Compute double integrals in polar coordinates

**Useful facts:** Suppose that \( f(x, y) \) is continuous on a region \( R \) in the plane \( z = 0 \).

1. If the region \( R \) is bounded by \( \alpha \leq \theta \leq \beta \) and \( a \leq r \leq b \), then
   \[
   \int_\alpha^\beta \int_a^b f(x, y) \, dx \, dy = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.
   \]

2. If the region \( R \) is bounded by \( \alpha \leq \theta \leq \beta \) and \( r_1(\theta) \leq r \leq r_2(\theta) \) (called a radially simple region), then
   \[
   \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.
   \]

**Example (1)** Find the volume of a sphere of radius \( a \) by double integration.

**Solution:** We can view that the center of the sphere is at the origin \((0, 0, 0)\), and so the equation of the sphere is \( x^2 + y^2 + z^2 = a^2 \). We then can compute the volume of the upper half part of the sphere and multiply our answer by 2.

\[
V = 2 \int_0^a \int_{\sqrt{a^2 - x^2}}^{\sqrt{a^2 - y^2}} \sqrt{a^2 - x^2 - y^2} \, dy \, dx.
\]

To compute this integral, we observe that the polar coordinates may be a better mechanism in this case. With polar coordinates, the function \( z = \sqrt{a^2 - x^2 - y^2} \) becomes \( z = \sqrt{a^2 - r^2} \), over the region \(-\pi \leq \theta \pi \) and \( 0 \leq r \leq a \). Therefore, using polar coordinates, we have (using \( u = a^2 - r^2 \) and \( 2rdr = -du \) to start with)

\[
V = 2 \int_{-\pi}^{\pi} \int_0^a \sqrt{a^2 - u} r \, dr \, d\theta = \int_{-\pi}^{\pi} \int_0^a u^{1/2} \, du \, d\theta = 2\pi \frac{2a^3}{3} = \frac{4\pi a^3}{3}.
\]

**Example (2)** Find the area of the region \( R \) bounded by one loop of \( r = 2 \cos 2\theta \).

**Solution:** In the interval \([-\pi, \pi]\) of \( \theta \), \( \cos 2\theta = 0 \) exactly at \( \theta = \pm \frac{\pi}{4} \) and \( \theta = \pm \frac{3\pi}{4} \). For one loop, this is the case when \( \alpha = -\frac{\pi}{4} \) and \( \beta = \frac{\pi}{4} \), while \( r_1 = 0 \) and \( r_2 = 2 \cos 2\theta \). Use the fact that \( \sin \frac{3\pi}{4} = 1 \) to get

\[
A = \int_\frac{\pi}{4}^{\frac{3\pi}{4}} \int_0^{2\cos 2\theta} r \, dr \, d\theta = \int_\frac{\pi}{4}^{\frac{3\pi}{4}} 2\cos^2 2\theta \, d\theta = \frac{\pi}{2}.
\]

**Example (3)** Find the area of the region \( R \) inside the smaller loop of \( r = 1 - 2 \sin \theta \).

**Solution:** In the interval \([-\pi, \pi]\) of \( \theta \), \( \sin \theta = \frac{1}{2} \) exactly at \( \theta = \pm \frac{\pi}{4} \) and \( \theta = \pm \frac{3\pi}{4} \). For the smaller loop, this is the case when \( \alpha = \frac{\pi}{4} \) and \( \beta = \frac{3\pi}{4} \), while \( r_1 = 0 \) and \( r_2 = 1 - 2 \sin \theta \). Thus

\[
A = \int_\frac{\pi}{4}^{\frac{3\pi}{4}} \int_0^{1 - 2\sin \theta} r \, dr \, d\theta = \int_\frac{\pi}{4}^{\frac{3\pi}{4}} (1 - 2\sin \theta)^2 \, d\theta = \frac{2\pi - 3\sqrt{3}}{2}.
\]

**Example (4)** Find the volume of the solid that lies below the surface \( z = x^2 + y^2 \) over the region \( R \) bounded by \( r = 2 \cos \theta \).
Solution: In the interval $[-\pi, \pi]$ of $\theta$, $\cos \theta = 0$ exactly at $\theta = \pm \frac{\pi}{2}$. This is the case when $\alpha = -\frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$, while $r_1 = 0$ and $r_2 = 2 \cos \theta$. Thus

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2 \cos \theta} r^3 dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2 \cos \theta)^4}{4} d\theta = 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2\theta)^2}{4} \theta = \frac{3\pi}{2}.$$

Example (5) Evaluate the double integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{4-x^2-y^2}} dy dx.$$

Solution: Change to polar coordinates. Then

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{4-x^2-y^2}} dy dx = \int_0^\frac{\pi}{2} \int_0^{1-r^2} \frac{1}{4-r^2} dr d\theta = \frac{\pi}{2} \int_0^1 \frac{1}{4-r^2} rd\theta = \frac{\pi(2-\sqrt{3})}{2}.$$

Example (6) Find the volume of the solid that lies below the surface $z = 1 + x$ and above the plane $z = 0$ over the region $R$ bounded by $r = 1 + \cos \theta$.

Solution: In the interval $[-\pi, \pi]$ of $\theta$, $\cos \theta = -1$ exactly at $\theta = \pm \pi$. This is the case when $\alpha = -\pi$ and $\beta = \pi$, while $r_1 = 0$ and $r_2 = 1 + \cos \theta$. Thus

$$V = \int_{-\pi}^\pi \int_0^{1+\cos \theta} (1 + r \cos \theta) rd\theta d\theta = \int_{-\pi}^\pi \left[ \frac{(1 + \cos \theta)^2}{2} + \frac{(1 + \cos \theta)^3}{3} \cos \theta \right] d\theta = \frac{1}{6} \int_{-\pi}^\pi (3 + 9 \cos^2 \theta + 2 \cos^4 \theta) d\theta = \frac{11 \pi}{4}.$$

Example (7) Find the volume of the solid bounded by the paraboloid $z = 12 - 2x^2 - y^2$ and $z = x^2 + 2y^2$.

Solution: The intersection of the two surfaces, when projected down to the $z = 0$ plane, is the common solution of both $z = 12 - 2x^2 - y^2$ and $z = x^2 + 2y^2$, which is a curve with equation $3x^2 + 3y^2 = 14$, or $x^2 + y^2 = 4$ on the plane $z = 0$. In terms of polar coordinates, the region $R$ bounded by this curve (a circle centered at the origin with radius 2) is also bounded by $-\pi \leq \theta \leq \pi$ and $0 \leq r \leq 2$. The top surface is $z = 12 - 2x^2 - y^2$ and the bottom one is $z = x^2 + 2y^2$. Thus

$$V = \int_{-\pi}^\pi \int_0^2 (12 - 2x^2 - y^2 - x^2 - 2y^2) rd\theta dr = 3 \int_{-\pi}^\pi \int_0^2 (4 - x^2 - y^2) rd\theta dr$$

$$= 3 \int_{-\pi}^\pi \int_0^2 (4 - r^2) rd\theta dr = 6 \pi \left[ \frac{2r^2 - r^4}{4} \right]_0^2 = 24 \pi.$$