Review Sheet:  Chapter 1

Content:  “Essential Calculus, Early Transcendentals,” James Stewart, 2007
Chapter 1: Functions and Limits

Concepts, Definitions, Laws, Theorems:

(def) A function, $f$, is a rule that assigns to each element $x$ in a set $A$ exactly one element, called $f(x)$, in a set $B$.

In this definition,  
domain of $f$ = the set $A$  
range of $f$ = the set of all possible values of $f(x)$ as $x$ varies in $A$

(def) An independent variable is a symbol that represents an arbitrary number in the domain of a function $f$.

(def) A dependent variable is a symbol that represents a number in the range of $f$.

(def) A graph of a function is the set of ordered pairs $(x, f(x))$ so long as $x \in domain(f)$.

(con) The vertical line test. A curve in the $xy$–plane is the graph of a function of $x$ if and only if no vertical line intersects the curve more than once.

(def) The absolute value of a number, $a$, denoted by $|a|$, is the distance from $a$ to 0 on the real number line.

In general, $|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$

(thm) $|a| \geq 0$ for every number $a$.

(def) For $f$ to be an even function, $f(x) = f(-x)$ for every number $x$ in its domain.

(def) For $f$ to be an odd function, $f(x) = -f(-x)$ for every number $x$ in its domain.

(def) A function $f$ is called increasing on an interval $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in $I$.

(def) A function $f$ is called decreasing on an interval $I$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in $I$.

(def) A function $P$ is called a polynomial if $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$.

In this definition, the letter $n$ is a non-negative integer (in math, we say $n \in \mathbb{Z}^+$ to denote this idea), and the numbers $\{a_0, a_1, a_2, \ldots, a_{n-1}, a_n\}$ are all constants called coefficients. The degree of a polynomial is the highest power of $x$ so long as the coefficient of the associated term is not 0.

(con) The domain of all polynomial functions is the set of all real numbers (in math, we say $x \in \mathbb{R}$ or $x \in (-\infty, \infty)$ to denote this idea). (The range will vary depending on the function.)

(con) Lines ($f(x) = mx + b$), Quadratic functions aka parabolas ($f(x) = ax^2 + bx + c$) and Cubic functions
(f(x) = ax^3 + bx^2 + cx + d) are all simple examples of polynomials.

(def) Functions that look like \( f(x) = x^a \) \((a = \text{constant})\) are called \textbf{power functions}.

(con) Power functions help us build other kinds of functions. For example, if in \( f(x) = x^a \) the exponent happens to be a positive integer, \( a = n \), we get pieces that make up polynomials. If, instead, the exponents are the reciprocals of \( n \), so \( a = \frac{1}{n} \), we get root functions. Other exponents give us different variations on this idea.

(def) \textbf{Rational functions} are the ratio of two polynomials. For example: \( f(x) = \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) are polynomials and \( q(x) \neq 0 \) for any \( x \) in the domain of \( f(x) \)

(con) It is often useful to remember that \(-1 \leq \sin x \leq 1\) \(\text{AND}\) \(-1 \leq \cos x \leq 1\).

(def) Functions that look like \( f(x) = a^x \), \((a > 0)\), are called \textbf{exponential functions}.

(con) The domain of all exponential functions is the set of all real numbers, and the range is \( f(x) \in (0, \infty) \).

(def) Functions that look like \( f(x) = \log_a x \), \((a > 0)\), are called \textbf{logarithmic functions}.

(con) The domain of all logarithmic functions is the set of \( x \in (0, \infty) \), and the range is \( f(x) \in \mathbb{R} \).

(con) Graph shifting: Suppose that \( c > 0 \) and some general function \( y = f(x) \)
\[ y = f(x) + c \] corresponds to a vertical shift upward of \( c \) units
\[ y = f(x) - c \] corresponds to a vertical shift downward of \( c \) units
\[ y = f(x - c) \] corresponds to a horizontal shift right of \( c \) units
\[ y = f(x + c) \] corresponds to a horizontal shift left of \( c \) units

(con) Graph scaling, stretching, and reflecting: Suppose that \( c > 1 \) and some general function \( y = f(x) \)
\[ y = cf(x) \] corresponds to a vertical stretch by a factor of \( c \)
\[ y = \frac{1}{c} f(x) \] corresponds to a vertical compression by a factor of \( c \)
\[ y = f(cx) \] corresponds to a horizontal compression by a factor of \( c \)
\[ y = f\left(\frac{x}{c}\right) \] corresponds to a horizontal stretch by a factor of \( c \)
\[ y = -f(x) \] corresponds to a reflection of \( y = f(x) \) about the \( x \)-axis
\[ y = f(-x) \] corresponds to a reflection of \( y = f(x) \) about the \( y \)-axis

(def) Given two functions, \( f \) and \( g \), the \textbf{composite function} \( f \circ g \) is defined by: \((f \circ g)(x) = f(g(x))\)

(def) We write \( \lim_{x \to a} f(x) = L \) \textbf{(the limit of} \( f(x) \text{ as} \ x \text{ approaches} \ a \text{ equals} \ L)\) if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) to be sufficiently close to \( a \) but not equal to \( a \).
(def) We write \( \lim_{x \to a} f(x) = L \) (the limit of \( f(x) \) as \( x \) approaches \( a \) from the left equals \( L \)) if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) to be sufficiently close to \( a \) but less than \( a \).

(Def) We write \( \lim_{x \to a} f(x) = L \) (the limit of \( f(x) \) as \( x \) approaches \( a \) from the right equals \( L \)) if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) to be sufficiently close to \( a \) but greater than \( a \).

(thm) \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} f(x) = L \) (LEFT = RIGHT)

(def) Analysis Version: We write \( \lim_{x \to a} f(x) = L \) (the limit of \( f(x) \) as \( x \) approaches \( a \) equals \( L \)) if for every \( \varepsilon > 0 \), there is a corresponding \( \delta > 0 \) so that
if \( 0 < |x - a| < \delta \) (if we bound a small circle of radius \( \delta \) around the point \( a \)) then
\[ |f(x) - L| < \varepsilon \] (the value \( L \) is within \( \varepsilon \) units of \( f(x) \)).

(law) Suppose that \( c \) is a constant and \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist, then the following are true:

Sum Law: \[ \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \]

Difference Law: \[ \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \]

Constant Multiple Law: \[ \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x) \]

Product Law: \[ \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \]

Quotient Law: \[ \lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{provided} \quad \lim_{x \to a} g(x) \neq 0 \]

Power Law: \[ \lim_{x \to a} [f(x)]^n = \left[ \lim_{x \to a} f(x) \right]^n \quad \text{where} \quad n \in \mathbb{Z}^+ \]

Root Law: \[ \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \quad \text{where} \quad n \in \mathbb{Z}^+ \]

(con) The limit of a constant is a constant. \[ \lim_{x \to a} c = c \]

Substitution works on power functions. \[ \lim_{x \to a} x^n = a^n \quad \text{where} \quad n \in \mathbb{Z}^+ \]

Substitution works on root functions. \[ \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{where} \quad n \in \mathbb{Z}^+ \quad \text{(don’t break domain rules)} \]

Substitution works on polynomials. \[ \lim_{x \to a} p(x) = p(a) \]

Substitution works on rational functions. \[ \lim_{x \to a} \left[ \frac{p(x)}{q(x)} \right] = \frac{p(a)}{q(a)} \quad \text{provided} \quad q(a) \neq 0 \]

Substitution works on sine and cosine. \[ \lim_{x \to a} \sin x = \sin a \quad \text{and} \quad \lim_{x \to a} \cos x = \cos a \]

(thm) If \( f(x) \leq g(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist, then
\[ \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \]
THE SQUEEZE THEOREM: If $f(x) \leq g(x) \leq h(x)$ when $x$ is near $a$ (except possibly at $a$) and 
\[ \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \] then 
\[ \lim_{x \to a} g(x) = L. \]

\[ \sin x \quad x \to 0 \quad \frac{\sin x}{x} = 1 \]

\[ \cos x - 1 \quad x \to 0 \quad \frac{\cos x - 1}{x} = 0 \]

A function $f$ is **continuous at a number** $a$ if 
\[ \lim_{x \to a} f(x) = f(a). \]

Continuity requires 3 things: 1. $f(a)$ must be defined. 2. $\lim_{x \to a} f(x)$ must exist. 3. $\lim_{x \to a} f(x) = f(a)$ The limit of $f(x)$ as $x$ approaches $a$ must equal the function value at $x = a$.

A function $f$ is **continuous from the right** at a number $a$ if 
\[ \lim_{x \to a^+} f(x) = f(a). \]

A function $f$ is **continuous from the left** at a number $a$ if 
\[ \lim_{x \to a^-} f(x) = f(a). \]

A function $f$ is **continuous on an interval** if it is continuous at every number in the interval.

If $f(x)$ and $g(x)$ are continuous at a point $x = a$ and $c$ is a constant, then the following functions are also continuous at $x = a$:
\[ (f + g)(x), \quad (f - g)(x), \quad cf(x), \quad (fg)(x), \quad \left( \frac{f}{g}(x) \right) \quad \text{provided} \quad g(a) \neq 0. \]

Any polynomial is continuous everywhere. i.e. Polynomials are continuous on $x \in \mathbb{R}$ (their domain). Any rational function is continuous everywhere it is defined (i.e. on its domain).

More generally: The following types of functions are continuous at every number in their domains: polynomials, rational functions, root functions, trigonometric functions.

If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$, then 
\[ \lim_{x \to a} f(g(x)) = f(b). \] You can also think about this in the following way: 
\[ \lim_{x \to a} f(g(x)) = f\left( \lim_{x \to a} g(x) \right) \]

If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then the composite function $(f \circ g)(x)$ is continuous at $a$.

**THE INTERMEDIATE VALUE THEOREM:** Suppose that $f$ is continuous on the closed interval $[a,b]$ and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a Number $c$ in $(a,b)$ such that $f(c) = N$. 

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The notation \( \lim_{x \to a} f(x) = \infty \) means that the values of \( f(x) \) can be made arbitrarily large (as large as we can imagine) by taking \( x \) sufficiently close to \( a \) (on either side of \( a \)) but not equal to \( a \). This indicates the presence of a **vertical asymptote** at \( x = a \).

The line \( x = a \) is called a **vertical asymptote** of the curve \( y = f(x) \) if at least one of the following statements is true:

\[
\begin{align*}
\lim_{x \to a^+} f(x) &= \infty \\
\lim_{x \to a^-} f(x) &= \infty \\
\lim_{x \to a^+} f(x) &= -\infty \\
\lim_{x \to a^-} f(x) &= -\infty \\
\lim_{x \to a^+} f(x) &= \pm \infty \\
\lim_{x \to a^-} f(x) &= \pm \infty
\end{align*}
\]

Let \( f \) be a function defined on some interval \((a, \infty)\). Then \( \lim_{x \to \infty} f(x) = L \) means that the values of \( f(x) \) can be made as close to \( L \) as we like by taking \( x \) sufficiently large. This indicates the presence of a **horizontal asymptote** of the curve \( y = f(x) \) at \( y = L \). This is likewise true if \( \lim_{x \to -\infty} f(x) = L \).

\[
\begin{align*}
\lim_{x \to \infty} \frac{1}{x^n} &= 0 \\
\lim_{x \to -\infty} \frac{1}{x^n} &= 0
\end{align*}
\]

provided \( n \) is a really big positive integer.