Section R1 C Lines and Their Equations

To define a line, you need at a minimum a single point and a slope.

However, The single point and slope can present themselves in a variety of ways.

\[
\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y\text{-direction}}{\text{Change in } x\text{-direction}} \quad \text{with} \quad (x_0, y_0) \quad \text{and} \quad (x_1, y_1)
\]

\[
m = \frac{y_1 - y_0}{x_1 - x_0} \quad \text{and order, although it must be consistent, is irrelevant.}
\]

Notice: \[m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{-y_0 + y_1}{x_0 - x_1} = \frac{y_0 - y_1}{x_0 - x_1}\]

**Example:** Fine the slope between the points (3, 4), (1, 5)

\[
m = \frac{5 - 4}{1 - 3} = \frac{1}{-2} = -\frac{1}{2} \quad \text{or} \quad m = \frac{4 - 5}{3 - 1} = -\frac{1}{2} = -\frac{1}{2}
\]

Order simply does not matter as long as you are consistent.

It is not enough, however, to use slope between points to say they are on the same line.

**Example:** \((3, 4), (1, 5)\) and \((2, 0), (4, -1)\)

From before: \[m = -\frac{1}{2} \quad \text{and} \quad m = \frac{0 - (-1)}{2 - 4} = \frac{1}{-2} = -\frac{1}{2}\]

But are these points on the same line?
Visualize:

While 2 points do define a line, they do not define every line of that slope.

What's going on? Parallel lines share the same slope. But they lines have different intersection points the y-axis (y-intercepts).

Standard Slope intercept form: \( y = mx + b \) where \( m \) is the slope and \( b \) is the y-intercept or simply the known point \((0, b)\).

But what if we do not know the y-intercept?

If we assume we know a single point \( (x_1, y_1) \) and slope \( m \), how do we get to the \( y = mx + b \) ?

Well, we know a line has infinitely many points. Lets just call any one of them \( (x, y) \). Now we call on the definition of a slope.

And so: \( m = \frac{\Delta y}{\Delta x} = \frac{y - y_1}{x - x_1} \) must be true.

Now we just use this equation of \( m \) to define a general line:

\[
m(x - x_1) = y - y_1
\]

Normally as: \( y - y_1 = m(x - x_1) \) and called the “point-slope form of a line”
\[ y - y_1 = mx - mx_1 \]

or, \[ y = mx + \left( \frac{y_1 - mx_1}{1} \right) \]

This part is the general 'b' form \[ y = mx + b \]

Follow through with the algebra and notice:

Back to our parallel lines example:

**Example**: Find equations that represent the lines formed by: \( (3, 4) \), \( (1, 5) \) and \( (2, 0) \), \( (4, -1) \)

We recall:

\[ m = -\frac{1}{2} \quad \text{parallel} \quad m = -\frac{1}{2} \]

\[ y - y_1 = m(x - x_1) \]

\[ y - 4 = -\frac{1}{2}(x - 3) \]

Use \( (3, 4) \)

\[ y - 4 = -\frac{1}{2}x + \frac{3}{2} \]

\[ y = -\frac{1}{2}x + \frac{11}{2} \]

\[ y - y_1 = m(x - x_1) \]

\[ y - 0 = -\frac{1}{2}(x - 2) \]

Use \( (2, 0) \)

\[ y - 0 = -\frac{1}{2}x + 1 \]

This corresponds with our old picture!

Also note:

\[ y - y_1 = m(x - x_1) \]

\[ y - 5 = -\frac{1}{2}(x - 1) \]

Use \( (1, 5) \)

\[ y - 5 = -\frac{1}{2}x + \frac{1}{2} \]

\[ y = -\frac{1}{2}x + \frac{11}{2} \]

\[ y - y_1 = m(x - x_1) \]

\[ y - (-1) = -\frac{1}{2}(x - 4) \]

Use \( (2, 0) \)

\[ y - (-1) = -\frac{1}{2}x + 2 \]

\[ y = -\frac{1}{2}x + 1 \]

We get the same thing!

Extension: write the y-intercept as a coordinate on each line:

\[
\begin{align*}
(3, 4) \text{ and } (1, 5) & \quad \left(0, \frac{11}{2}\right) \\
(2, 0) \text{ and } (4, -1) & \quad (0, 1)
\end{align*}
\]

Extension: What coordinates are on each of these lines when \( x = 10? \)
\[ y = -\frac{1}{2} x + \frac{11}{2} \]
\[ y = -\frac{1}{2} (10) + \frac{11}{2} \]
\[ y = -5 + \frac{11}{2} \]
\[ y = -\frac{10}{2} + \frac{11}{2} = \frac{1}{2} \]

so, \( (10, \frac{1}{2}) \)

\[ y = -\frac{1}{2} x + 1 \]
\[ y = -\frac{1}{2} (10) + 1 \]
\[ y = -5 + 1 \]
\[ y = -4 \]

so, \( (10, -4) \)

Special Cases

**Horizontal Lines**

\[ m = \frac{\Delta y}{\Delta x} = \frac{b - b}{a - 0} = \frac{0}{a} \]

\[ m = 0 \]

\[ y = m \cdot x + b \]

So, \( y = 0x + b \)

\[ y = b \]

\[ x \text{-value} = \text{anything} \]
\[ y \text{-value} = \text{anything} \]

Which makes visual sense because the \( y \) has **constant value**.

**Vertical Lines**

\[ m = \frac{\Delta y}{\Delta x} = \frac{c - 0}{a - a} = \frac{c}{0} = \infty \]

normally we say \( m = \text{undefined} \)

How do we write \( y = ( \text{undefined} ) x + b \)?

We notice that the \( y \)-value can be anything but the \( x \)-value is always the same. As before we write \( x = a \) which makes visual sense. The \( x \) has **constant value**.