Problem Set 1.1, page 8

1. The combinations give (a) a line in $\mathbb{R}^3$ (b) a plane in $\mathbb{R}^3$ (c) all of $\mathbb{R}^3$.

2. $v + w = (2, 3)$ and $v - w = (6, -1)$ will be the diagonals of the parallelogram with $v$ and $w$ as two sides going out from $(0, 0)$.

3. This problem gives the diagonals $v + w$ and $v - w$ of the parallelogram and asks for the sides. The opposite of Problem 2. In this example $v = (3, 3)$ and $w = (2, -2)$.

4. $3v + w = (7, 5)$ and $c v + d w = (2c + d, c + 2d)$.

5. $u + v = (-2, 3, 1)$ and $u + v + w = (0, 0, 0)$ and $2u + 2v + w = (c)$ add first answers $= (-2, 3, 1)$. The vectors $u, v, w$ are in the same plane because a combination gives $(0, 0, 0)$. Stated another way: $u = -v - w$ is in the plane of $v$ and $w$.

6. The components of every $cv + dw$ add to zero. $c = 3$ and $d = 9$ give $(3, 3, -6)$.

7. The nine combinations $c(2, 1) + d(0, 1)$ with $c = 0, 1, 2$ and $d = (0, 1, 2)$ will lie on a lattice. If we took all whole numbers $c$ and $d$, the lattice would lie over the whole plane.

8. The other diagonal is $v - w$ (or else $w - v$). Adding diagonals gives $2v$ (or $2w$).

9. The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$. Three possible parallelograms!

10. $i - j = (1, 1, 0)$ is in the base ($xy$ plane). $i + j + k = (1, 1, 1)$ is the opposite corner from $(0, 0, 0)$. Points in the cube have $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

11. Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

12. A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and $24$ two-dimensional faces and $32$ edges in Worked Example 2.4 A.

13. Sum = zero vector. Sum = $-2:00$ vector = $8:00$ vector. $2:00$ is $30^\circ$ from horizontal $= (\cos \frac{\pi}{6}, \sin \frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

14. Moving the origin to $6:00$ adds $j = (0, 1)$ to every vector. So the sum of twelve vectors changes from $0$ to $12j = (0, 12)$.

15. The point $\frac{3}{4} v + \frac{1}{4} w$ is three-fourths of the way to $v$ starting from $w$. The vector $\frac{1}{4} v + \frac{1}{4} w$ halfway to $u = \frac{1}{2} v + \frac{1}{2} w$. The vector $v + w$ is $2u$ (the far corner of the parallelogram).

16. All combinations with $c + d = 1$ are on the line that passes through $v$ and $w$. The point $V = -2v + 2w$ is on that line but it is beyond $w$.

17. All vectors $cv + cw$ are on the line passing through $(0, 0)$ and $u = \frac{1}{2} v + \frac{1}{2} w$. That line continues out beyond $v + w$ and back beyond $(0, 0)$. With $c \geq 0$, half of this line is removed, leaving a ray that starts at $(0, 0)$.

18. The combinations $cv + dw$ with $0 \leq c \leq 1$ and $0 \leq d \leq 1 fill the parallelogram with sides $v$ and $w$. For example, if $v = (1, 0)$ and $w = (0, 1)$ then $cv + dw$ fills the unit square.

19. With $c \geq 0$ and $d \geq 0$ we get the infinite “cone” or “wedge” between $v$ and $w$. For example, if $v = (1, 0)$ and $w = (0, 1)$, then the cone is the whole quadrant $x \geq 0, y \geq 0$. Question: What if $w = -v$? The cone opens to a half-space.
Solutions to Exercises

20 (a) \( \frac{1}{3}u + \frac{1}{3}v + \frac{1}{3}w \) is the center of the triangle between \( u \), \( v \) and \( w \); \( \frac{1}{3}u + \frac{1}{3}w \) lies between \( u \) and \( w \) (b) To fill the triangle keep \( c \geq 0, d \geq 0, e \geq 0 \), and \( c + d + e = 1 \).

21 The sum is \( (u - u) + (w - v) + (u - w) \) = zero vector. Those three sides of a triangle are in the same plane!

22 The vector \( \frac{1}{2}(u + v + w) \) is outside the pyramid because \( c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1 \).

23 All vectors are combinations of \( u, v, w \) as drawn (not in the same plane). Start by seeing that \( cu + dv \) fills a plane, then adding \( ew \) fills all of \( \mathbb{R}^3 \).

24 The combinations of \( u \) and \( v \) fill one plane. The combinations of \( v \) and \( w \) fill another plane. Those planes meet in a line: only the vectors \( cu \) are in both planes.

25 (a) For a line, choose \( u = v = w = \) any nonzero vector (b) For a plane, choose \( u \) and \( v \) in different directions. A combination like \( w = u + v \) is in the same plane.

26 Two equations come from the two components: \( c + 3d = 14 \) and \( 2c + d = 8 \). The solution is \( c = 2 \) and \( d = 4 \). Then \( 2(1, 2) + 4(3, 1) = (14, 8) \).

27 The combinations of \( i = (1, 0, 0) \) and \( i + j = (1, 1, 0) \) fill the \( xy \) plane in \( xyz \) space.

28 There are 6 unknown numbers \( v_1, v_2, v_3, w_1, w_2, w_3 \). The six equations come from the components of \( v + w = (4, 5, 6) \) and \( v - w = (2, 5, 8) \). Add to find \( 2v = (6, 10, 14) \) so \( v = (3, 5, 7) \) and \( w = (1, 0, -1) \).

29 Two combinations out of infinitely many that produce \( b = (0, 1) \) are \(-2u + v = 1/2w = 1/2v \). No, three vectors \( u, v, w \) in the \( x \)-\( y \) plane could fail to produce \( b \) if all three lie on a line that does not contain \( b \). Yes, if one combination produces \( b \) then two (and infinitely many) combinations will produce \( b \). This is true even if \(u = 0 \); the combinations can have different \( cu \).

30 The combinations of \( v \) and \( w \) fill the plane unless \( v \) and \( w \) lie on the same line through \( (0, 0) \). Four vectors whose combinations fill 4-dimensional space: one example is the “standard basis” \((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) \).

31 The equations \( cu + dv + ew = b \) are

\[
\begin{align*}
2c - d &= 1 \\
-c + 2d - e &= 0 \\
-d + 2e &= 0
\end{align*}
\]

So \( d = 2e \), \( c = 3/4 \), \( d = 2/4 \), \( e = 1/4 \).

Problem Set 1.2, page 19

1 \( u \cdot v = -1.8 + 3.2 = 1.4, u \cdot w = -4.8 + 4.8 = 0, v \cdot w = 24 + 24 = 48 = w \cdot v \).

2 \( \|u\| = 1 \) and \( \|v\| = 5 \) and \( \|w\| = 10 \). Then \( 1.4 < (1)(5) \) and \( 48 < (5)(10) \), confirming the Schwarz inequality.

3 Unit vectors \( v/\|v\| = (\frac{\sqrt{5}}{6}, \frac{\sqrt{6}}{6}) \) and \( w/\|w\| = (\frac{\sqrt{5}}{6}, \frac{\sqrt{6}}{6}) \). The cosine of \( \theta \) is \( \frac{v \cdot w}{\|v\| \cdot \|w\|} = \frac{24}{25} \). The vectors \( w, u, -w \) make \( 0^\circ, 90^\circ, 180^\circ \) angles with \( w \).

4 (a) \( v \cdot (w - v) = -1 \) (b) \( (v + w) \cdot (v - w) = v \cdot v + w \cdot w - v \cdot w - w \cdot v \) (c) \( v \cdot (v + 2w) = v \cdot v + 2v \cdot w = 1 + 2 = 3 \) (d) \( (v - 2w) \cdot (v + 2w) = v \cdot v + 2v \cdot w - 2v \cdot w - 4w \cdot w = 1 - 4 = -3 \).
5 \( u_1 = v / \| v \| = (3, 1) / \sqrt{10} \) and \( u_2 = w / \| w \| = (2, 1, 2) / 3 \). \( U_1 = (1, -3) / \sqrt{10} \) is perpendicular to \( u_1 \) (and so is \( (−1, 3) / \sqrt{10} \)). \( U_2 \) could be \((−1, −2, 0) / \sqrt{5}\). There is a whole plane of vectors perpendicular to \( u_2 \), and a whole circle of unit vectors in that plane.

6 All vectors \( w = (c, 2c) \) are perpendicular to \( v \). All vectors \((x, y, z) \) with \( x + y + z = 0 \) lie on a plane. All vectors perpendicular to \( (1, 1, 1) \) and \( (1, 2, 3) \) lie on a line.

7 (a) \( \cos \theta = v \cdot w / \| v \| \cdot \| w \| = 1 / (2)(1) \) so \( \theta = 60^\circ \) or \( \pi / 2 \) radians (b) \( \cos \theta = 0 \) so \( \theta = 90^\circ \) or \( \pi / 2 \) radians (c) \( \cos \theta = 2 / (2)(2) = 1 / 2 \) so \( \theta = 60^\circ \) or \( \pi / 3 \) (d) \( \cos \theta = −1 / \sqrt{2} \) so \( \theta = 135^\circ \) or \( 3 \pi / 4 \).

8 (a) False: \( v \) and \( w \) are any vectors in the plane perpendicular to \( u \) (b) True: \( u \cdot (v + 2w) = u \cdot v + 2u \cdot w = 0 \) (c) True, \( \| u - v \| = \| (u - v) \cdot (u - v) \) splits into \( u \cdot u + v \cdot v - 2u \cdot v = 2 \) when \( u \cdot v = v \cdot u = 0 \).

9 If \( v_2u_2 / v_1w_1 = −1 \) then \( v_2u_2 = −v_1w_1 \) or \( v_1w_1 + v_2u_2 = v \cdot w = 0 \): perpendicular!

10 Slopes \( 2 \) and \( −1/2 \) multiply to give \( −1 \): then \( v \cdot w = 0 \) and the vectors (the directions) are perpendicular.

11 \( v \cdot w \neq 0 \) means angle \( > 90^\circ \); these \( w \)'s fill half of 3-dimensional space.

12 (1, 1) perpendicular to \((1, 5) - c(1, 1) \) if \( 6 - 2c = 0 \) or \( c = 3 \); \( v \cdot (w - cv) = 0 \) if \( c = v \cdot w / \| v \| \). Subtracting \( c v \) is the key to perpendicular vectors.

13 The plane perpendicular to \((1, 0, 1) \) contains all vectors \((c, d, −c) \). In that plane, \( v = (1, 0, −1) \) and \( w = (0, 1, 0) \) are perpendicular.

14 One possibility among many: \( u = (1, −1, 0, 0), v = (0, 0, 1, −1), w = (1, 1, −1, 1) \) and \((1, 1, 1) \) are perpendicular to each other. “We can rotate those \( u, v, w \) in their 3D hyperplane.”

15 \( \frac{1}{2} (x + y) = (2 + 8) / 2 = 5 \); \( \cos \theta = 2 \sqrt{16} / \sqrt{10} \sqrt{10} = 8 / 10 \).

16 \( \| v \|^2 = 1 + 1 + \cdots + 1 = 9 \) so \( \| v \| = 3 \); \( u = v / 3 = (\frac{1}{3}, \ldots, \frac{1}{3}) \) is a unit vector in 9D; \( w = (1, 1, 0, 0, 0) / \sqrt{2} \) is a unit vector in the 8D hyperplane perpendicular to \( v \).

17 \( \cos \alpha = 1 / \sqrt{2}, \cos \beta = 0, \cos \gamma = −1 / \sqrt{2} \). For any vector \( v, \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = (v_1^2 + v_2^2 + v_3^2) / \| v \|^2 = 1 \).

18 \( \| v \|^2 = 4^2 + 2^2 = 20 \) and \( \| w \|^2 = (−1)^2 + 2^2 = 5 \). Pythagoras is \((3, 4)^2 = 25 = 20 + 5 \).

19 Start from the rules (1), (2), (3) for \( v \cdot w = w \cdot v \) and \( u \cdot (v + w) \) and \( (cv) \cdot w \). Use rule (2) for \( (v + w) \cdot (v + w) = (v + w) \cdot v + (v + w) \cdot w \). By rule (1) this is \( v \cdot (v + w) + w \cdot (v + w) \). Rule (2) again gives \( v \cdot v + v \cdot w + w \cdot v + w \cdot w = v \cdot v + 2v \cdot w + w \cdot w \). Notice \( v \cdot w = w \cdot v \)! The main point is to be free to open up parentheses.

20 We know that \((v - w) \cdot (v - w) = v \cdot v - 2v \cdot w + w \cdot w \). The Law of Cosines writes \( \| v \| \cdot \| w \| \cdot \cos \theta \) for \( v \cdot w \). When \( \theta < 90^\circ \) this \( v \cdot w \) is positive, so in this case \( v \cdot v + w \cdot w \) is larger than \( \| v \|^2 \).

21 \( 2v \cdot w \leq 2 \| v \| \cdot \| w \| \) leads to \( \| v + w \|^2 = v \cdot v + 2v \cdot w + w \cdot w \leq \| v \|^2 + 2 \| v \| \cdot \| w \| + \| w \|^2 \). This is \((\| v \| + \| w \|)^2 \). Taking square roots gives \( \| v + w \| \leq \| v \| + \| w \| \).

22 \( v_1^2 w_1^2 + 2v_1 v_1 w_2 w_2 + v_2^2 w_2^2 \leq v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2 \) is true (cancel 4 terms) because the difference is \( v_1^2 w_1^2 + v_2^2 w_2^2 - 2v_1 v_1 w_1 w_2 = (v_1 w_2 - v_2 w_1)^2 \geq 0 \).
23 cos \( \beta = \|w_1\|/\|w\| \) and sin \( \beta = w_2/\|w\| \). Then 
\[
\cos(\beta - \alpha) = \cos \beta \cos \alpha + \sin \beta \sin \alpha = v_1w_1/\|v\|\|w\| + v_2w_2/\|v\|\|w\| = v \cdot w/\|v\|\|w\|. 
\]
This is \( \cos \theta \) because \( \beta - \alpha = \theta \).

24 Example 6 gives \( |u_1||U_1| \leq 1/2(u_1^2 + U_1^2) \) and \( |u_2||U_2| \leq 1/2(u_2^2 + U_2^2) \). The whole line 
gets .96 \leq (6)(.8) + (.8)(.6) \leq 1/2(6.25 + 8.25) + 1/2(8.25 + .625) = 1. True: .96 < 1.

25 The cosine of \( \theta \) is \( x/\sqrt{x^2 + y^2} \), near side over hypotenuse. Then \( |\cos \theta|^2 \) is not greater than 1: \( x^2/(x^2 + y^2) \leq 1 \).

26 The vectors \( \mathbf{w} = (x, y) \) with \((1, 2) \cdot \mathbf{w} = x + 2y = 5 \) lie on a line in the \( xy \) plane. 
The shortest \( \mathbf{w} \) on that line is \((1, 2) \). (The Schwarz inequality \( \|\mathbf{w}\| \leq \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\| \) is an equality when \( \cos \theta = 0 \) and \( \mathbf{w} = (1, 2) \) and \( \|\mathbf{w}\| = \sqrt{5} \).) 

27 The length \( \|\mathbf{v} - \mathbf{w}\| \) is between 2 and 8 (triangle inequality when \( \|\mathbf{v}\| = 5 \) and \( \|\mathbf{w}\| = 3 \)). The dot product \( \mathbf{v} \cdot \mathbf{w} \) is between \(-15\) and 15 by the Schwarz inequality.

28 Three vectors in the plane could make angles greater than 90° with each other: for example \((1, 0), (-1, 4), (-1, -4)\). Four vectors could not do this (360° total angle).

29 For a specific example, pick \( \mathbf{v} = (1, 2, -3) \) and then \( \mathbf{w} = (-3, 1, 2) \). In this example 
\[
\cos \theta = \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\|\mathbf{w}\| = -7/\sqrt{14}\sqrt{14} = -1/2 \text{ and } \theta = 120°.
\]
This always happens when \( x + y + z = 0 \):
\[
\mathbf{v} \cdot \mathbf{w} = xz + xy + yz = -1/2(x + y + z)^2 - 1/2(x^2 + y^2 + z^2)
\]
This is the same as \( \mathbf{v} \cdot \mathbf{w} = 0 - 1/2 \|\mathbf{v}\|\|\mathbf{w}\|. \) Then \( \cos \theta = 1/2 \).

30 Wikipedia gives this proof of geometric mean \( G = \sqrt[3]{xyz} \leq \) arithmetic mean \( A = (x + y + z)/3 \). First there is equality in case \( x = y = z \). Otherwise \( A \) is somewhere between the three positive numbers, say for example \( z < A < y \).

Use the known inequality \( g \leq a \) for the two positive numbers \( x \) and \( y + z - A \). Their mean \( a = 1/2(x + y + z - A) \) is \( 1/2(3A - A) = \) same as \( A \)!

So \( a \geq g \) says that \( A^3 \geq g^2 A = x(y + z - A)A \). But \( (y + z - A)A = (y - A)(A - z) + yz > yz \).

Substitute to find \( A^3 > xyz = G^3 \) as we wanted to prove. Not easy!

There are many proofs of \( G = (x_1x_2 \cdots x_n)^{1/n} \leq A = (x_1 + x_2 + \cdots + x_n)/n \). In calculus you are maximizing \( G \) on the plane \( x_1 + x_2 + \cdots + x_n = n \). The maximum occurs when all \( x \)’s are equal.

31 The columns of the 4 by 4 “Hadamard matrix” (times \( 1/2 \)) are perpendicular unit vectors:
\[
\frac{1}{2}H = \frac{1}{2} \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

32 The commands \( V = \text{randn}(3, 30); D = \text{sqrt(diag}(V' * V)) \); \( U = V \setminus D \); will give 30 random unit vectors in the columns of \( U \), then \( u' * U \) is a row matrix of 30 dot products whose average absolute value may be close to \( 2/\pi \).
Problem Set 1.3, page 29

1. $2x_1 + 3x_2 + 4x_3 = (2, 5, 9)$. The same vector $b$ comes from $S$ times $x = (2, 3, 4)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row } 1) \cdot x \\ (\text{row } 2) \cdot x \\ (\text{row } 2) \cdot x \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

2. The solutions are $y_1 = 1, y_2 = 0, y_3 = 0$ (right side = column 1) and $y_1 = 1, y_2 = 3, y_3 = 5$. That second example illustrates that the first $n$ odd numbers add to $n^2$.

$$\begin{align*}
y_1 &= B_1 \\
y_1 + y_2 &= B_2 & \text{gives } y_2 &= -B_1 + B_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} B_1 \\
y_1 + y_2 + y_3 &= B_3 & \text{gives } y_3 &= -B_2 + B_3 = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} B_3 \\
\end{align*}$$

The inverse of $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$: independent columns in $A$ and $S$!

3. The combination $0w_1 + 0w_2 + 0w_3$ always gives the zero vector, but this problem looks for other zero combinations (then the vectors are dependent, they lie in a plane):

$$w_2 = (w_1 + w_3)/2$$

so one combination that gives zero is $\frac{1}{2}w_1 - w_2 + \frac{1}{2}w_3$.

4. The rows of the $3 \times 3$ matrix in Problem 4 must also be dependent: $r_2 = \frac{1}{2}(r_1 + r_3)$.

The column and row combinations that produce $0$ are the same: this is unusual.

5. $c = 3 \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$ has column 3 = 2 (column 1) + column 2

$c = -1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ has column 3 = column 1 + column 2

$c = 0 \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ has column 3 = 3 (column 1) - column 2

7. All three rows are perpendicular to the solution $x$ (the three equations $r_1 \cdot x = 0$ and $r_2 \cdot x = 0$ and $r_3 \cdot x = 0$ tell us this). Then the whole plane of the rows is perpendicular to $x$ (the plane is also perpendicular to all multiples $cx$).

$$x_1 - 0 = b_1 \quad x_1 = b_1$$

$$x_2 - x_1 = b_2 \quad x_2 = b_1 + b_2$$

$$x_3 - x_2 = b_3 \quad x_3 = b_1 + b_2 + b_3$$

$$x_4 - x_3 = b_4 \quad x_4 = b_1 + b_2 + b_3 + b_4$$

8. The cyclic difference matrix $C$ has a line of solutions (in 4 dimensions) to $Cx = 0$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{when } x = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$
Problem Set 2.1, page 40

10 \[ z_2 - z_1 = b_1 \quad z_1 = -b_1 - b_2 - b_3 \]
\[ 0 - z_3 = b_3 \quad z_2 = -b_2 - b_3 \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1}b \]

11 The forward differences of the squares are \((t + 1)^2 - t^2 = 2t + 1\). Differences of the \(n\)th power are \((t + 1)^n - t^n = nt^n + \cdots\). The leading term is the derivative \(nt^{n-1}\). The binomial theorem gives all the terms of \((t + 1)^n\).

12 Centered difference matrices of even size seem to be invertible. Look at eqns. 1 and 4:
\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
\]
where the columns are
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
x_1 - x_3 \\
x_2 - x_4 \\
x_3 - x_1 \\
x_4 - x_2
\end{bmatrix}
\]
First solve for each column:
\[
\begin{align*}
x_1 &= b_1 \\
x_2 &= x_2 \\
x_3 &= x_3 \\
x_4 &= x_4
\end{align*}
\]

13 Odd size: The five centered difference equations lead to \(b_1 + b_3 + b_5 = 0\).
\[
\begin{align*}
x_2 &= b_1 \\
x_3 - x_1 &= b_2 \\
x_4 - x_2 &= b_3 \\
x_5 - x_3 &= b_4 \\
-x_4 &= b_5
\end{align*}
\]
Add equations 1, 3, 5. The left side of the sum is zero. The right side is \(b_1 + b_3 + b_5\). There cannot be a solution unless \(b_1 + b_3 + b_5 = 0\).

14 An example is \((a, b) = (3, 6)\) and \((c, d) = (1, 2)\). The ratios \(a/c\) and \(b/d\) are equal. Then \(ad = bc\). Then (when you divide by \(bd\)) the ratios \(a/b\) and \(c/d\) are equal!

Problem Set 2.1, page 40

1 The columns are \(i = (1, 0, 0)\) and \(j = (0, 1, 0)\) and \(k = (0, 0, 1)\) and \(b = (2, 3, 4) = 2i + 3j + 4k\).

2 The planes are the same: \(2x = 4x = 2, 2y = 3y = 9 = 9, 4z = 16 = 4\). The solution is the same point \(X = x\). The columns are changed; but same combination.

3 The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.

4 If \(z = 2\) then \(x + y = 0\) and \(x - y = z\) give the point \((1, -1, 2)\). If \(z = 0\) then \(x + y = 6\) and \(x - y = 4\) produce \((5, 1, 0)\). Halfway between those is \((3, 0, 1)\).

5 If \(x, y, z\) satisfy the first two equations they also satisfy the third equation. The line \(L\) of solutions contains \(v = (1, 1, 0)\) and \(w = (\frac{1}{2}, 1, \frac{1}{2})\) and \(u = \frac{1}{2}v + \frac{1}{2}w\) and all combinations \(cv + dw\) with \(c + d = 1\).

6 Equation 1 + Equation 2 = Equation 3 is now 0 = -4. Line misses plane; no solution.

7 Column 3 = Column 1 makes the matrix singular. Solutions \((x, y, z) = (1, 1, 0)\) or \((0, 1, 1)\) and you can add any multiple of \((-1, 0, 1)\); \(b = (4, 6, c)\) needs \(c = 10\) for solvability (then \(b\) lies in the plane of the columns).

8 Four planes in 4-dimensional space normally meet at a point. The solution to \(Ax = (3, 3, 3, 2)\) is \(x = (0, 0, 1, 2)\) if \(A\) has columns \((1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 0, 0), (1, 1, 1, 1)\). The equations are \(x + y + z + t = 3, y + z + t = 3, z + t = 3, t = 2\).

9 (a) \(Ax = (18, 5, 0)\) and (b) \(Ax = (3, 4, 5, 5)\).