Research Prospectus

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Abstract

We study properties of Class 2 and edge-chromatic critical graphs. Vizing conjectured that for an edge-chromatic critical graph G of maximum degree Δ, $|E(G)| \geq \frac{1}{2}(|V(G)|(\Delta - 1) + 3)$ and $\alpha(G) \leq \frac{|V(G)|}{2}$. Vizing also conjectured that every planar graph of maximum degree at least 6 is Class 1. We discuss partial results to these three conjectures and the extension of the planar graph conjecture to embeddings of graphs on other surfaces, with an emphasis on results derived from the discharging method in conjunction with adjacency lemmas for edge-chromatic critical graphs. We present problems for future research with regard to the size and independence number of edge-chromatic critical graphs and the classification of embedded graphs.

1 Introduction

A proper k-edge-coloring of a graph $G = (V, E)$ is a function $c : E \to \{1, 2, \ldots, k\}$ such that for any two incident edges e and f, $c(e) \neq c(f)$. The chromatic index $\chi'(G)$ is the smallest k for which G admits a proper k-edge-coloring. Although chromatic index is well-defined and has been studied for multigraphs, all graphs considered here are simple.

Edge-coloring of graphs arose in attempts in the late 1800s to prove the 4-Color Problem for vertex-colorings of planar graphs. Tait [28], in 1880, recognized that the 4-Color Problem was equivalent to a statement about the edge-coloring of certain cubic graphs. A major breakthrough for edge-coloring as a problem in its own right came in 1965, when Vizing [29] proved that for any simple graph G with maximum degree Δ, $\chi'(G) \in \{\Delta, \Delta+1\}$. If a graph G has chromatic index $\chi'(G) = \Delta$, then G is called Class 1; otherwise, $\chi'(G) = \Delta + 1$ and G is called Class 2. It is natural, then, to try to classify graphs based on chromatic index.

In 1981, Holyer [8] showed that the problem of classification of graphs by chromatic index is NP-complete for an arbitrary graph. For some classes of graphs, the problem has been solved. In 1916, König [11] proved that every bipartite graph is Class 1. More generally, consider a Class 1 graph G, i.e., G can be edge-colored using Δ colors. Since each color class of G induces a matching and the size of a matching is at most half the order of a graph, it follows immediately that the size of a Class 1 graph G is at most $\Delta(G)\lfloor\frac{|V(G)|}{2}\rfloor$. Any graph with more than $\Delta(G)\lfloor\frac{|V(G)|}{2}\rfloor$ edges (called an overfull graph) must then be Class 2. This implies immediately that all regular graphs on an odd number of vertices, including complete graphs with odd order, must be Class 2.

A graph G is called critical if G is Class 2, but for any edge e, the graph $G - e$ is Class 1. A critical graph G of maximum degree Δ is called Δ-critical. Much research has occurred on critical graphs, as they have much more structure than Class 2 graphs in general. In particular, many lemmas regarding the degree of adjacent vertices have been proven. Most important of these is Vizing's Adjacency Lemma (VAL) [29]:

Lemma 1.1 *Let G be a Δ-critical graph. Let uv be an edge of G. Then, u has at least max$\{2, \Delta - d(v) + 1\}$ neighbors of maximum degree.*

VAL only gives information about the first neighborhood of a vertex. There are two main general lemmas involving the second neighborhood of a vertex:

Lemma 1.2 *(Sanders and Zhao [24]) Let G be a Δ-critical graph and x, y be two adjacent vertices. If $d(x) < \Delta$ and $d(y) < \Delta$, then x is adjacent to at least*

$\Delta - d(y) + 1$ vertices z satisfying the following: $z \neq y$; z is adjacent to at least $2\Delta - (d(x) + d(y))$ vertices different from x of degree at least $2\Delta - (d(x) + d(y)) + 2$; and if z is not adjacent to y, then z is adjacent to at least $2\Delta - (d(x) + d(y)) + 1$ vertices different from x of degree at least $2\Delta - (d(x) + d(y)) + 2$.

Lemma 1.3 *(Luo and Zhao [18]) Let G be a Δ-critical graph. Let $xy \in E(G)$ with $4 \leq d = d(x) \leq \Delta - 2$ and $4 \leq s = d(y)$. If for an integer $k \geq 0$, either y is adjacent to $d + s - \Delta - 2 - k$ vertices $u \neq x$ of degree at most $2\Delta - (d + s) + 1$, where $d + s - \Delta - 2 - k \geq 1$; or y is adjacent to $d + s - \Delta - 2 - k$ vertices $u \neq x$ of degree at most $2\Delta - (d + s) + 1$, where $d + s - \Delta - 2 - k \geq 2$, then*

(1) There exist at least $\Delta - d + 1$ neighbors $y' \neq x$ of y satisfying the following: (i) $d(y') \geq \Delta - k$; (ii) if $d(y) \neq \Delta$, then y' is adjacent to at least $\Delta - k - 2$ vertices distinct from x, y of degree at least $\Delta - k$; (iii) if $d(y) \neq \Delta$ and y' is not adjacent to x, then y' is adjacent to at least $\Delta - k - 1$ vertices distinct from y of degree at least $\Delta - k$; (iv) if $d(y) = \Delta$, then replace at least $\Delta - k$ in (ii) and (iii) by at least $\Delta - k - 1$;

(2) There exist at least $d - k - 1$ neighbors $x' \neq y$ of x satisfying the following: (i) $d(x') \geq \Delta - k$; (ii) if $d(y) \neq \Delta$, then x' is adjacent to at least $\Delta - k - 2$ vertices distinct from x, y of degree at least $\Delta - k$; (iii) if $d(y) \neq \Delta$ and x' is not adjacent to y, then x' is adjacent to at least $\Delta - k - 1$ vertices distinct from x of degree at least $\Delta - k$; (iv) if $d(y) = \Delta$, then replace at least $\Delta - k$ in (ii) and (iii) by at least $\Delta - k - 1$.

Regarding Class 2 and critical graphs, in the late 1960s, Vizing made three still-unresolved conjectures.

It is clear that deleting edges from a graph cannot increase chromatic index. Thus, any Class 2 graph with maximum degree Δ must have a Δ-critical subgraph, obtained by deleting edges. Further deletion of edges results in a Class 1 graph. A natural questions arises: how many edges must a critical graph have?

Conjecture 1 *(Vizing's Size Conjecture, 1968 [30])*
If $G = (V, E)$ is a Δ-critical graph, then $|E| \geq \frac{1}{2}(|V|(\Delta - 1) + 3)$.

For a graph $G = (E, V)$, a subset S of V is independent if the graph induced by S has no edges. The independence number $\alpha(G)$ of a graph is the maximum size of an independent set of G. Since independence number is related to matchings, and any color class in an edge-coloring of a graph is a matching, it seems that the independence number should be relatively low in critical graphs.

Conjecture 2 *(Vizing's Independence Number Conjecture, 1968 [30])*
If G is a critical graph, then $\alpha(G) \leq \frac{|V(G)|}{2}$.

Planar graphs are a very interesting class of graphs to consider.

Conjecture 3 *(Vizing's Planar Graph Conjecture, 1965 [29])*
Every planar graph with maximum degree at least 6 is Class 1.

Mel'nikov [21] extended Vizing's idea for an upper bound on the maximum degree of Class 2 graphs on spheres to general surfaces. For a surface Σ, define $\Delta(\Sigma) = \max\{\Delta | \ G$ is a Δ-critical graph that can be embedded in $\Sigma\}$. In 1890, Heawood [7] showed that for any surface Σ with Euler characteristic $\chi \neq 2$ (i.e., any surface other than the sphere), the chromatic number for a graph embedded on that surface is at most $\lfloor \frac{7+\sqrt{49-24\chi}}{2} \rfloor$. This upper bound is now defined as the Heawood number, $H(\chi) = \lfloor \frac{7+\sqrt{49-24\chi}}{2} \rfloor$. The 4-Color Theorem (Appel and Haken, [1]) showed that the Heawood number for a sphere is an upper bound for chromatic number of planar graphs. As an implication of Vizing's Size Conjecture, we have the further conjecture for general graph embeddings.

Conjecture 4 *Let Σ be a surface with Euler characteristic χ. Then, $\Delta(\Sigma) \in \{H(\chi), H(\chi) - 1\}$. In particular, Vizing's Planar Graph Conjecture may be rephrased as $\Delta(S) = 5$, where S is a sphere.*

All of these conjectures are very difficult and remain open, but there has been much progress in partial results. My proposed research will work toward further characterization of Class 2 and critical graphs, with regards to the parameters of graph embeddings, size, and independence number.

2 Progress So Far

This section is a summary of progress so far toward Vizing's conjectures.

2.1 Size

Vizing's Size Conjecture is of particular interest because of its consequences to Vizing's Planar and Indepence Number Conjectures. Supposing the Size Conjecture is true, this implies improved general bounds for $\Delta(\Sigma)$ as well as implies the Planar Graph Conjecture is correct for planar graphs G with $\Delta(G) \geq 7$. Further, it would imply that the Independence Number Conjecture is true asymptotically.

Vizing's Size Conjecture has been verified for Δ-critical graphs with $\Delta \in \{3, 4, 5, 6\}$ (see [4],[9],[10], [14]). For graphs with small maximum degree, Li and Li [12], Luo and Zhang [13], and Zhao [35] have published results, summarized in part below.

Theorem 2.1 *Let G be a Δ-critical graph with n vertices.*

$$|E(G)| \geq \begin{cases} \frac{5}{2}n & \Delta = 6 \\ \frac{17}{6}n & \Delta = 7 \\ \frac{25}{8}n & \Delta = 8 \\ \frac{17}{5}n & \Delta = 9 \\ \frac{37}{10}n & \Delta = 10 \\ \frac{8}{2}n & \Delta = 11 \\ \frac{23}{5}n & \Delta = 12 \end{cases}$$

Many results have been published regarding general lower bounds (see [3], [6], [35]). The best currently known bound is below:

Theorem 2.2 *(Woodall, [31])*

Let G be a Δ-critical graph with m edges and n vertices. Then,

$$m \geq \begin{cases} \frac{n\Delta}{3} + \frac{n}{3} & \Delta \geq 2 \\ \frac{n\Delta}{3} + \frac{n}{2} & \Delta \geq 8 \\ \frac{n\Delta}{3} + \frac{2n}{3} & \Delta \geq 15 \end{cases}$$

Although generally it is very difficult to determine the precise number of edges required for a graph to be critical, Plantholt ([22], [23]) established exact criteria for graphs G with $\Delta(G) = |V(G)| - 1$ and $\Delta(G) = |V(G)| - 2$.

Theorem 2.3
(1) Let G be a graph with $|V(G)| = 2s + 1$ and $\Delta(G) = 2s$ (i.e., G is a subgraph of K_{2s+1} with a spanning star). G is Class 2 if and only if $|E(G)| \geq 2s^2 + 1$.
(2)Let G be a graph with $|V(G)| = 2s + 2$ and $\Delta(G) = 2s$. G is Class 2 if and only if G has a vertex v such that $|E(G - v)| \geq 2s^2 + 1$.

2.2 Independence Number

Vizing's Independence Number Conjecture was verified for graphs with many edges (including for overfull graphs) by Grünewald and Steffen [5]. Luo and Zhao [15] verified the conjecture for graphs G with $|V(G)| \leq 2\Delta(G)$. It is interesting to note that although the conjecture is obviously true for 2-critical graphs (specifically, odd cycles), the conjecture remains open even for 3-critical graphs.

The first result for an arbitrary Δ-critical graph was in 2000:

Theorem 2.4 *(Brinkmann et al. [2]) If G is a Δ-critical graph, then $\alpha(G) < \frac{2}{3}|V(G)|$. In particular, if $\Delta \leq 10$, we have the following table of values.*

$$\alpha(G) \leq \begin{cases} \frac{7}{13}n \approx 0.538n & \Delta = 3 \\ \frac{5}{9}n \approx 0.556n & \Delta = 4 \\ \frac{13}{23}n \approx 0.565n & \Delta = 5 \\ \frac{4}{7}n \approx 0.571n & \Delta = 6 \\ \frac{29}{50}n \approx 0.580n & \Delta = 7 \\ \frac{17}{29}n \approx 0.586n & \Delta = 8 \\ \frac{13}{22}n \approx 0.591 & \Delta = 9 \\ \frac{22}{37}n \approx 0.595n & \Delta = 10 \end{cases}$$

Luo and Zhao [17] improved this bound for graphs with $7 \leq \Delta \leq 19$.

Theorem 2.5 *Let G be a Δ-critical graph with $\Delta \leq 19$. Then,*

$$\alpha(G) \leq \begin{cases} \frac{19}{33}n \approx 0.576n & \Delta = 7 \\ \frac{11}{19} \approx 0.579n & \Delta = 8 \\ \frac{25}{43} \approx 0.581n & \Delta = 9 \\ \frac{7}{12} \approx 0.583n & \Delta = 10 \\ \frac{79}{134} \approx 0.590n & \Delta = 11 \\ \frac{22}{37} \approx 0.595n & \Delta = 12 \\ \frac{97}{167} \approx 0.599n & \Delta = 13 \\ \frac{53}{88} \approx 0.602n & \Delta = 14 \\ \frac{23}{38} \approx 0.605n & \Delta = 15 \\ \frac{31}{51} \approx 0.608n & \Delta = 16 \\ \frac{133}{218} \approx 0.610n & \Delta = 17 \\ \frac{71}{116} \approx 0.612n & \Delta = 18 \\ \frac{151}{246} \approx 0.614n & \Delta = 19 \end{cases}$$

If 2010, Luo and Zhao [19] improved Brinkmann's general bound.

Theorem 2.6 *If G is a Δ-critical graph with $\Delta \geq 6$, then $\alpha(G) \leq \frac{5}{8}|V(G)|$.*

The general result was further improved in 2011:

Theorem 2.7 *(Woodall, [32]) If G is a Δ-critical graph then $\alpha(G) \leq \frac{3}{5}|V(G)|$.*

2.3 Edge coloring of graphs on surfaces

Vizing [29] was able to prove that if a planar graph G has maximum degree at least 8, then G is Class 1. In 2000, Zhang [34] and, independently in 2001, Sanders and Zhao [24] improved the gap in Vizing's Planar Graph Conjecture by showing that if G is a planar graph with maximum degree 7, then G is Class 1.

In 1970, Mel'nikov [21] found general bounds for $\Delta(\Sigma)$ in the case where the Euler characteristic of Σ is non-positive. His bounds result only from Vizing's Adjacency Lemma and Euler's formula.

Theorem 2.8 *Let Σ be a surface with Euler characteristic χ. If $\chi \leq 0$, then*

$$\Delta(\Sigma) \leq max\{\lfloor\frac{11+\sqrt{25-24\chi}}{2}\rfloor, \lfloor\frac{8+2\sqrt{52-18\chi}}{3}\rfloor\}$$

Beginning in 2003, using the discharging method in conjunction with newer adjacency lemmas, exact results for $\Delta(\Sigma)$ have been found for surfaces Σ with Euler characteristic $\chi(\Sigma) \in \{0,-1,-2,-3,-4,-5\}$:

Theorem 2.9 *(Sanders and Zhao, [26]) If $\chi(\Sigma) = 0$, then $\Delta(\Sigma) = 6$.*

Theorem 2.10 *(Luo, et al, [16], [18], [20])*

$$\Delta(\Sigma) = \begin{cases} 7 & \chi(\Sigma) = -1 \\ 8 & \chi(\Sigma) \in \{-2,-3,-4\} \\ 9 & \chi(\Sigma) = -5 \end{cases}$$

We have found the further exact result:

Theorem 2.11 *If $\chi(\Sigma) \in \{-6,-7\}$, then $\Delta(\Sigma) = 10$.*

3 Future Work

Exact values for $\Delta(\Sigma)$ of a suface Σ with $\chi(\Sigma) \in \{0,-1,-2,-3,-4,-5,-6,-7\}$ are now known.

Problem 1

If $\chi(\Sigma) = -8$, *then* $\Delta(\Sigma) = 10$. *If* $\chi(\Sigma) \in \{-9, -10\}$, *then* $\Delta(\Sigma) = 11$.

Although there has been much work on determining $\Delta(\Sigma)$ for surfaces of a specific Euler characteristic, no work has been published improving the general bounds. Mel'nikov's bounds result from arguments using only Vizing's Adjacency Lemma and counting arguments. Since that time, many other adjacency lemmas have been proven, which have been effective in conjunction with the discharging method.

Problem 2

Improve Mel'nikov's bounds for $\Delta(\Sigma)$ *using discharging methods and newer adjacency lemmas. Specifically, find a constant* k *such that* $\Delta(\Sigma) \leq H(\chi) + k$, *where* $H(\chi)$ *is the Heawood number of the surface.*

As mentioned in Woodall's 2009 paper [32], his result that $\alpha(G) \leq \frac{3}{5}|V(G)|$ was best possible, in that there is a class of graphs satisfying all known adjacency lemmas, and for which the independence number is asymptotically $\frac{3}{5}|V(G)|$.

Problem 3

Verify Vizing's Independence Number Conjecture for small values of Δ.

Problem 4

Improve Woodall's upper bound for $\alpha(G)$ *in critical graphs.*

Plantholt([22], [23]) determined precisely the class of graphs with maximum degree at least two less than the order, with the characterization dependent only on size.

Problem 5

Let G *be a* Δ*-critical graph of order* $\Delta + 3$ *or* $\Delta + 4$. *Determine the minimum size of* G *in order for* G *to be Class 2.*

Problem 6

Verify Vizing's Size Conjecture for $\Delta = 7$.

Problem 7

Determine some constant c *such that if* $\Delta(G) \geq cn$, *then* $|E(G)| \geq \frac{1}{2}(|V(G)|(\Delta(G)-1)+3)$.

References

[1] Appel, K. and Haken, W., A proof of the four color theorem, *Discrete Math* **16** (1976) 179180.

[2] G.Brinkmann, S.A. Choudum, S. Grünewald and E. Steffen, Bounds for the independence number of critical graphs, *Bull. London Math. Soc.* **32** (2000) 137-140.

[3] L. Clark and D. Haile, Remarks on the size of critical edge-chromatic graphs, *Discrete Math* **171** (1997) 287-293.

[4] S. Fiorini and R.J. Wilson, Edge colourings of graphs, *Pitman, San Francisco* (1997).

[5] S. Grünewald and E. Steffen, Independent sets and 2-factors in edge-chromatic-critical graphs, *J. of Graph Theory* **45** (2004) 113-118.

[6] D. Haile, Bounds on the size of critical edge-chromatic graphs, *Ars Combinatoria* **53** (1999) 85-96.

[7] Heawood, P. J., Map Colour Theorem, *Quart. J. Math.* **24** (1890) 332-338.

[8] I. Holyer, The NP-completness of edge-coloring, *SIAM J. Comput.* **10** (1981) 718-720.

[9] .T. Jakobsen, On critical graphs with chromatic index 4, *Discrete Math.*, 9 (1974) 265-276.

[10] K. Kayathri, On the size of edge-chromatic critical graphs, *Graphs and Combinatorics*, 10 (1994) 139-144.

[11] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, *Math. Ann.* **77** (1916) 453-465.

[12] S. Li and X. Li, Edge coloring of graphs with small maximum degrees, *Discrete Math* **309** (2009) 4843-4852.

[13] R. Luo and C.Q. Zhang, Edge coloring of graphs with small average degrees, *Discrete Math* **275** (2004) 207-218.

[14] R. Luo, L.Y. Miao and Y. Zhao, The size of edge chromatic critical graphs with maximum degree 6, *J. Graph Theory* **60** (2009) 149-171.

[15] R. Luo and Y. Zhao, A note on Vizing's independence number conjecture of edge chromatic critical graphs, *Discrete Mathematics* **306** (2006) 1788-1790.

[16] R. Luo and Y. Zhao, Finding the exact bound of the maximum degree of class two graphs embeddable in a surface of characteristic $\epsilon \in \{-1, -2, -3\}$, *J. of Combin. Theory, Ser. B* **98** (2008) 707-720.

[17] R. Luo and Y. Zhao, An application of Vizing and Vizing-like adjacencylemmas to Vizings independence number conjecture of edge chromatic critical graphs, *Discrete Math* **309** (2009) 29252929.

[18] R. Luo and Y. Zhao, Finding $\Delta(\Sigma)$ for a surface Σ of characteristic $\chi(\Sigma) = -5$, *J. Graph Theory* **68** (2011) 148-168.

[19] R. Luo and Y. Zhao, A new upper bound for the independence number of edge chromatic critical graphs, *J. Graph Theory* **68** (2011) 202-212

[20] R. Luo, Z.K. Miao and Y. Zhao, Finding $\Delta(\Sigma)$ for a surface Σ of characteristic $\chi(\Sigma) = -4$, *J. Graph Theory* **83** (2016) 277-302.

[21] L.S. Mel'nikov, The chromatic class and location of a graph on a closed surface, *Mat. Zametki* **7** (1970) 671-68 /*Math. Notes* **7** (1970) 405-411.

[22] M. Plantholt, The chromatic index of graphs with a spanning star, *J. Graph Theory* **5** (1981) 45?53.

[23] M. Plantholt, The chromatic index of graphs with large maximum degree, *Discrete Math* **47** (1983) 91?96.

[24] D. Sanders and Y. Zhao, Planar graphs of maximum degree seven are class I, *J. Combin. Theory Ser. B* **83** (2001) 201-212.

[25] D.P. Sanders and Y. Zhao, On the size of edge chromatic critical graphs, *J. Combin. Theory Ser. B* **86** (2002) 408-412.

[26] D. Sanders and Y. Zhao, Coloring edges of graphs embedded in a surface of characteristic zero, *J. Combin. Theory Ser. B* **87** (2003) 254-263.

[27] M. Steibitz, D. Scheide, B. Toft, L. Favrholdt, Graph Edge Coloring, Wiley.

[28] P.G. Tait, On the coloring of maps. *Proc. Roy. Soc. Edinburgh Sect. A*, 10 (1880) 501-503, 729.

[29] V.G. Vizing, Critical graphs with a given chromatic class (Russian), *Diskret. Analiz.* **5** (1965) 9-17.

[30] V.G. Vizing, Some unsolved problems in graph theory, *Uspekhi Mat. Nauk* **23** (1968) 117-134, *Russian Math. Surveys* **23** (1968) 125-142.

[31] D.R. Woodall, The average degree of an edge-chromatic critical graph II, *J. of Graph Theory* **56** (2007) 194-218.

[32] D.R. Woodall, The independence number of an edge-chromatic critical graph, *J. of Graph theory* **66** (2011) 98-103.

[33] Z. Yan and Y. Zhao, Edge colorings of embedded graphs, *Graphs and Combinatorics* **16** (2000) 245-256.

[34] L. Zhang, Every planar graph with maximum degree 7 is of class 1, *Graphs and Combinatorics* **16** (2000) 467-495.

[35] Y. Zhao, New lower bounds for the size of edge chromatic critical graphs, *J. of Graph Theory*, 46 (2004) 81-92.