Directions: Show all work. Answers without work generally do not earn points. This test has 60 points but is scored out of 50 (higher scores capped at 50).

1. [2 parts, 5 points each] Max flow/Min Cut.

(a) Find a flow of maximum value in the following network.

(b) Find a cut of minimum capacity in the same network (repeated below).
2. [4 points] A supermarket stocks 5 different vegetables (broccoli, carrots, lettuce, onions, and peas). At the same time, 5 customers (Beth, Eric, Jerry, Kim, and Sam) enter the store in search of vegetables. Unfortunately, due to limited supplies, it is not possible for more than 1 person to purchase any given vegetable. Sam and Jerry each like 2 vegetables. Kim and Beth each like 3 vegetables. Eric likes all 5 vegetables. Is it guaranteed that everyone can purchase a vegetable they like? Either show that this is the case or find a counterexample.

Not Guaranteed:

Since $E_2 = \{E, C, L3\}$, there is no perfect matching.

3. [3 parts, 2 points each] Let $G$ be the following bipartite graph.

(a) Find $R(\{x_1, x_3, x_4\})$.

$$\{y_1, y_2, y_3, y_4, y_5\}$$

(b) What is the deficiency of $\{x_2, x_3, x_4, x_6\}$?

$$\delta(S) = |S| - |R(S)| = 4 - |\{y_3, y_5\}| = 4 - 2 = 2$$

(c) What can you conclude from part (b) about the size of a maximum matching in $G$?

At least two vertices are in $\{x_1, x_3, x_4, x_6\}$ are unmatched, so a maximum matching has size at most 4.
4. Stable Matchings.

(a) [7 points] Given a set \{1, 2, 3, 4, 5\} of men and a set \{a, b, c, d, e\} of women with the following preference lists, find a stable matching.

\[
\begin{align*}
\text{Men propose:} & & \quad & \text{Women propose:} \\
\text{epdae} & 1 & b & \quad \text{cbdae} & 1 & a \\
\text{bcdae} & 2 & b & \quad \text{bcdae} & 2 & b \\
\text{dcbea} & 3 & c & \quad \text{dcbea} & 3 & c \\
\text{bcdae} & 4 & d & \quad \text{bcdae} & 4 & d \\
\text{ppded} & 5 & e & \quad \text{ebacd} & 5 & e \\
\end{align*}
\]

(b) [3 points] Which (if any) of the matched pairs are common to all stable matchings? (The preference lists are repeated below.)

\[
\begin{align*}
\text{Men propose:} & & \quad & \text{Women propose:} \\
\text{epdae} & 1 & b & \quad \text{cbdae} & 1 & a \\
\text{bcdae} & 2 & b & \quad \text{bcdae} & 2 & b \\
\text{dcbea} & 3 & c & \quad \text{dcbea} & 3 & c \\
\text{bcdae} & 4 & d & \quad \text{bcdae} & 4 & d \\
\text{ppded} & 5 & e & \quad \text{ebacd} & 5 & e \\
\end{align*}
\]

All of the matched pairs are common to all stable matchings, because there is only one stable matching.
5. **[4 points]** Is it possible for a stable matching to pair two people who are each others least desirable partners? Either argue that this is impossible or provide an example where this occurs.

   \[ \begin{array}{ccc}
   a & b & \hat{1} \\
   a & b & \hat{2} \\
   \end{array} \]

   In this situation, the only stable matching is \( \{a, 2b\} \), and 2 and b are each others least desirable partners.

6. **[4 parts, 1.5 points each]** Which of the following are groups? Answer yes or no for each; no justification required. Here, \( \mathbb{R} \) is the set of real numbers, \( \mathbb{R}^+ \) is the set of positive real numbers, + denotes standard arithmetic addition, and \( \times \) denotes standard arithmetic multiplication.

   (a) \((\mathbb{R}, +)\)
   
   \[ \boxed{\text{Yes}} \]

   (b) \((\mathbb{R}^+, +)\)
   
   \[ \boxed{\text{No}} \]

   (c) \((\mathbb{R}, \times)\)
   
   \[ \boxed{\text{No}} \text{ because } 0 \text{ has no multiplicative inverse} \]

   (d) \((\mathbb{R}^+, \times)\)
   
   \[ \boxed{\text{Yes}} \]
7. [2 points] If the received transmission is \( r = 00110110 \) and the error pattern is \( e = 00110011 \), what transmission was sent?

\[
\begin{array}{c}
\text{r} & 00110110 \\
\text{e} & 00110011 \\
\hline
\text{c} & 00000101
\end{array}
\]

8. Let \( W = \mathbb{Z}_2^5 \), so our messages are bitstrings of length 5. Consider the parity bit encoding scheme \( E: W \to \mathbb{Z}_2^6 \) given by \( E(x_1 \cdots x_5) = x_1 \cdots x_5 y \) where \( y = x_1 + \cdots + x_5 \mod 2 \). The decoding function \( D(x_1 \cdots x_5 y) \) first checks whether the parity bit \( y \) is correct; if it is, then the decoding function returns \( x_1 \cdots x_5 \) as the transmitted message. Otherwise, the decoding function reports an error. The transmission channel flips bits with probability \( p = 0.1 \).

(a) [2 points] If a message \( x \in W \) is sent without any encoding, what is the probability that it is received and decoded properly?

\[
(1-p)^5 = (0.9)^5 \approx 0.59
\]

(b) [2 points] A message is sent using the encoding scheme. If the received message is 011011, what does the decoding function do?

Since \( 1 = 0 + 1 + 1 + 0 + 1 \mod 2 \), the parity bit is correct, and the decoder returns the message 01101.

(c) [4 points] A message is sent using the encoding scheme. There are three possibilities: either the message is correctly decoded, there is an undetected error in transmission, or there is a detected error in transmission. Find the probabilities of each of these 3 cases.

\[
\text{Correct: No bit flips} - \quad (1-p)^5 = (0.9)^5 \approx 0.59
\]

\[
\text{Undetected error: 2, 4, or 6 flips:} \quad \binom{6}{2} p^2 (1-p)^4 + \binom{6}{4} p^4 (1-p)^2 + \binom{6}{6} p^6 (1-p)^0 \\
= 15 \cdot (0.9)^4 + 15 \cdot (0.9)^2 + (0.1)^6 \approx 0.0996
\]

\[
\text{Detected Error: 1, 3, or 5 flips:} \quad \binom{6}{1} p (1-p)^5 + \binom{6}{3} p^3 (1-p)^3 + \binom{6}{5} p^5 (1-p) \\
= 6 \cdot (0.1)(0.9)^5 + 20 \cdot (0.9)^3 + 6 \cdot (0.9)^5 \approx 0.369
\]
9. [2 points] List the elements in $S(1001, 1)$.

\[ \{1001, 0001\}, \{101, 1011, 1000, 2\} \]

10. [2 points] Let $x \in \mathbb{Z}_2^{12}$. How many elements are in $S(x, 4)$? You may leave your answer in terms of binomial coefficients.

\[ \binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} + \binom{12}{4} \]

11. Consider the encoding function $E: \mathbb{Z}_2^3 \to \mathbb{Z}_2^6$ given by

\[
E(00) = 000000 \quad \quad \quad \quad \quad \quad E(01) = 001111 \\
E(10) = 111100 \quad \quad \quad \quad \quad \quad E(11) = 110011 \\
\text{Minimum distance } 4
\]

(a) [2 points] If our goal is to detect errors, how many errors can we tolerate?

To detect: \[ \leq 3 \text{ errors} \]

(b) [1 point] If our goal is to detect errors and 110111 is received, what should the decoding function do?

Return [Error]

(c) [2 points] If our goal is to correct errors, how many errors can we tolerate?

\[
2k + 1 \leq 4 \quad \implies \quad k \leq \frac{3}{2} \quad \Rightarrow \quad \text{Tolerate at most 1 error.}
\]

\[
2k \leq 3
\]

(d) [1 point] If our goal is to correct errors and 101111 is received, what should the decoding function do?

101111 is within distance 1 of 001111,

So decoder returns 01.