Finding minimum areas

Strategy to tackle the problem

(1) Identify the variable to be minimized (here will be the area $A$), and assign symbols to other given quantities. (Give the variables names).

(2) Identify the relation that allow us to write down an equality (referred as the primary equation) expressing $A$ in terms of other quantities. (Usually such a relationship can be found from facts in geometry).

(3) Reduce the primary equation into one that express $A$ in terms of a single independent variable. (This may involve the use of a secondary equation). Determine the domain. (Where can the independent variable take values from?)

(4) Use calculus (derivative, critical numbers, comparison or appropriate discussions we learn from calculus).

Example 1 A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are each $1\frac{1}{2}$ inches. The margins on each side are 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

Solution: (1) Let $A$ denote the area and let $x$ and $y$ denote the length (top-bottom) and the width (side-side) of the printed area of the page, respectively. Then the length of the page will be $1\frac{1}{2} + x + 1\frac{1}{2} = x + 3$. Similarly, the width of the page is $y + 2$.

(2) Express the area of the page in terms of $x$ and $y$. From geometry, the primary equation is $A = (x + 3)(y + 2)$. (3) We recognize that we need to express $y$ in terms of $x$. As the page should have 24 square inches of printed area, it gives a secondary equation $24 = xy$, and so $y = \frac{24}{x}$. Substitute $y = \frac{24}{x}$ into the primary equation, we
have

\[ A = A(x) = (x + 3) \left( \frac{24}{x} + 2 \right) = 30 + 2x + \frac{72}{x}. \]

Since \( x \) is a length, \( x > 0 \). Thus the domain of the function \( A(x) \) is \( x > 0 \), or \([0, \infty)\).

(4) Now we have successfully model the problem into one that finds the absolute maximum of a function \( A(x) \) on a closed interval \([0, \infty)\). To apply calculus to find the maximum volume, we first compute the derivative \( A'(x) = 2 - 27/x^2 \). Set \( A'(x) = 0 \) to get the critical numbers \( x = 6 \) (the other solution \( x = -6 \) is not in \([0, \infty)\)). Apply the first derivative test to conclude that \( x = 6 \) and \( y = \frac{24}{6} = 4 \). Thus the dimensions of the page are \( x + 3 = 9 \) and \( y + 2 = 6 \).