Algebras with inner MB-representation

Abstract

We investigate algebras of sets, and pairs \( \langle A, I \rangle \) consisting of an algebra \( A \) and an ideal \( I \subset A \), that possess an inner MB-representation. We compare inner MB-representability of \( \langle A, I \rangle \) with several properties of \( \langle A, I \rangle \) considered by Baldwin. We show that \( A \) is inner MB-representable if and only if \( A = S(A \setminus \mathcal{H}(A)) \), where \( S(\cdot) \) is a Marczewski operation defined below and \( \mathcal{H} \) consists of sets that are hereditarily in \( A \). We study the question of uniqueness of the ideal in that representation.

1 The implications

Let \( X \) be a nonempty set and let \( F \) be a nonempty family of nonempty subsets of \( X \). Following the idea of Burstin and Marczewski we define:

\[
S(F) = \{ A \subset X : (\forall P \in F)(\exists Q \in F)(Q \subset A \cap P \text{ or } Q \subset P \setminus A) \}
\]

and

\[
S_0(F) = \{ A \subset X : (\forall P \in F)(\exists Q \in F)(Q \subset P \setminus A) \}.
\]
Then \(S(\mathcal{F})\) is an algebra of subsets of \(X\) and \(S_0(\mathcal{F})\) is an ideal on \(X\). (See [BBRW].) For an ideal \(\mathcal{I}\) on \(X\) an algebra \(\mathcal{A}\) of subsets of \(X\) such that \(\mathcal{I} \subset \mathcal{A}\) we say that

- the pair \(\langle \mathcal{A}, \mathcal{I} \rangle\) (respectively, the algebra \(\mathcal{A}\)) has inner MB-representation provided there exists an \(\mathcal{F} \subset \mathcal{A}\) such that \(\mathcal{A} = S(\mathcal{F})\) and \(\mathcal{I} = S_0(\mathcal{F})\) (respectively, \(\mathcal{A} = S(\mathcal{F})\));

- the pair \(\langle \mathcal{A}, \mathcal{I} \rangle\) has density property provided \(\mathcal{I} = S_0(\mathcal{A} \setminus \mathcal{I})\);

- the pair \(\langle \mathcal{A}, \mathcal{I} \rangle\) (respectively, the algebra \(\mathcal{A}\)) is topological provided there exists a topology \(\tau\) on \(X\) such that \(\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle\) (respectively, \(\mathcal{A} = S(\mathcal{F})\)), where \(\mathcal{F} = \tau \setminus \{\emptyset\}\);

- the pair \(\langle \mathcal{A}, \mathcal{I} \rangle\) has the hull property provided for every \(U \subset X\) there is a \(V \in \mathcal{A}\) such that \(U \subset V\) and for every \(W \in \mathcal{A}\) if \(U \subset W\) then \(V \setminus W \in \mathcal{I}\);

- the pair \(\langle \mathcal{A}, \mathcal{I} \rangle\) is complete provided the quotient algebra \(\mathcal{A}/\mathcal{I}\) is complete;

- the pair \(\langle \mathcal{A}, \mathcal{I} \rangle\) has the splitting property provided for every \(C \subset D \subset \mathcal{A}\), if \(D\) is an antichain (i.e., \(A \cap B \in \mathcal{I}\) for every distinct \(A, B \in D\)) then there exists a mapping \(D \ni D \mapsto I_D \in \mathcal{I}\) such that \(C \setminus I_C\) and \(D \setminus I_D\) are disjoint for every \(C \in \mathcal{C}\) and \(D \in \mathcal{D} \setminus \mathcal{C}\).

In the graph from Theorem 2 each of these properties is denoted, respectively, as: inner, dense, top, hull, comp, and split.

We start here with the following simple characterization of pairs with inner MB-representation. (Compare also [Wr, lemma 1].)

**Proposition 1** A pair \(\langle \mathcal{A}, \mathcal{I} \rangle\) has an inner MB-representation if and only if \(\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})\).

**Proof.** If \(\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})\) then \(\mathcal{A} \setminus \mathcal{I} \subset \mathcal{A} \setminus S_0(\mathcal{A} \setminus \mathcal{I})\), since we always have \(\mathcal{F} \cap S_0(\mathcal{F}) = \emptyset\). So, \(S_0(\mathcal{A} \setminus \mathcal{I}) \subset \mathcal{I}\). The other inclusion is obvious. Thus, \(\langle \mathcal{A}, \mathcal{I} \rangle\) has an inner MB-representation.

Conversely, assume that \(\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle\) for some \(\mathcal{F} \subset \mathcal{A}\). By [BBRW, prop. 1.2] to prove that \(S(\mathcal{A} \setminus \mathcal{I}) = S(\mathcal{F})\) it is enough to show that the families \(\mathcal{A} \setminus \mathcal{I}\) and \(\mathcal{F}\) are mutually coinitial, that is, every element of each of these families contains an element from the other.

Clearly, \(\mathcal{F} \subset \mathcal{A} \setminus S_0(\mathcal{F}) = \mathcal{A} \setminus \mathcal{I}\), so every element of \(\mathcal{F}\) contains an element from \(\mathcal{A} \setminus \mathcal{I}\). Conversely, if \(\mathcal{A} \in \mathcal{A} \setminus \mathcal{I}\) then there exists an \(F \in \mathcal{F}\) with \(F \subset \mathcal{A}\), since \(A \notin \mathcal{I} = S_0(\mathcal{F})\).
**Theorem 2** We have the following implications between the properties of a pair \( \langle A, I \rangle \).

\[
\text{hull} \quad \rightarrow \quad \text{inner} \quad \rightarrow \quad \text{dense}
\]

\[
\text{top} \quad \rightarrow \quad \text{hull & comp} \quad \leftrightarrow \quad \text{dense & comp} \quad \leftrightarrow \quad \text{split & inner} \quad \rightarrow \quad \text{dense & split}
\]

\[
\text{comp} \quad \rightarrow \quad \text{split}
\]

**Diagram**

Moreover, none of the implications can be reversed, with possible exception of "top \( \Rightarrow \) hull & comp."

**Proof.** The facts that every topological pair is complete and has the hull property are well known and easy to see. Indeed, if \( \langle A, I \rangle \) is a topological pair generated by a topology \( \tau \) on \( X \) then \( I \) consists of all nowhere dense sets (with respect to \( \tau \)) and \( A \) consists of open sets (with respect to \( \tau \)) modulo \( I \). (See [BR].) Then, for each \( E \subset X \), the closure \( \text{cl}(E) \) plays a role of its hull. Since an open set \( U \) can be expressed as \( U = V \setminus E \) where \( V \) is regular open and \( E \) is nowhere dense (see e.g. [O, thm. 4.5]), the quotient algebra \( A/I \) is isomorphic to the Boolean algebra of regular open sets, which is complete (see e.g. [K]). Hence \( A/I \) is complete.

The implication "inner \( \Rightarrow \) dense" results immediately from Proposition 1 and the definitions. All other implications follow from the following implications proved in Baldwin’s paper [Ba]: "hull \( \Rightarrow \) inner," "comp \( \Rightarrow \) split," "split & inner \( \Rightarrow \) comp,” and “dense & comp \( \Rightarrow \) hull.”

The fact that the implications “top \( \Rightarrow \) hull” and “top \( \Rightarrow \) comp” cannot be reversed follows from Baldwin’s examples from [Ba], where he shows that the properties hull and complete are independent of each other.

An example showing that “dense & split” does not imply “inner” is described in Example 3. This takes care of nonreversability of the implications “split & inner \( \Rightarrow \) dense & split,” “inner \( \Rightarrow \) dense,” and “comp \( \Rightarrow \) split.”

Example 4 shows that the implications “hull \( \Rightarrow \) inner” cannot be reversed.

The following example answers a question of Baldwin [Ba, question 2] whether every pair with density and splitting properties must be inner. Also,
Baldwin had the example of a family with a splitting property which is not complete only under the assumption of the continuum hypothesis, while the example below is in ZFC.

**Example 3** If $X$ is an infinite set, $A$ is an algebra of subsets of $X$ which are either finite or cofinite, and $\mathcal{I}=\{\emptyset\}$ then the pair $\langle A,\mathcal{I} \rangle$ has density and splitting properties but is neither inner nor complete.

**Proof.** The pair $\langle A,\mathcal{I} \rangle$ has density property since $S_0(A\setminus\{\emptyset\})=\{\emptyset\}=\mathcal{I}$. It does not have inner MB-representation by Proposition 1 and the fact that $S(A\setminus\{\emptyset\})=\mathcal{P}(X)$. The splitting property is satisfied trivially, since $\mathcal{I}=\{\emptyset\}$.

The pair $\langle A,\mathcal{I} \rangle$ is not complete by the implications from Theorem 2. ■

The following example answers a question of Baldwin [Ba, question 1] whether every pair with inner MB-representation must have a hull property.

In what follows we use the standard set theoretic notation as in [Ci]. Let $X$ be an infinite set of cardinality $\kappa$. We say that a family $\mathcal{F}_0 \subseteq [X]^\kappa$ is almost disjoint provided $|F_1 \cap F_2| < \kappa$ for every distinct $F_1,F_2 \in \mathcal{F}_0$. 

**Example 4** There exists a maximal almost disjoint family $\mathcal{F}_0 \subseteq [X]^\kappa$ such that for $\mathcal{F}=\{F \triangle A: F \in \mathcal{F}_0 \& A \in [X]^{<\kappa}\}$ the pair $\langle S(\mathcal{F}),S_0(\mathcal{F}) \rangle$ has inner MB-representation but neither is complete nor it has the hull property.

**Proof.** In [BC, fact 4] it was proved that for every $\mathcal{F}$ as in the theorem the algebra $S(\mathcal{F})$ contains $\mathcal{F}$ (so it has inner MB-representation) and $S_0(\mathcal{F})=[X]^{<\kappa}$.

Let $\langle A,B \rangle$ be a partition of $X$ into the sets of cardinality $\kappa$ and let $\mathcal{G} \subseteq [X]^\kappa$ be a partition of $X$ into $\kappa$ many sets such that $|G \cap A|=|G \cap B|=\kappa$ for every $G \in \mathcal{G}$. Let $\mathcal{F}_0 \subseteq [X]^\kappa$ be a maximal almost disjoint family extending $\mathcal{G}$ such that for every $F \in \mathcal{F}_0$ either $F \subseteq A$ or $F \subseteq B$. Such an $\mathcal{F}_0$ exists by the Zorn lemma. It is easy to see that $\mathcal{F}_0$ is a maximal almost disjoint family in $[X]^\kappa$.

To see that $\langle S(\mathcal{F}),S_0(\mathcal{F}) \rangle$ does not have the hull property notice that $A \subseteq X$ does not have a hull. Indeed, take a $V \in S(\mathcal{F})$ containing $A$. Then for every $G \in \mathcal{G} \subseteq \mathcal{F}$ there is an $F_G \in \mathcal{F}$ contained in $G$ such that $F_G$ is either disjoint or contained in $V$. Thus, $F_G = G \setminus A_G$ for some $A_G \in [X]^{<\kappa}$, since elements of $\mathcal{F}_0$ are almost disjoint. This implies also that $F_G = G \setminus A_G$ must be a subset of $V$, since it cannot be disjoint with $V \supseteq A$. In other words, for every $G \in \mathcal{G}$ there exists an $x_G \in G \setminus (V \setminus A)$. So, $Y = \{x_G: G \in \mathcal{G}\} \subseteq [B]^{<\kappa}$, and by the maximality, there exists an $F \in \mathcal{F}_0$ such that $|F \cap Y|=\kappa$. Then, for $W = V \setminus F \in S(\mathcal{F})$ we have $A \subseteq W \subseteq V$, while $V \setminus W = F \cap Y \notin [X]^{<\kappa} = S_0(\mathcal{F})$. Thus, there is no hull for $A$ with respect to $\langle S(\mathcal{F}),S_0(\mathcal{F}) \rangle$. ■

**Problem 5** Is every complete pair $\langle A,\mathcal{I} \rangle$ with the hull property topological?
2 Notes on algebras with inner MB-representations

According to Proposition 1 if a pair \( \langle A, \mathcal{I} \rangle \) has inner MB-representation then it has a canonical one — by a family \( \mathcal{F} = A \setminus \mathcal{I} \). But what if we only consider inner MB-representability of an algebra \( A \)? If \( A \) has an inner MB-representation, say \( A = S(\mathcal{F}) \), then by Proposition 1 for \( \mathcal{I} = S_0(\mathcal{F}) \) we have \( A = S(A \setminus \mathcal{I}) \). Is there a canonical ideal \( \mathcal{I} \) with this property? Is such an ideal unique?

To give a positive answer to the first of these questions we need the following fact. Note that, in general, \( \mathcal{F}_2 \subset \mathcal{F}_1 \) does not imply \( S(\mathcal{F}_2) \subset S(\mathcal{F}_1) \). For instance, if \( X = \{0, 1, 2\} \), \( \mathcal{F}_2 = \{\{0\}\} \), and \( \mathcal{F}_1 = \{\{0\}, \{1, 2\}\} \) then \( \{2\} \in S(\mathcal{F}_2) \setminus S(\mathcal{F}_1) \).

**Lemma 6** If \( \mathcal{I}_1 \subset \mathcal{I}_2 \) are ideals contained in an algebra \( A \) then we have \( S(A \setminus \mathcal{I}_2) \subset S(A \setminus \mathcal{I}_1) \).

**Proof.** Let \( A \in S(A \setminus \mathcal{I}_2) \). To show that \( A \in S(A \setminus \mathcal{I}_1) \) take a \( P \in A \setminus \mathcal{I}_1 \). We need to find a \( Q \in A \setminus \mathcal{I}_1 \) for which

\[
\text{either } Q \subset P \cap A \text{ or } Q \subset P \setminus A. \tag{1}
\]

If \( P \in A \setminus \mathcal{I}_2 \) then clearly there is a \( Q \in A \setminus \mathcal{I}_2 \subset A \setminus \mathcal{I}_1 \) satisfying (1). So assume that \( P \notin A \setminus \mathcal{I}_2 \). Then \( P \in \mathcal{I}_2 \setminus \mathcal{I}_1 \). So, \( P \cap A \) and \( P \setminus A \) belong to \( \mathcal{I}_2 \) and at least one of them does not belong to \( \mathcal{I}_1 \). This set can be taken as \( Q \), since \( \mathcal{I}_2 \setminus \mathcal{I}_1 \subset A \setminus \mathcal{I}_1 \).

For an algebra \( A \) of subsets of \( X \), the ideal of hereditary sets in \( A \) is defined as \( \mathcal{H}(A) = \{ A \in A : \mathcal{P}(A) \subset A \} \).

**Proposition 7** Let \( \mathcal{I} \) be an ideal on a set \( X \), let \( A \) be an algebra on \( X \) and assume that \( \mathcal{I} \subset A = S(\mathcal{A} \setminus \mathcal{I}) \neq \mathcal{P}(X) \). Then for every ideal \( \mathcal{J} \) such that \( \mathcal{I} \subset \mathcal{J} \subset \mathcal{H}(A) \) we have \( A = S(A \setminus \mathcal{J}) \).

**Proof.** Notice that any ideal \( \mathcal{J} \subset A \) is a proper subset of \( A \) since \( A \neq \mathcal{P}(X) \). It is easy to see that for any such ideal we have \( A \subset S(A \setminus \mathcal{J}) \). Indeed, if \( A \in A \) and \( P \in A \setminus \mathcal{J} \) then either \( Q = P \setminus A \) belongs to \( A \setminus \mathcal{J} \) or \( Q = P \cap A \) belongs to \( A \setminus \mathcal{J} \). Now, by Lemma 6, we have

\[
A \subset S(A \setminus \mathcal{H}(A)) \subset S(A \setminus \mathcal{J}) \subset S(A \setminus \mathcal{I}) = A.
\]

This finishes the proof.

The proposition implies immediately the following corollary, which shows, in particular, that the ideal \( \mathcal{I} = \mathcal{H}(A) \) is canonical ideal in representation \( A = S(A \setminus \mathcal{I}) \).
Corollary 8 An algebra $A \not= \mathcal{P}(X)$ has an inner MB-representation if and only if $A = S(A \setminus H(A))$.

Notice that Corollary 8 immediately implies [BBC, thm. 13], since conditions (I) and (II) from that theorem say that $H(A) = A \cap [X]^{<\kappa}$ while (III) says that $S(A \setminus H(A)) \setminus A \not= \emptyset$. In particular, Corollary 8 implies easily that the following algebras do not have inner MB-representation:

- The algebra $\mathcal{B}$ of Borel subset of $\mathbb{R}$, since $S(\mathcal{B} \setminus H(\mathcal{B})) = S(\mathcal{B} \setminus [\mathbb{R}]^{\leq \omega})$ is a classical Marczewski’s algebra. (Compare [BBC, cor. 14].)
- The interval algebra $A$ (i.e., generated by all intervals $[a, b)$, where $a, b \in \mathbb{R}$), since $H(A) = \{\emptyset\}$ and so $S(A \setminus H(A))$ is an algebra of subsets of $\mathbb{R}$ with nowhere dense boundary. (Compare [BBC, prop. 12].)
- The algebra $A$ generated by all open intervals $(a, b)$ $(a, b \in \mathbb{R})$, since $H(A) = [\mathbb{R}]^{<\omega}$ and so $S(A \setminus H(A))$ is an algebra of subsets of $\mathbb{R}$ with nowhere dense boundary.

Next, we will address the question of uniqueness of the ideal in the representation $A = S(A \setminus H(A))$. We will start with the following proposition.

Proposition 9 Let $A$ be an algebra, let $I \subset J \subset A$ be ideals, and $Y \in A$.

(a) If every $P \subset Y$ from $A \setminus J$ contains a subset in $I \setminus J$ then $\mathcal{P}(Y) \subset S(A \setminus J)$.

(b) If $I \cap \mathcal{P}(Y) = J \cap \mathcal{P}(Y)$ then $S(A \setminus I) \cap \mathcal{P}(Y) = S(A \setminus J) \cap \mathcal{P}(Y)$.

Proof. (a): Let $A \in \mathcal{P}(Y)$ and take $P \in A \setminus J$. We need to find a $Q \in A \setminus J$ for which

either $Q \subset P \cap A$ or $Q \subset P \setminus A$.

If $P \in I \setminus J$ then either $P \cap A$ or $P \setminus A$ belongs to $I \setminus J$, so we may take this set as a $Q$. So, assume that $P \in A \setminus I$ then there is a $P_0 \in I \setminus J$ contained in $P$. Thus, as before, either $P_0 \cap A$ or $P_0 \setminus A$ belongs to $I \setminus J$ and we may take this set as a $Q$.

Part (b) is obvious.

For an algebra $A \subset \mathcal{P}(X)$ and the ideals $I$ and $J$ such that $J \subset I \subset A$ a set $Y \in A$ will be called $(I, J)$-special if $I \cap \mathcal{P}(X \setminus Y) = J \cap \mathcal{P}(X \setminus Y)$ and each set $P \subset Y$ such that $P \in A \setminus J$ has a subset in $I \setminus J$.

From Proposition 9 we easily derive the following corollary.
Corollary 10 Let $A$ be an algebra on $X$ and let $J \subset I \subset A$ be ideals. If $Y \in A$ is an $\langle I, J \rangle$-special set then

$$S(A \setminus J) = \{C \cup D: C \in \mathcal{P}(Y) \& D \in \mathcal{P}(X \setminus Y) \cap S(A \setminus J)\}.$$  

From Proposition 9(a) applied to $Y = \mathbb{R}$ we obtain immediately the following facts.

- If $\mathcal{L}$ is the algebra of Lebesgue measurable subsets of $\mathbb{R}, \mathcal{N}$ is the ideal of measure zero sets, and $\mathcal{N}_0$ is the ideal generated by $F_\sigma$ sets from $\mathcal{N}$ then $S(\mathcal{L} \setminus J) = \mathcal{P}(\mathbb{R})$ for every ideal $J$ contained either in $\mathcal{N}_0$ or in $\mathcal{N} \cap [\mathbb{R}]^{<2^\omega}$.
- If $\mathcal{B}$ is the algebra of subsets of $\mathbb{R}$ with the Baire property and $\mathcal{M}$ is the ideal of meager sets, then $S(\mathcal{B} \setminus J) = \mathcal{P}(\mathbb{R})$ for every ideal $J$ contained either in $\mathcal{N}_0$ or in $\mathcal{M} \cap [\mathbb{R}]^{<2^\omega}$.

From Corollary 10 we immediately see that, most of the time, $\mathcal{H}(A)$ is not the only ideal $I$ for which $A = S(A \setminus I)$. The easiest way to see it is to notice the following conclusion from Corollary 10.

Corollary 11 If $A$ is an algebra on $X, J \subset I \subset A$ are ideals, $A = S(A \setminus I)$ and there exists a $Y \in I$ such that $I \cap \mathcal{P}(X \setminus Y) = J \cap \mathcal{P}(X \setminus Y)$, then $S(A \setminus I) = S(A \setminus J)$.

Finally we note that the existence of an $\langle I, J \rangle$-special set is by no means necessary for the conclusion of Corollary 11.

Example 12 There exists an algebra $A$ and an ideal $J \subset \mathcal{H}(A)$ for which $A = S(A \setminus J)$ while there is no $\langle \mathcal{H}(A), J \rangle$-special set $Y \in \mathcal{H}(A)$.

Proof. In the papers [R] and [NR] the authors investigated the family $\mathcal{D}$ of perfect subsets of $[\omega]^{<\omega}$, where $[\omega]^{<\omega}$ is endowed with the Ellentuck topology, that is, the topology generated by the sets $[x, y] = \{z \in [\omega]^{\omega}: x \subset z \subset y\}$, where $x \in [\omega]^{\omega}$ and $y \in [\omega]^{\omega}$. A subset of $[\omega]^{\omega}$ is called a chain if it consists of sets incomparable with respect to inclusion. A chain is called a Sorgenfrey chain if its subspace topology is homeomorphic to the Sorgenfrey topology on $(0, 1]$. It is shown in [NR, thm. 3.4] that if $P \in \mathcal{D}$ does not contain a countable perfect set then $P$ contains a perfect uncountable Sorgenfrey chain.

Let $\mathcal{G}$ be the family of all perfect Sorgenfrey chains and let $A = S(\mathcal{D})$. By [NR, thm. 3.5] and [R, cor. 1.10], we have $A = S(\mathcal{D}) = S(\mathcal{G})$ and $J = S_0(\mathcal{D}) \subset S_0(\mathcal{G}) = \mathcal{H}(A)$. We will show that

(a) $A = S(A \setminus J)$, and
(b) \( \mathcal{A} = S(A \setminus \mathcal{H}(A)) \), but

(c) there is no \( \langle \mathcal{H}(A), \mathcal{J} \rangle \)-special set \( Y \in \mathcal{H}(A) \).

To prove (a) observe that \( D \subseteq S(D) \) since, for any two perfect sets \( P \) and \( Q \), at least one of the sets \( P \cap Q, P \setminus Q \) has a perfect part. Now, from \( D \subseteq S(D) \) and \( D \cap S_0(D) = \emptyset \) it follows that \( D \) and \( A \setminus \mathcal{J} = S(D) \setminus S_0(D) \) are mutually coinitial which, by [BBRW, prop. 1.2], implies (a). The clause (b) results from (a) and Proposition 7.

To prove (c), by way of contradiction assume that there is a \( \langle \mathcal{H}(A), S_0(D) \rangle \)-special set \( Y \in \mathcal{H}(A) \). Then \( \mathcal{H}(A) \cap \mathcal{P}(\omega^\omega \setminus Y) = S_0(D) \cap \mathcal{P}(\omega^\omega \setminus Y) \). Since \( \mathcal{H}(A) = S_0(\mathcal{G}) \), we have

\[
S_0(\mathcal{G}) \cap \mathcal{P}(\omega^\omega \setminus Y) = S_0(D) \cap \mathcal{P}(\omega^\omega \setminus Y). \tag{2}
\]

Next observe that

(d) each set from \( D \cap \mathcal{P}(\omega^\omega \setminus Y) \) contains a set from \( \mathcal{G} \).

Indeed, let \( D \in D \cap \mathcal{P}(\omega^\omega \setminus Y) \). Since \( D \subseteq S(D) \setminus S_0(D) \), it follows from \( S(D) = S(\mathcal{G}) \) and (2) that

\[
D \in (S(D) \setminus S_0(D)) \cap \mathcal{P}(\omega^\omega \setminus Y) = (S(\mathcal{G}) \setminus S_0(\mathcal{G})) \cap \mathcal{P}(\omega^\omega \setminus Y).
\]

Hence by [BBRW, prop 1.1(4)], there is a \( G \in \mathcal{G} \) such that \( G \subseteq D \) as desired.

Since \( \mathcal{G} \) consists of uncountable sets, from (d) we derive that no countable perfect set in \( \omega^\omega \) is contained in \( \omega^\omega \setminus Y \). From [NR] it follows that each nonempty open set in \( \omega^\omega \) contains a set from \( \mathcal{G} \). Thus \( Y \), which is in \( \mathcal{H}(A) = S_0(\mathcal{G}) \), has the empty interior. Consequently, \( \omega^\omega \setminus Y \) is dense and so, by [R, thm. 1.5], it contains a countable perfect set \( Q \). However, this contradicts the previous observation.

\[ \blacksquare \]

References


