Find maximum and minimum values of a function over a closed interval

**Facts:** Let \( f(x) \) be a function on \([a, b]\) and \( c \) is a point in the interval \([a, b]\).

1. If for any point \( x \) in \([a, b]\), \( f(x) \geq f(c) \) (respectively, \( f(x) \leq f(c) \)), then \( f(c) \) is the **absolute (or global) minimum value** (respectively, **absolute (or global) local maximum value**) of \( f(x) \) on \([a, b]\).
2. If \( a < c < b \), and for any point \( x \) in an open interval containing \( c \), \( f(x) \geq f(c) \) (respectively, \( f(x) \leq f(c) \)), then \( f(c) \) is a **local minimum value** of \( f(x) \) (respectively, **local maximum value**) on \([a, b]\).
3. If \( f(x) \) is continuous on \([a, b]\) and differentiable in \((a, b)\), a point \( c \) in \([a, b]\) is a **critical point** of \( f(x) \) if either \( f'(c) \) does not exist, or \( f'(x) = 0 \).
4. **Important:** If \( f(x) \) is continuous on \([a, b]\) and differentiable in \((a, b)\), and if for some \( c \) in \((a, b)\), \( f(c) \) is a local maximum or local minimum, then \( c \) must be a critical point. Any absolute maximum or minimum must take place at critical points inside the interval or at the boundaries point \( a \) or \( b \).

**Example 1** State whether the function \( f(x) = |x - 2| \) attains a maximum value or a minimum value in the interval \((1, 4]\).

**Solution:** Apply the definition of absolute value to get

\[
 f(x) = \begin{cases} 
 2 - x & \text{if } 2 \leq x \leq 4, \\
 2x - 2 & \text{if } 1 < x < 2. 
\end{cases}
\]

Thus the graph of this function consists of two pieces of lines, and so the minimum value \( f(2) = 0 \) @ \( x = 2 \), and the maximum value is \( f(4) = 2 \) @ \( x = 4 \).

**Example 2** Find the maximum value and the minimum value attained by \( f(x) = \frac{1}{x(1-x)} \) in the interval \([2, 3]\).

**Solution:** Note that the domain of \( f(x) \) does not contain \( x = 0 \) and \( x = 1 \), and these points are not in the interval \([2, 3]\).

(Step 1) Find critical points. Compute

\[
 f'(x) = -\frac{1 - 2x}{x^2(1-x)^2} = \frac{2x - 1}{x^2(1-x)^2}.
\]
Therefore, the only possible critical point is \( x = \frac{1}{2} \). As this point is not in the interval \([2, 3]\), it is not a critical point.

(Step 2) Compute \( f(x) \) at the critical point(s) and at the boundaries of the closed interval.

\[
\begin{align*}
  f(2) &= \frac{1}{2(1 - 2)} = -\frac{1}{2}, \\
  f(3) &= \frac{1}{3(1 - 3)} = -\frac{1}{6}.
\end{align*}
\]

(Step 3) Compare the data resulted in Step 2 to make conclusions.

\( f(x) \) attains its absolute maximum value \( f(3) = -\frac{1}{6} \) @ \( x = 3 \) and \( f(x) \) attains its absolute minimum value \( f(2) = -\frac{1}{2} \) @ \( x = 2 \).

**Example 3**  Find the maximum value and the minimum value attained by \( f(x) = x^2 + \frac{1}{6} \) in the interval \([1, 3]\).

**Solution:** Note that the domain of \( f(x) \) does not contain \( x = 0 \), and this point is not in the interval \([1, 3]\).

(Step 1) Find critical points. Compute

\[
f'(x) = 2x - \frac{16}{x^2} = \frac{2x^3 - 16}{x^2}.
\]

Set \( f'(x) = 0 \). As a fraction equals zero if and only if its numerator equals zero, we have \( 2x^3 - 16 = 0 \), and so the only possible critical point is \( x = 2 \). As this point is in the interval \([1, 3]\), it is a critical point.

(Step 2) Compute \( f(x) \) at the critical point(s) and at the boundaries of the closed interval.

\[
\begin{align*}
  f(1) &= 1 + \frac{16}{1} = 17, \\
  f(2) &= 2^2 + \frac{16}{2^2} = 8, \\
  f(3) &= 3^2 + \frac{16}{3^2} = 9 + \frac{16}{9} = \frac{97}{9}.
\end{align*}
\]

(Step 3) Compare the data resulted in Step 2 to make conclusions.

Note that \( 8 < \frac{97}{9} < 17 \), and so \( f(x) \) attains its absolute maximum value \( f(1) = 17 \) @ \( x = 1 \) and \( f(x) \) attains its absolute minimum value \( f(2) = 8 \) @ \( x = 2 \).
Example 4 Find the maximum value and the minimum value attained by \( f(x) = x^{\frac{1}{2}} - x^{\frac{3}{2}} \) in the interval \([0, 4]\).

Solution: Note that the domain of \( f(x) \) does not contain any negative number, and so the function is continuous on \([0, 4]\).

(Step 1) Find critical points. Compute

\[
\frac{df}{dx} = \frac{1}{2} x^{-1/2} - \frac{3}{2} x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{1 - 3x}{2\sqrt{x}}.
\]

Set \( f'(x) = 0 \). As a fraction equals zero if and only if its numerator equals zero, we have \( 1 - 3x = 0 \), and so the \( x = \frac{1}{3} \) is a critical point. Since \( f'(x) \) does not exist at \( x = 0 \), but \( f(x) \) is (right) continuous at \( x = 0 \), both \( \frac{1}{3} \) and 0 are critical points.

(Step 2) Compute \( f(x) \) at the critical point(s) and at the boundaries of the closed interval.

\[
\begin{align*}
    f(0) &= 0 - 0 = 0, \\
    f\left(\frac{1}{3}\right) &= \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{1}{2\sqrt{3}}, \\
    f(4) &= 4^{\frac{1}{2}} - 4^{\frac{3}{2}} = 2 - 8 = -6.
\end{align*}
\]

(Step 3) Compare the data resulted in Step 2 to make conclusions.

Note that \(-6 < 0 < \frac{1}{2\sqrt{3}}\), and so \( f(x) \) attains its absolute maximum value \( f\left(\frac{1}{3}\right) = \frac{1}{2\sqrt{3}} \) \(@ x = \frac{1}{3} \) and \( f(x) \) attains its absolute minimum value \( f(4) = -6 \) \(@ x = 4 \).