Compute partial derivatives with Chain Rule

**Formulae:** These are the most frequently used ones:

1. If \( w = f(x, y) \) and \( x = x(t) \) and \( y = y(t) \) such that \( f, x, y \) are all differentiable. Then
   \[
   \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}.
   \tag{1}
   \]

2. If \( w = f(x_1, x_2, \cdots, x_m) \) and for each \( i, \ (1 \leq i \leq n), \ x_i = x_i(t_1, t_2, \cdots, t_n) \) such that \( f, x_1, \cdots, x_m \) are all differentiable. Then
   \[
   \frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial t_i}.
   \tag{2}
   \]

**Example (1):** Given \( w = \ln(u + v + z) \), with \( u = \cos^2 t, \ v = \sin^2 t \) and \( z = t^2 \), find \( dw/dt \) both by using the chain rule and by expressing \( w \) explicitly as a function of \( t \) before differentiating.

**Solution:** First we apply Chain Rule (1):
\[
\frac{dw}{dt} = \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = \frac{1}{u + v + z} (-2 \cos t \sin t + 2 \sin t \cos t + 2t) = \frac{2t}{u + v + z}.
\]

Next, we express \( w \) as a function of \( t \) by substituting \( u, v \) and \( z \) into \( w \) to get \( w = \ln(\cos^2 t + \sin^2 t + t^2) = \ln(1 + t^2) \). Therefore, \( dw/dt = 2t/(1 + t^2) \).

**Example (2):** Given \( w = yz + zx + xy, \ x = s^2 - t^2, \ y = s^2 + t^2 \) and \( z = s^2 t^2 \), find \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \).

**Solution:** This is a partial derivative problem and so we apply Chain Rule (2).
\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} = (z + y)(2s) + (z + x)2s + (x + y)2t^2
\]
\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} = -(z + y)(2t) + (z + x)2t + (x + y)2s^2 t
\]

**Example (3):** Given \( p = f(x, y, z), \ x = x(u, v), \ y = y(u, v) \) and \( z = z(u, v) \), write the chain rule formulas giving the partial derivatives of the dependent variable \( p \) with respect to each independent variable.

**Solution:** This is a partial derivative problem and so we apply Chain Rule (2).
\[
\frac{\partial p}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}
\]
\[
\frac{\partial p}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v}
\]
Example (4): Given \( x^3 + y^3 + z^3 = xyz \), find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) as functions of \( x, y \) and \( z \).

Solution: As \( z \) is an implicit function of \( x \) and \( y \), implicit differentiation must be used. Just view \( z = z(x, y) \) everywhere \( z \) occurs when we differentiate both sides of the equation.

(Step 1) View \( z = z(x, y) \) and differentiate both sides of the equation with respect to \( x \) to get
\[
3x^2 + 3z^2 \frac{\partial z}{\partial x} = yz + xy \frac{\partial z}{\partial x}.
\]

(Step 2) Solve for \( \frac{\partial z}{\partial x} \) in the resulting equation above.
\[
\frac{\partial z}{\partial x} = \frac{yz - 3x^2}{3z^2 - xy}.
\]

Do the same for \( y \).

(Step 1) View \( z = z(x, y) \) and differentiate both sides of the equation with respect to \( y \) to get
\[
3y^2 + 3z^2 \frac{\partial z}{\partial y} = xz + xy \frac{\partial z}{\partial y}.
\]

(Step 2) Solve for \( \frac{\partial z}{\partial y} \) in the resulting equation above.
\[
\frac{\partial z}{\partial y} = \frac{xz - 3y^2}{3z^2 - xy}.
\]

Example (5): Given \( x^3 + y^3 + z^3 = xyz \) as the equation of a surface, find an equation of the plane tangent to this surface at the point \( P(1, -1, -1) \).

Solution 1: View \( z = z(x, y) \), and so the vector \( n = (z_x, z_y, -1) \) at \( P \) will be a normal vector of the tangent plane. From Example (4) above, we have, at \( P(1, -1, -1) \),
\[
z_x = \left. \frac{yz - 3x^2}{3z^2 - xy} \right|_{(x,y,z)=(1,-1,-1)} = \frac{(-1)(-1) - 3}{3(-1)^2 - 1(-1)} = -\frac{2}{4} = -\frac{1}{2},
\]
\[
z_y = \left. \frac{xz - 3y^2}{3z^2 - xy} \right|_{(x,y,z)=(1,-1,-1)} = \frac{1(-1) - 3}{3(-1)^2 - 1(-1)} = -\frac{4}{4} = -1.
\]
Hence the equation of the tangent plane is
\[
-\frac{1}{2}(x - 1) - (y + 1) - (z + 1) = 0, \text{ or } (x - 1) + 2(y + 1) + 2(z + 1) = 0.
\]

Solution 2: One can also set \( F(x, y, z) = x^3 + y^3 + z^3 - xyz \) and view the equation of the surface is \( F(x, y, z) = 0 \). In this case, the vector \( u = (F_x, F_y, F_z) \) at \( P(1, -1, -1) \) can be a normal vector of the tangent plane. We compute the partial derivatives:
\[
F_x = \frac{\partial F}{\partial x} = 3x^2 - yz, \quad F_y = \frac{\partial F}{\partial y} = 3y^2 - xz, \quad F_z = \frac{\partial F}{\partial z} = 3z^2 - xy.
\]
Therefore, \( \mathbf{u} = (2, 4, 4) \) and so an equation of the tangent plane is

\[
2(x - 1) + 4(y + 1) + 4(z + 1) = 0, \quad \text{or} \quad (x - 1) + 2(y + 1) + 2(z + 1) = 0.
\]

**Example (6)**: Suppose that \( w = f(u) \) and that \( u = x + y \). Show that \( \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} \).

**Solution**: Notice that \( \frac{\partial u}{\partial x} = 1 = \frac{\partial u}{\partial y} \). By Chain Rule (2),

\[
\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial w}{\partial u} = \frac{\partial w}{\partial y}.
\]

**Example (7)**: Suppose that \( w = f(x, y) \) and that \( x = r \cos \theta \) and \( y = r \sin \theta \). Show that

\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2}.
\]

**Solution**: Apply Chain Rule (2) to compute the first order of partial derivatives

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} = w_x \cos \theta + w_y \sin \theta
\]

\[
\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta} = -w_x r \sin \theta + w_y r \cos \theta.
\]

Then, we apply Chain Rule (2) again to compute the second order of partial derivatives (making use of \( w_{xy} = w_{yx} \)),

\[
\frac{\partial^2 w}{\partial r^2} = w_{xx} \cos^2 \theta + w_{xy} \cos \theta \sin \theta + w_{yx} \sin \theta \cos \theta + w_{yy} \sin^2 \theta = w_{xx} \cos^2 \theta + (w_{xy} + w_{yx}) \cos \theta \sin \theta + w_{yy} \sin^2 \theta,
\]

and

\[
\frac{\partial^2 w}{\partial \theta^2} = w_{xx} r^2 \sin^2 \theta - w_{xy} r^2 \sin \theta \cos \theta - w_{yx} r^2 \cos \theta + w_{yy} r^2 \cos^2 \theta + w_{yz} r^2 \cos \theta \sin \theta - w_{xz} r \cos \theta - w_{yz} r \sin \theta.
\]

It follows that

\[
\frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = \frac{w_{xx} \sin^2 \theta + w_{yy} \cos^2 \theta - (w_{xy} + w_{yx}) \sin \theta \cos \theta - w_x \cos \theta + w_y \sin \theta}{r}.
\]

\[
\frac{1}{r} \frac{\partial w}{\partial r} = \frac{w_x \cos \theta + w_y \sin \theta}{r}.
\]

Now add equations (3), (4) and (5) side by side and apply \( \sin^2 \theta + \cos^2 \theta = 1 \) to get the conclusion.