Concepts, Definitions, Laws, Theorems, Formulae:

(thm) If \( a > 0 \) and \( a \neq 1 \), then \( f(x) = a^x \) is a continuous function with domain = all reals, and range = \((0, \infty)\). In particular, \( a^x > 0 \) for all \( x \). If \( a, b > 0 \) and \( x, y \) are real numbers, then:

\[
\begin{align*}
 a^{x+y} &= a^x a^y \\
 a^{x-y} &= \frac{a^x}{a^y} \\
 (a^x)^y &= a^{xy} \\
 (ab)^x &= a^x b^x
\end{align*}
\]

(con) If \( a > 1 \), then \( \lim_{x \to \infty} a^x = \infty \) and \( \lim_{x \to -\infty} a^x = 0 \)  
(Easiest to remember with picture)

(con) If \( 0 < a < 1 \), then \( \lim_{x \to \infty} a^x = 0 \) and \( \lim_{x \to -\infty} a^x = \infty \)  
(Easiest to remember with picture)

(con) The \( x \)-axis is a horizontal asymptote of \( f(x) = a^x, a > 0 \).

(con) All of the properties we learn about the exponential function extend to the natural exponential function where specifically the letter \( a \) is replaced by the number \( e \).

(def) A function, \( f \), is called a one to one function if it never takes on the same value twice. That means:

\[ f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2 \]

(con) A function is one to one if and only if no horizontal line intersects its graph more than once.

(def) Let \( f \) be a one to one function with domain \( A \) and range \( B \). Then its inverse function \( f^{-1} \) has domain \( B \) and range \( A \) and is defined by: \( f^{-1}(y) = x \) whenever \( f(x) = y \) for any \( y \) in \( B \).

(con) domain of \( f^{-1} = \) range of \( f \)

(con) range of \( f^{-1} = \) domain of \( f \)

(con) Some nice properties of the inverse function:

\[
\begin{align*}
 f^{-1}(f(x)) &= x \quad \text{for every} \quad x \in A \\
 f(f^{-1}(x)) &= x \quad \text{for every} \quad x \in B
\end{align*}
\]

(thm) If \( f \) is a one to one continuous function defined on an interval, then its inverse function \( f^{-1} \) is also continuous.
If \( f \) is a one to one differentiable function with inverse function \( f^{-1} \) and \( f'(f^{-1}(a)) \neq 0 \), then the inverse function is differentiable at \( a \) and \( (f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} \).

One can write \( \log_a x = y \), the logarithmic function with base \( a \) of \( x \) equals \( y \), as an equivalence to the particular exponential function \( a^y = x \) because the functions are functional inverses of each other.

\[ \log_a(a^x) = x \text{ for every real number } x \text{ and } a^{\log_a x} = x \text{ for every positive } x. \]

The logarithm laws:
\[
\log_a(xy) = \log_a x + \log_a y \\
\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y \\
\log_a x^r = r \log_a x
\]

If \( a > 1 \), then \( \lim_{x \to \infty} \log_a x = \infty \) (Easiest to remember with picture)

If \( a > 1 \), then \( \lim_{x \to 0^+} \log_a x = -\infty \) (Easiest to remember with picture)

The \( y \)-axis is a vertical asymptote of \( y = \log_a x \).

Change of Base formula:
\[ \log_a x = \frac{\log_b x}{\log_b a} \]

Derivatives of logarithms:
\[ \frac{d}{dx} (\log_a x) = \frac{1}{x \ln a} \]

Derivatives of exponential functions:
\[ \frac{d}{dx} (a^x) = a^x \ln(a) \]

Exponential growth and decay problems are governed by the equation: \( y(t) = y(0)e^{kt} \)

Be able to derive and use the derivatives of inverse trig functions:
\[
\begin{align*}
\frac{d}{dx} (\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}}, \quad x \in (-1, 1) \\
\frac{d}{dx} (\csc^{-1} x) &= \frac{-1}{x\sqrt{x^2 - 1}} \\
\frac{d}{dx} (\cos^{-1} x) &= \frac{-1}{\sqrt{1-x^2}}, \quad x \in (-1, 1) \\
\frac{d}{dx} (\sec^{-1} x) &= \frac{1}{x\sqrt{x^2 - 1}} \\
\frac{d}{dx} (\tan^{-1} x) &= \frac{1}{1+x^2}, \quad x \in \mathbb{R} \\
\frac{d}{dx} (\cot^{-1} x) &= \frac{-1}{1+x^2}
\end{align*}
\]
L'Hopital Rule

Suppose \( f \) and \( g \) are differentiable and \( g'(x) \neq 0 \) near \( a \). Furthermore, suppose that \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = 0 \) or that \( \lim_{x \to a} f(x) = \pm \infty \) and \( \lim_{x \to a} g(x) = \pm \infty \).

then: \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \) provided the limit exists.

To use L'Hopital rule, you must be in the form of \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).

To calculate limits involving other indeterminate forms (things like \( 1^\infty, \infty - \infty, \infty \cdot 0, etc \), you must massage the algebra to L'Hopital structure, sometimes using logarithms to get where you need to go.

Know the definitions of hyperbolic functions:

\[
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{1}{\tanh x} \\
\csc h x &= \frac{1}{\sinh x} \\
\sech x &= \frac{1}{\cosh x} \\
\end{align*}
\]

(Be able to visualize their graphs.)

Be able to derive and use the derivatives of hyperbolic trig functions:

\[
\begin{align*}
\frac{d}{dx}(\sinh x) &= \cosh x \\
\frac{d}{dx}(\cosh x) &= \sinh x \\
\frac{d}{dx}(\tanh x) &= \text{sech}^2 x \\
\frac{d}{dx}(\coth x) &= -\text{csch}^2 x \\
\frac{d}{dx}(\csc h x) &= -\frac{1}{\sinh x} \\
\frac{d}{dx}(\sech x) &= -\frac{1}{\cosh x} \\
\end{align*}
\]

If needed, I will provide you with the derivatives of the inverse hyperbolic functions:

\[
\begin{align*}
\frac{d}{dx}(\sinh^{-1} x) &= \frac{1}{\sqrt{1 + x^2}} \\
\frac{d}{dx}(\cosh^{-1} x) &= \frac{1}{\sqrt{x^2 - 1}} \\
\frac{d}{dx}(\tanh^{-1} x) &= \frac{1}{1 - x^2} \\
\frac{d}{dx}(\coth^{-1} x) &= \frac{1}{1 - x^2} \\
\frac{d}{dx}(\csc h^{-1} x) &= -\frac{1}{x \sqrt{x^2 + 1}} \\
\frac{d}{dx}(\sech^{-1} x) &= -\frac{1}{x \sqrt{1 - x^2}}
\end{align*}
\]