1. (5pts) State the fundamental theorem of calculus.

If \( f \) is continuous on \([a,b]\), then the function \( g \) defined by

\[
g(x) = \int_a^x f(t) \, dt, \quad a \leq x \leq b
\]

is an antiderivative of \( f \).

That is, \( g'(x) = f(x) \) for \( a < x < b \).

2. (5pts) Given \( y = \int \cos dt \), find \( y'(x) \).

\[
y'(x) = \frac{d}{dx} \int_0^x \cos t \, dt = \frac{d}{du} \int_c^u \cos \, dt \cdot \frac{du}{dx} = \cos u \cdot \frac{du}{dx} = 2x \cos(x^2)
\]

\[
u = x^2, \quad \frac{du}{dx} = 2x
\]

3. (5pts) Given that \( f(x) \) is a continuous function on \( a \leq x \leq b \) except at \( x = c \) with \( a < c < b \), how would you properly evaluate an integral such as \( \int_a^b f(x) \, dx \)?

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]

\[
= \lim_{t \to c^-} \int_a^t f(x) \, dx + \lim_{s \to c^+} \int_s^b f(x) \, dx
\]

From Questions 4 through 6, choose 2 problems to answer fully. Clearly indicate which 2 to grade.

4. (10pts) Without the aid of tables, show all the steps necessary to integrate: \( \int_0^{\pi/2} x^2 \sin 2x \, dx \)

\[
\int_0^{\pi/2} x^2 \sin 2x \, dx = \left[ x^2 \left( -\frac{1}{2} \cos 2x \right) \right]_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} x \cos 2x \, dx
\]

\[
u = x^2, \quad \frac{du}{dx} = 2x \quad \mu = x, \quad \frac{dv}{dx} = \sin 2x \quad dx
\]

\[
u_x = \frac{x^2}{2}, \quad \mu_x = \frac{x}{2}, \quad \frac{du}{dx} = 2x, \quad \frac{dv}{dx} = 2 \cos 2x
\]

\[
= -\frac{1}{2} \left( \frac{\pi}{2} \right)^2 \cos(\pi) + \frac{1}{2} \left( \frac{\pi}{2} \right) \sin(\pi) + \frac{1}{4} \cos(0) - \left( 0 + 0 + \frac{1}{4} \cos(0) \right)
\]

\[
= -\frac{1}{2} \left( \frac{\pi^2}{4} \right)(-1) + \frac{1}{2} \left( \frac{\pi}{2} \right)(0) + \frac{1}{4}(-1) - \frac{1}{4}(1) = \frac{\pi^2}{8} - \frac{1}{2}
\]
5. (10 pts) **Without the aid of tables**, show all the steps necessary to integrate: \[ \int \frac{x^2}{\sqrt{9-x^2}} \, dx \]

\[ \cos \theta = \frac{x}{3} \quad \Rightarrow \quad \theta = \cos^{-1} \left( \frac{x}{3} \right) \]
\[ x = 3 \cos \theta \quad \Rightarrow \quad dx = -3 \sin \theta \, d\theta \]
\[ x^2 = 9 \cos^2 \theta \]
\[ \sqrt{9-x^2} = \sqrt{9-9\cos^2 \theta} = 3\sqrt{1-\cos^2 \theta} = 3 \sin \theta \]

Substitute:
\[ \int \frac{x^2}{\sqrt{9-x^2}} \, dx = -9 \int \cos^2 \theta \, d\theta = -9 \int \frac{1 + \cos 2\theta}{2} \, d\theta \]
\[ = -\frac{9}{2} \int d\theta - \frac{9}{2} \int \cos 2\theta \, d\theta = -\frac{9}{2} \theta - \frac{9}{2} \left( \frac{1}{2} \sin 2\theta \right) = -\frac{9}{2} \theta - \frac{9}{4} (2) \sin \theta \cos \theta \]
\[ = -\frac{9}{2} \cos^{-1} \left( \frac{x}{3} \right) - \frac{9}{2} \left( \frac{\sqrt{9-x^2}}{3} \right) \frac{x}{3} = -\frac{9}{2} \cos^{-1} \left( \frac{x}{3} \right) - \frac{1}{2} x \sqrt{9-x^2} + C \]
(Other variations possible with \( \sin \theta = \frac{x}{3} \))

6. (10 pts) **Without the aid of tables**, show all the steps necessary to integrate: \[ \int \frac{x+4}{x^3 + 3x^2 - 10x} \, dx \]

\[ \frac{x+4}{x^3 + 3x^2 - 10x} = \frac{x+4}{x(x^2+3x-10)} = \frac{x+4}{x(x+5)(x-2)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2} \]

\[ x + 4 = A(x+5)(x-2) + B(x)(x-2) + C(x)(x+5) \]

\[ x = 0 \quad \Rightarrow \quad 4 = A(5)(-2) + B(0)(-2) + C(0) \quad \Rightarrow \quad 4 = -10A \quad \Rightarrow \quad A = -\frac{2}{5} \]

\[ x = -5 \quad \Rightarrow \quad -1 = A(0) + B(-5)(-7) + C(0) \quad \Rightarrow \quad -1 = +35B \quad \Rightarrow \quad B = -\frac{1}{35} \]

\[ x = 2 \quad \Rightarrow \quad 6 = A(0) + B(0) + C(2)(7) \quad \Rightarrow \quad 6 = 14C \quad \Rightarrow \quad C = \frac{3}{7} \]

So:
\[ \int \frac{x+4}{x^3 + 3x^2 - 10x} \, dx = \int \left( \frac{-\frac{2}{5}}{x} + \frac{-\frac{1}{35}}{x+5} + \frac{\frac{3}{7}}{x-2} \right) \, dx \]
\[ = -\frac{2}{5} \ln |x| - \frac{1}{35} \ln |x+5| + \frac{3}{7} \ln |x-2| + C \]
7. (5pts) Given the picture below, write an integral that would appropriately describe the area of the shaded region.

\[ A = \int_c^d (q(y) - f(y)) \, dy \]

From Questions 8 through 9, choose 1 problem to answer fully. Clearly indicate which to grade.

8. (10pts) Given the following functional information, sketch the region and use the disk/washer method to set up an integral describing the volume of the rotated area.

\[ y = x^2, \text{ and } x = y^2, \text{ about } x = -1 \]

\[ V = \int_0^1 \left( A_{\text{OUT}} - A_{\text{IN}} \right) \, dy \]

\[ V = \pi \int_0^1 \left( (1 + \sqrt{y})^2 - (1 + y^2)^2 \right) \, dy \]
9. (10 pts) Given the following functional information, sketch the region and use the **shell method** to set up an integral describing the volume of the rotated area.

\[ y = \sqrt{x - 1}, \quad y = 0, \quad x = 5, \quad \text{about } y = 3 \]

\[
V = \int 2\pi rf(y) \, dy
\]

\[
r = 3 - y \quad \text{span on } y = 0 \rightarrow 2
\]

\[
f(y) = y^2 + 1
\]

\[
V = \int_0^2 2\pi (3-y)(\text{top-bottom}) \, dy
\]

\[
= \int_0^2 2\pi (3-y) [5 - (y^2 + 1)] \, dy
\]

10. (5pts) Given \( \sum_{n=0}^{\infty} a_n \), what does \( \lim_{n \to \infty} a_n = 1 \) mean?

\[
\sum_{n=0}^{\infty} a_n \text{ Diverges by test for divergence.}
\]

11. (5pts) If \( f(x) \) has a power series representation: \( f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \) with \( |x-a| < R \) (\( R \) = radius of convergence), what is the formula you should use to represent the coefficients, \( c_n \)?

\[
c_n = \frac{f^{(n)}(a)}{n!} \quad \text{where } f^{(n)}(a) \text{ is the n^{th} deriv evaluated at } a.
\]

**From Questions 12 through 14, choose 2 problems to answer fully. Clearly indicate which 2 to grade.**

12. (5pts) Determine if the following series converges or diverges.

\[
\sum_{n=1}^{\infty} \left( \frac{n}{3n+1} \right)^n
\]

Root Test: \( \lim_{n \to \infty} \sqrt[n]{\left( \frac{n}{3n+1} \right)^n} = \lim_{n \to \infty} \frac{n}{3n+1} \cdot \frac{1}{n} = \lim_{n \to \infty} \frac{1}{3 + \frac{1}{n}} = \frac{1}{3} < 1 \)

\( \therefore \) by Root Test, abs. convergent \( \therefore \) convergent.
13. (5 pts) Determine if the following series converges or diverges. \[ \sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1} \]

\[ \int_{2}^{\infty} \frac{\ln(x+1)}{x+1} \, dx = \lim_{t \to \infty} \int_{2}^{t} \frac{\ln(x+1)}{x+1} \, dx = \lim_{t \to \infty} \left[ \frac{1}{2} \ln(x+1)^2 \right]_{2}^{t} = \lim_{t \to \infty} \frac{1}{2} \ln(t+1)^2 - \frac{1}{2} \ln(3)^2 = \infty \]

by \( \ln(t+1)^2 \) is divergent.

14. (5 pts) Determine if the following series converges or diverges. \[ \sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)} \]

\[ \lim_{n \to \infty} \frac{10n+1}{n(n+1)(n+2)} = \lim_{n \to \infty} \frac{10n^3 + n^2}{10n^3 + 30n^2 + 20n} = \lim_{n \to \infty} \frac{10 + \frac{1}{n}}{10 + \frac{30}{n} + \frac{20}{n^2}} = 1 > 0 \]

So, by LCT, \( \sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)} \) is convergent b/c \( \sum_{n=1}^{\infty} \frac{10}{n^2} \) is convergent.

From Questions 15 through 16, choose 1 problem to answer fully. Clearly indicate which 1 to grade.

15. (10 pts) Find the radius of convergence and interval of convergence for the following series: \( \sum_{n=1}^{\infty} \frac{x^n}{n^3} \)

\[ \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)^{3n+1}} \cdot \frac{n^3}{x^n} \right| = \lim_{n \to \infty} \left| \frac{1}{3} \cdot \left| x \right| \right| = \frac{1}{3} \left| x \right| \leq 1 \]

Test Interval: \(-3 < x < 3\)

\( x = -3: \) \( \sum_{n=1}^{\infty} \frac{(-3)^n}{n^3} \)

\( x = +3: \) \( \sum_{n=1}^{\infty} \frac{3^n}{n^3} \)

Alt. Harmonic Conv.

So \( 1, C: -3 \leq x < 3 \) with \( R = 3 \).
16. (10 pts) Create a power series representation for the function: \( f(x) = \ln(4 + x) \)

Note: \( \frac{d}{dx} \left( \ln(x) \right) = \frac{1}{x} \)

\[
\frac{1}{4+x} = \frac{1}{4(1+\frac{1}{4}x)} = \frac{1}{4} \left( \frac{1}{1+\frac{1}{4}x} \right) = \frac{1}{4} \left( \frac{1}{1-\left(-\frac{1}{4}\right)x} \right)
\]

I know: \( \frac{1}{1-x} = \sum_{n=1}^{\infty} x^n \) \w/ \( R=1 \) So: \( \frac{1}{4} \left( \frac{1}{1-\left(-\frac{1}{4}\right)x} \right) = \frac{1}{4} \sum_{n=0}^{\infty} \left( -\frac{1}{4}x \right)^n \)

\[
= \sum_{n=0}^{\infty} \left( -\frac{1}{4} \right)^n \left( \frac{1}{4} \right)^n x^n = \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{4} \right)^{n+1} x^n \quad \text{with} \quad R=4
\]

To get back to \( \ln(4+x) \): by FTC \( \int \frac{1}{4} \ln(4+x) = \ln(4+x) \)

So: \( \ln(4+x) = \int \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{4} \right)^{n+1} x^n = \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{4} \right)^{n+1} x^{n+1} + C \)

17. (5pts) Given the polar curve: \( r = \frac{ed}{1+e \cos \theta} \), with \( e = \frac{3}{2} \), describe in words what specific shape you would expect the curve to follow.

Hyperbola opening on x-axis.

18. (5 pts) Sketch the following polar curve: \( r = 2 \cos \theta + 1 \)
19. (10 pts) Given the parametric equations: $x = t + \ln t$, $y = t - \ln t$.

Find the values of $t$ where the curve is concave upward.

From here, we know that $t > 0$ because of the logs. A domain restriction.

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - \frac{1}{t}}{1 + \frac{1}{t}} = \frac{t-1}{t+1}
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{\frac{t+1-t+1}{(t+1)^2}}{1 + \frac{1}{t}} = \frac{t}{t+1}
\]

Tricky part:

\[
\frac{d^2y}{dx^2} = \frac{2t}{(t+1)^3} < 0 \quad \text{for} \quad t > 0
\]

We ignore this part b/c we know the log has to keep $t > 0$

20. (10 pts) Set up the integral that describes the area that lies inside $r = 1$ and outside $r = 1 - \cos \theta$.

Points of intersection:

\[
\begin{align*}
1 &= 1 - \cos \theta \\
0 &= -\cos \theta \\
0 &= \cos \theta \\
\theta &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}
\end{align*}
\]

\[
A = 2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{1} r^2 \, dr \, d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left( r^3 \right)_{0}^{1} \, d\theta = \int_{0}^{\pi/2} \left( 1^3 - (1 - \cos \theta)^3 \right) \, d\theta
\]