Topics to be covered on a PhD entrance exam in topology, Spring 2000

- Examples of topological spaces.
- Separation axioms ($T_0$, $T_1$, Hausdorff, regular, and normal spaces).
- Metric space topology (completeness, equivalent forms of compactness).
- Continuity.
- Connected spaces.
- Compactness.

Suggested reference books.

- Dugundji, *Topology*, Allyn & Bacon. (Chapters I-IX and XI.)
- Kelly, *General Topology*, D. van Nostrand. (Chapters: all except II, VI, and Appendix.)
- Gemignani, *Elementary Topology*, Addison-Wesley. (Chapters: all except XI.)
NAME (print): ______________________

Topology Ph.D. Entrance Exam, August 2000

In the exercises that follow $\overline{A}$ stands for the closure of $A$, and $A \setminus B$ for the set difference: $A \setminus B = \{x \in A: x \notin B\}$.

Ex. 1. (a) Define a $T_0$ topological space.

(b) Show that a topological space $X$ is a $T_0$-space if and only if $\{x\} \neq \{y\}$ for every distinct $x, y \in X$.

Ex. 2. A topological space $X$ is said to be completely regular provided that for each $p \in X$ and closed set $A$ in $X$ such that $p \notin A$, there is a continuous function $f: X \to [0, 1]$ such that $f(p) = 0$ and $f[A] = \{1\}$.

Prove that any subspace of a completely regular space is completely regular.

Ex. 3. Let $X$ be a topological space and let $A$ and $B$ be non-empty proper closed subsets of $X$ such that $X = A \cup B$. Show that $X \setminus (A \cap B)$ is not connected.

Ex. 4. (a) Give an example of sets $A_i$ $(i = 1, 2, 3, \ldots)$ in a topological space for which

$$\bigcup_{i=1}^{\infty} A_i \neq \bigcup_{i=1}^{\infty} \overline{A_i}.$$ 

(b) Show that for any family $\{A_i: i = 1, 2, 3, \ldots\}$ of subsets of a topological space $X$ the following formula holds:

$$\bigcap_{i=1}^{\infty} A_i = \overline{\bigcup_{i=1}^{\infty} A_i} \cup \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i.$$ 

Ex. 5. Let $S = \langle \mathbb{R}, \tau_S \rangle$ be a Sorgenfrey line, $D(\mathbb{N}) = \langle \mathbb{N}, \tau_D \rangle$ be a discrete topology on $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $D(\mathbb{R}) = \langle \mathbb{R}, \tau_D \rangle$ be a discrete topology on $\mathbb{R}$.

Show that there is a continuous mapping from $S$ onto $D(\mathbb{N})$ but that there is no continuous mapping from $S$ onto $D(\mathbb{R})$.

Ex. 6. Let $X$ be a normal space and let $U_1$ and $U_2$ be open subsets of $X$ such that $X = U_1 \cup U_2$. Show that there are open sets $V_1$ and $V_2$ such that $\overline{V_1} \subset U_1$, $\overline{V_2} \subset U_2$, and $X = V_1 \cup V_2$. 

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Ex. 1. Let \( \langle X_0, \tau_0 \rangle \) and \( \langle X_1, \tau_1 \rangle \) be connected topological spaces. Show that \( X_0 \times X_1 \) with the product topology is connected.

Ex. 2. Consider the real line \( \mathbb{R} \) with the topology \( \tau \) generated by the family of intervals:
\[
\mathcal{F} = \{ [a, b): a \in \mathbb{Q} \& b \in \mathbb{R} \& a < b \},
\]
where \( \mathbb{Q} \) stands for the set of rational numbers. Let \( X \) be the product of \( \langle \mathbb{R}, \tau \rangle \) with itself (with the product topology). Prove or disprove that \( X \) is normal.

Ex. 3. Prove or find a counterexample for the statement:

A compact subset of a topological space \( \langle X, \tau \rangle \) is closed in \( X \).

Ex. 4. Let \( \tau \) be the usual topology on the real line \( \mathbb{R} \). Answer one of the following two questions.

(a) Does there exists a topology \( \tau_0 \subset \tau \) such that \( \langle \mathbb{R}, \tau_0 \rangle \) is homeomorphic to figure eight (i.e., two circles tangent at a point)?

(b) Does there exists a topology \( \tau_0 \subset \tau \) such that \( \langle \mathbb{R}, \tau_0 \rangle \) is homeomorphic to the unit circle \( S^1 = \{ (x, y) \in \mathbb{R}^2: x^2 + y^2 = 1 \} \)?

Ex. 5. Let \( \langle X, \tau \rangle \) and \( \langle Y, \tau' \rangle \) be the topological spaces and let \( f: X \to Y \) be a function. Consider the graph \( G(f) = \{ (x, f(x)): x \in X \} \) of \( f \) as a subspace of the cartesian product \( X \times Y \) (with the product topology). Prove or disprove each the following.

(a) If \( f \) is continuous, then \( G(f) \) is homeomorphic to \( X \).

(b) If \( G(f) \) is homeomorphic to \( X \), then \( f \) is continuous.
Ex. 1. Let $\mathbb{R}^2$ be the euclidean plane (i.e., with natural topology). Let

$$X = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \cup \{ (x, 0) \in \mathbb{R}^2 : -1 \leq x \leq 1 \},$$

$$Y = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \cup \{ (x, 0) \in \mathbb{R}^2 : -1 \leq x \leq 2 \}.$$

Are $X$ and $Y$ homeomorphic? Give reasons for your answer.

Ex. 2. Prove that every compact metric space has a countable base for its topology.

Ex. 3. Let $\langle X, d \rangle$ be a compact metric space, and let $f : X \to X$ satisfy

$$d(f(x_1), f(x_2)) < d(x_1, x_2) \text{ for all distinct } x_1, x_2 \in X.$$

Show that there is a point $p \in X$ such that $f(p) = p$.

Ex. 4. A topological space $X$ is said to have countable pseudo character provided every singleton in $X$ is a $G_\delta$-set (i.e., it is a countable intersection of open sets). Show that every compact Hausdorff space with countable pseudo character is first countable, that is, it has a countable local base at every point $x \in X$.

Ex. 5. Let $\mathcal{F}$ be the family of all non-zero polynomials of the form

$$w(x, y) = a_0 x^2 + a_1 y^2 + a_2 xy + a_3 x + a_4 y + a_5$$

with rational coefficients and for every $w \in \mathcal{F}$. Let

$$E_w = \{ (x, y) \in \mathbb{R}^2 : w(x, y) = 0 \}.$$

Show that the plane $\mathbb{R}^2$ is not covered by the sets $E_w$ with $w \in \mathcal{F}$, that is, that $\mathbb{R}^2 \neq \bigcup_{w \in \mathcal{F}} E_w$. 

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