1]$, and also $\|u_n\|_{L^2[0,1]} \leq 1$ for all $n$. Prove that
\[ \lim_{n \to \infty} \|u_n\|_{L^1[0,1]} = 0. \]
1. Is \( f(x) = x^3 \) uniformly continuous in \((-5, 1)\)? in \((1, \infty)\)? Prove your conclusion.

2. Let \( f_n(x) = \frac{nx}{n + x} \).
   (a) Find \( f(x) = \lim_{n \to \infty} f_n(x) \);
   (b) Does \( f_n \) converge to \( f \) uniformly \((0, 1)\)? Prove your conclusion.
   (c) Does \( f_n \) converge to \( f \) uniformly \((1, \infty)\)? Prove your conclusion.

3. If \( \lim_{n \to \infty} s_n \leq 0 \), which of the following statements are true? Explain your conclusion.
   (a) \( \exists N > 0 \) such that \( \forall n > N, s_n \leq 0 \).
   (b) \( \forall N > 0, \exists n > N \) such that \( s_n \leq 0 \).
   (c) \( \exists \varepsilon > 0 \) such that \( \forall N > 0, \exists n > N \) with \( s_n > -\varepsilon \).
   (d) \( \lim_{n \to \infty} s_n^2 > (\lim_{n \to \infty} s_n)^2 \).

4. Let \( g(x) \) be defined as follows
   \[
g(x) = \int_{-\infty}^{\infty} \frac{\cos^2(xy)}{1+y^2}dy.
   \]
   (a) Prove \( g(x) \) is continuous in \((-\infty, \infty)\).
   (b) Is \( g(x) \) uniformly continuous in \((-\infty, \infty)\)? absolutely continuous in \((-\infty, \infty)\)?
   Prove your conclusion.

5. Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of real numbers satisfying \( \sum_{n=1}^{\infty} |a_n| < \infty \). Define \( f : [-1, 1] \to \mathbb{R} \) by \( f(x) = \sum_{n=1}^{\infty} a_n x^n \). Show that \( f \) is a function of bounded variation.

6. Suppose that \( f : [0, \infty) \to [0, \infty) \) is measurable and that \( \int_{0}^{\infty} f(x) < \infty \). Prove that
   \[
   \lim_{n \to \infty} \int_{0}^{\infty} \frac{x^n f(x)}{1 + x^n} dx = \int_{1}^{\infty} f(x) dx.
   \]

7. Suppose that \( f(x) \) is a uniformly continuous and Lebesgue integrable function on \( \mathbb{R} \).
   Show that
   \[
   \lim_{|x| \to \infty} f(x) = 0.
   \]

8. Let \( \{u_n\}_{n=1}^{\infty} \) be a sequence of Lebesgue measurable functions on \([0, 1]\) and assume
   \( \lim_{n \to \infty} u_n(x) = 0 \) a.e. on \([0, 1] \), and also \( \|u_n\|_{L^2[0,1]} \leq 1 \) for all \( n \). Prove that
   \[
   \lim_{n \to \infty} \|u_n\|_{L^1[0,1]} = 0.
   \]
1. (a) Prove by definition that $x^2$ is uniformly continuous in $(-7, 2)$.
   (b) Prove by definition that $x^2$ is not uniformly continuous in $[0, \infty)$.

2. If $\lim_{n \to \infty} s_n = 0$, which of the following statements are true? Explain your conclusion.
   (a) $\exists N > 0$ such that $\forall n > N$, $s_n \leq 0$.
   (b) $\forall N > 0$, $\exists n > N$ such that $s_n \leq 0$.
   (c) $\forall \epsilon > 0$, $\exists N > 0$, such that $\forall n > N$, $s_n > -\epsilon$.
   (d) $\exists \epsilon > 0$ such that $\forall N > 0$, $\exists n > N$ with $s_n > -\epsilon$.

3. Let $f(x)$ be a function of bounded variation defined on $[0,1]$. Prove that the set of all discontinuity points of $f(x)$ is countable.

4. Let $f(x)$ be a measurable function defined in $\mathbb{R}$ and satisfying:
   (a) $f(0) = 1$;
   (b) If $f(x_0) > 0$, then $\exists \delta > 0$ such that $f(x) > 0$ in $(x_0 - \delta, x_0 + \delta)$;
   (c) If $f(x_n) > 0$ and $x_n \to a$, then $f(a) > 0$.
Is it true $f(x) > 0 \forall x \in \mathbb{R}$? Prove your conclusion.

5. Suppose that $\{f_n\}$ is a sequence of real valued measurable functions defined on the interval $[0,1]$ and suppose that
   $$\lim_{n \to \infty} f_n(x) = f(x) \text{ a.e. on } [0,1].$$
   Let $p > 1$, $M > 0$, and that $\|f_n\|_{L^p(\mathbb{R})} \leq M$ for all $n$.
   (a) Prove that $f \in L^p(\mathbb{R})$ and that $\|f\|_{L^p(\mathbb{R})} \leq M$.
   (b) Prove that $\lim_{n \to \infty} \|f - f_n\|_{L^p(\mathbb{R})} = 0$.

6. Suppose that $f \in L^1(\mathbb{R})$ is a uniformly continuous function. Show that
   $$\lim_{|x| \to \infty} f(x) = 0.$$
7. For $1 \leq p < \infty$ define
\[
\|f\|_p = (\int_E |f|^p)^{1/p}
\]
and denote $L^p(E)$ to be the set of functions $f$ for which $\int_E |f|^p < \infty$. For each statement below, either prove it or show a counter example.

(a) If $E = [0, 1]$, then there is a constant $c \geq 0$ for which
\[
\|f\|_2 \leq c \|f\|_1 \quad \text{for all } f \in L^1(E).
\]

(b) Suppose $E = [1, \infty)$. If $f$ is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$.

8. (a) For a Lebesgue integrable function $f$ on $[0,1]$ define $F(x) = \int_0^x f(t)dt$ for $x \in [0, 1]$. Prove that $F$ is absolutely continuous on $[0,1]$.

(b) Give an explicit example of a continuous function $F$ of bounded variation on $[0,1]$ that is not absolutely continuous on $[0,1]$. It is not necessary to verify your assertion.
1. Let $f$ be a monotone function defined on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$. Show that $f$ has at most countable discontinuous points.

2. Answer the following questions and prove your conclusion.
   (a) Is $x^2$ uniformly continuous in the open interval $(-5, 2)$?
   (b) Is $x^2$ uniformly continuous in $[0, \infty)$?

3. If $\lim_{n \to \infty} s_n > 0$, which of the following statements are true? Prove your conclusion.
   (a) $\exists N > 0$ such that $\forall n > N, s_n > 0$.
   (b) $\exists \epsilon > 0$ such that $\forall N > 0, \exists n > N$ with $s_n > \epsilon$.
   (c) $\forall \epsilon > 0, \exists N > 0$, such that $\forall n > N, s_n > -\epsilon$.
   (d) $\lim_{n \to \infty} s_n^2 = (\lim_{n \to \infty} s_n)^2$.

4. Let $f \in L^1(-\infty, \infty)$, and $g(x)$ be defined as follows
   
   \[ g(x) = \int_{-\infty}^{\infty} \cos^2(xy)f(y)dy. \]
   (a) Prove $g(x)$ is continuous in $(-\infty, \infty)$.
   (b) Is $g(x)$ uniformly continuous in $(-\infty, \infty)$? absolutely continuous in $(-\infty, \infty)$? Prove your conclusion.

5. Let $f$ and $g$ be real valued measurable functions on $[0,1]$ with the property that $g$ is differentiable at every $x$ on $[0,1]$ and
   
   \[ g'(x) = (f(x))^2. \]
   Show that $f \in L^1[0,1]$.

6. (a) Let $m$ be the Lebesgue measure on $\mathbb{R}$ and let $E \in \mathbb{R}$ be measurable. Also, suppose $f \in L^1(E)$ and that $f > 0$ a.e. on $E$. Show that
   
   \[ \lim_{n \to \infty} \int_E |f(x)|^{1/n}dx = m(E). \]
   (b) Let $E = [a,b]$, where $a$ and $b$ are real numbers satisfying $a < b$. With the same assumptions on $f$ as in (a), show that
   
   \[ \lim_{n \to \infty} \left( \int_a^b |f(x)|^{1/n}dx \right)^n = \infty \text{ if } b - a > 1 \]
   and
   
   \[ \lim_{n \to \infty} \left( \int_a^b |f(x)|^{1/n}dx \right)^n = 0 \text{ if } b - a < 1. \]
7. Suppose that $f : [0, \infty) \to [0, \infty)$ is measurable and that $\int_0^{\infty} f(x) < \infty$. Prove that

$$\lim_{n \to \infty} \int_0^{\infty} \frac{x^n f(x)}{1 + x^n} \, dx = \int_1^{\infty} f(x) \, dx$$

8. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions on $[0, 1]$ and assume $\lim_{n \to \infty} u_n(x) = 0$ a.e. on $[0, 1]$, and also $\|u_n\|_{L^2[0,1]} \leq 1$ for all $n$. Prove that

$$\lim_{n \to \infty} \|u_n\|_{L^1[0,1]} = 0.$$
Solve exactly 6 out of the 8 problems.

1. Answer the following questions and prove your conclusion.
   - If $A$ is a closed subset of $[0,1]$ and $A \neq [0,1]$. Is it possible that $mA = 1$?
   - If $B$ is an open subset of $[0,1]$ and dense in $[0,1]$. Is it possible that $mB < 1$?

2. Let $\{p_k\}$ be all the rational numbers in $[0,1]$, and $H(x)$ be the Heaviside function defined as
   \[
   H(x) = \begin{cases} 
   1, & x > 0, \\
   0, & x \leq 0.
   \end{cases}
   \]
   Define $f(x)$ in $[0,1]$ as
   \[
   f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} H(x - p_k).
   \]
   - Prove by definition (in $\epsilon - \delta$ language) that $f(x)$ is continuous at all irrational numbers, and discontinuous at all rational numbers in $[0,1]$.
   - Determine and explain: (i). Is $f(x)$ Riemann integrable in $[0,1]$? (ii). Is $f(x)$ Lebesgue integrable? (iii) Is $f(x)$ of bounded total variation?

3. Let $f$ be Lebesgue integrable in $[0,b]$ with $b < \infty$. Prove
   \[
   \lim_{n \to \infty} \int_{0}^{b} f(x) \cos nx \, dx = 0.
   \]
   - Is it also true if $b = \infty$? Prove your conclusion.

4. If $f$ is absolutely continuous on $[0,1]$ and $f(x) > 0$ on $[0,1]$. Prove by definition that $1/f$ is absolutely continuous on $[0,1]$.
   - Is $f(x) = \sqrt{x}$ absolutely continuous on $[0,1]$? In $[1, \infty)$? Prove your conclusion.

5. Prove or disprove that for every $\varepsilon > 0$ and for every $f \in L^\infty[a,b]$, where $a$ and $b$ are finite, there is a $g \in C[a,b]$ such that
   \[
   \text{ess sup}_{a \leq x \leq b} |f - g| < \varepsilon.
   \]

6. Compute the Lebesgue integral
   \[
   \int_{0}^{1} \liminf_{n \to \infty} x e^x \sin^2(\pi nx) \, dx.
   \]
7. Prove or disprove the following statement.

(a) Let \( g \) be an integrable function on \([0; 1]\). Then there is a bounded measurable function \( f \) such that
\[
\int_0^1 fg = \|g\|_1 \|f\|_\infty.
\]

(b) Let \( \{f_n\} \) be a sequence of functions in \( L^1[0, 1] \), which converge almost everywhere to a function \( f \) in \( L^1 \), and suppose that there is a constant \( M \) such that \( \|f_n\|_1 \leq M \) all \( n \). Then for each function \( g \) in \( L^\infty \) we have
\[
\int_0^1 fg = \lim_{n \to \infty} \int_0^1 f_n g.
\]

8. Let \( \{a_n\}_{n=1}^\infty \) be a sequence of real numbers satisfying \( \sum_{n=1}^\infty |a_n| < \infty \). Define \( f : [-1, 1] \to \mathbb{R} \) by \( f(x) = \sum_{n=1}^\infty a_n^2 x^n \). Show that \( f \) is a function of bounded variation in \([-1, 1]\).
Entrance Exam, Real Analysis  
April 22, 2015

Solve exactly 6 out of the 8 problems.

1. Let $A \subset \mathbb{R}$ be an open set with the following property: $\forall \{x_n\} \subset A$, $\exists$ a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in A$.
Which of the following statements is true?
$mA = 0$? $mA = \ln 2$? $mA = \sqrt{\pi}$? $mA = \infty$? $mA$ is not uniquely determined and could be any positive number?
Prove your conclusion.

2. Answer the following questions and prove your conclusion.

- Let $A \subset [0,1]$ be an open set. If $\exists \delta < 1$ such that $mA \leq \delta$, is it possible that $A$ is dense in $[0,1]$?
- Let $B \subset [0,1]$ and $B \neq [0,1]$. If $\forall \epsilon > 0$, $mB > 1 - \epsilon$, is it possible that $B$ is closed?

3. Let $f \in L^1(-\infty, \infty)$, and $g(x)$ be defined as follows
$$g(x) = \int_{-\infty}^{\infty} \cos(xy)f(y)dy.$$  

- Prove $g(x)$ is continuous in $(-\infty, \infty)$.
- Is $g(x)$ uniformly continuous in $(-\infty, \infty)$? Prove your conclusion.

4. Let $f(x) > 0$ and be absolutely continuous on $[0,1]$. Let $g(x) = \frac{1}{\sqrt{f(x)}}$. Prove $g(x)$ is absolutely continuous on $[0,1]$.

5. (a) For a bounded Lebesgue integrable function $f$ on $[0,1]$ define $F(x) = \int_0^x f(t)dt$ for $x \in [0,1]$. Prove that $F$ is absolutely continuous on $[0,1]$.

(b) Give an explicit example of a continuous function $F$ of bounded variation on $[0,1]$ that is not absolutely continuous on $[0,1]$. It is not necessary to verify your assertion.
6. (a) Suppose that $f \in L^1[0,1]$ and let $m$ denote Lebesgue measure. Prove that for $c > 0$

$$m\{|f(x)| \geq c\} \leq \frac{1}{c} \int_0^1 |f(x)| \, dx.$$ 

(b) Suppose that $\{f_n\} \subseteq L^p[0,1]$ is a sequence of functions satisfying $\|f_n\|_{L^p[0,1]} \leq M$, where $1 < p < \infty$. Prove that

$$\lim_{c \to \infty} \int_{|f_n(x)| \geq c} |f_n(x)| \, dx = 0,$$

uniformly in $n$.

7. Assume that $n \geq 1$ is an integer and let $f \in \cap_{n=1}^\infty L^n[0,1]$. Prove that if

$$\sum_{n=1}^\infty \|f\|_{L^n[0,1]} < \infty,$$

then $f = 0$ a.e.

8. Suppose that $\{f_n\}$ is a sequence in $L^1(R)$ with $\|f_n\|_{L^1(R)} \leq 1$ for all $n$ and

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ a.e.}$$

(a) Prove that $f \in L^1(R)$ and that $\|f\|_{L^1(R)} \leq 1$.

(b) Show that

$$\lim_{n \to \infty} (\|f - f_n\|_{L^1(R)} - \|f_n\|_{L^1(R)} + \|f\|_{L^1(R)}) = 0.$$

Hint: The following inequality might be useful. For any numbers $a$ and $b$

$$0 \leq |a - b| - |a| + |b| \leq 2|b|$$
1. Given function $\phi(x)$ with $\phi(1) = 1$ and $\phi'(x) = e^{-x^2}$. The plane curve $\Gamma$ is defined by the equation

$$y = \phi(1 + xy + y^2).$$

Find the equation for the tangent line to the curve $\Gamma$ at the point $(x, y) = (-1, 1)$.

2. Does $f_n(x) = x^n \sin \left(\frac{1-x}{x}\right)$ converge uniformly on $(0,1)$? Prove your conclusion.

3. Let $\{x_n = p_n/q_n \in \mathbb{Q}\}$ be a sequence of rational numbers and $x_n \to \alpha \in \mathbb{R} \setminus \mathbb{Q}$ ($\alpha$ is irrational). Prove $q_n \to \infty$.

4. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$, and $f_n$ be a sequence of nonnegative Lebesgue integrable functions on $E$. If $f_n \to f$ a.e. on $E$, is it always true that

$$\int_E f dx \leq \lim_{n \to \infty} \int_E f_n dx?$$

Prove your conclusion.

5. Let $a$ and $b$ are real numbers satisfying $a < b$. Prove or disprove the following.

(a) If $f(x)$ is a differentiable function on $a < x < b$, it is absolutely continuous on $a < x < b$.

(b) If $f(x)$ is absolutely continuous on $a \leq x \leq b$, it is Lipschitz on $a \leq x \leq b$.

6. A family $\mathcal{F}$ of measurable functions on $E$ of finite measure is said to be uniformly integrable over $E$ provided that for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$ and for each measurable set $A \subseteq E$ with $m A < \delta$, $\int_A |f| < \varepsilon$. Either prove or disprove the following statements.

(a) If $\mathcal{G}$ is the family of measurable functions $f$ on $[0,1]$, each of which is integrable over $[0,1]$ and has $\int_0^1 |f| < 1$, then $\mathcal{G}$ is uniformly integrable over $[0,1]$.

(b) If $\{f_n\}$ is a sequence of nonnegative, measurable, and integrable functions that converges pointwise a.e. on $E$ to $h = 0$, then $\{f_n\}$ is uniformly integrable.
7. For $1 \leq p < \infty$ define

$$\|f\|_p = \left( \int_E |f|^p \right)^{1/p}$$

and denote $L^p(E)$ to be the set of functions $f$ for which $\int_E |f|^p < \infty$. Either prove the statement or show a counter example.

(a) If $E = [0, 1]$, then there is a constant $c \geq 0$ for which

$$\|f\|_2 \leq c \|f\|_1 \text{ for all } f \in L^1(E).$$

(b) Suppose $E = [1, \infty)$. If $f$ is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$.

8. Prove or disprove that for every $\varepsilon > 0$ and for every $f \in L^\infty[a, b]$, where $a$ and $b$ are finite, there is a $g \in C[a, b]$ such that

$$\text{ess sup}_{a \leq x \leq b} |f - g| < \varepsilon.$$
1. Given function $\phi(x)$ with $\phi(1) = 5$ and $\phi'(x) = e^{-x^2}$. The plane curve $\Gamma$ is defined by the equation

$$y = \phi(1 + \sin 3(xy)).$$

Find the equation for the tangent line to the curve $\Gamma$ at the point $(x, y) = (0, 5)$.

2. Does $f_n(x) = x^n \sin\left(\frac{1-x}{x}\right)$ converge uniformly on $(0,1)$? Prove your conclusion.

3. Evaluate the integral

$$I = \int_0^1 dy \int_y^1 e^{-x^2} dx.$$

4. Let $f$, $g$ be absolutely continuous in $(0,1)$. Is it true that the product $fg$ is also absolutely continuous in $(0,1)$? Prove your conclusion.

5. Let $\{f_n\}$ be a sequence of nonnegative integrable functions on $E$. For each of (a), (b), and (c) either prove the statement or show a counter example.

(a) If $\lim_{n \to \infty} \int_E f_n = 0$, then $\{f_n\} \to 0$ in measure.

(b) If $\{f_n\} \to 0$ in measure, then $\lim_{n \to \infty} \int_E f_n = 0$.

(c) If $\{f_n\} \to 0$ almost everywhere on $E$, then $\lim_{n \to \infty} \int_E f_n = 0$.

6. For $f$ in $C(E)$, where $C(E)$ is the set of continuous functions on $E$, define

$$\|f\|_p = \left(\int_E |f|^p\right)^{1/p}, \quad \|f\|_{\sup} = \sup_{x \in E} |f(x)|.$$

Here $1 \leq p < \infty$. For each of (a), (b), and (c) either prove the statement or show a counter example.

(a) If $E = [0,1]$ and $1 \leq p < q < \infty$, then there is a constant $c \geq 0$ for which

$$\|f\|_p \leq c \|f\|_q \quad \text{for all } f \in C(E).$$

(b) If $E = [0,1]$, then there is a constant $c \geq 0$ for which

$$\|f\|_{\sup} \leq c \|f\|_1 \quad \text{for all } f \in C(E).$$

(c) Suppose $E = [1, \infty)$. If $f \in C(E)$ is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$. 
7. Prove or disprove that for every $\varepsilon > 0$ and for every $f \in L^\infty[a, b]$, there is a $g \in C[a, b]$ such that
\[
\text{ess sup}_{a \leq x \leq b} |f - g| < \varepsilon.
\]

8. (a) Suppose $E \subseteq \mathbb{R}$ has Lebesgue measure zero. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is increasing and absolutely continuous, then $f(E)$ has Lebesgue measure zero. Here, $f(E) = \{ y \mid y = f(x) \text{ for some member } x \text{ of } E \}$.

(b) Would the same conclusion hold if we remove “increasing” in the assumption on $f$? In other words, if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then would $f(E)$ have Lebesgue measure zero? Explain your reasoning.