Errata: MB-representations and topological algebras

Theorem 5(b) from [BC] says the following:

Let $|X| = \kappa \geq \omega$, $\mathcal{F}_0 \subset [X]^{<\kappa}$ be an almost disjoint family, and $\mathcal{F} = \{F \triangle A : F \in \mathcal{F}_0 \& A \in [X]^{<\kappa}\}$. If $|\mathcal{F}_0| > \kappa$ then the algebra $S(\mathcal{F}) = \{A \subset X : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset A \cap P \text{ or } Q \subset P \setminus A)\}$ is not topological, that is, $A \neq S(\tau \setminus \{\emptyset\})$ for any topology $\tau$ on $X$.

Unfortunately, the printed proof has a gap. (It shows only that the pair $\langle S(\mathcal{F}), S\!_0(\mathcal{F}) \rangle$ is not topological.) A correct proof for this result follows.

Proof. By way of contradiction suppose that there exists a topology $\tau$ on $X$ such that $S(\mathcal{F}) = S(\tau_0)$, where $\tau_0 = \tau \setminus \{\emptyset\}$.

Notice that for every $F \in \mathcal{F}$ we have $F \in S(\mathcal{F}) = S(\tau_0)$ and

$$U \notin S(\mathcal{F}) \text{ for every } U \in [F]^{<\kappa} \text{ with } |F \setminus U| = \kappa. \quad (1)$$

In particular, $\mathcal{P}(F) \not\subset S(\tau_0)$, so $F$ does not belong to $S_0(\tau_0)$, which is defined as $\{A \subset X : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset P \setminus A)\}$. Thus,

$$\text{int}_\tau(F) \neq \emptyset \text{ for every } F \in \mathcal{F}. \quad (2)$$

For every $F \in \mathcal{F}_0$ let $V_F$ be a maximal pairwise disjoint subfamily of $\tau \cap [F]^{<\kappa}$ and notice that $|\bigcup V_F| < \kappa$. Indeed, otherwise we could find a subfamily $V$ of $V_F$ with $|\bigcup V| = |F \setminus \bigcup V| = \kappa$. But then $U = \bigcup V \in \tau \subset S(\tau_0) = S(\mathcal{F})$ would contradict $(1)$. So, $F \setminus \bigcup V_F \in \mathcal{F}$ and $V_F = \text{int}_\tau(F \setminus \bigcup V_F)$ is nonempty by $(2)$. To finish the proof it is enough to notice that $\{V_F : F \in \mathcal{F}_0\}$ is a family of nonempty pairwise disjoint subsets of $X$, contradicting the fact that $|\mathcal{F}_0| > \kappa = |X|$.

References


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