Convert equations from one coordinate system to another: I

Useful Facts

<table>
<thead>
<tr>
<th>Cylindrical</th>
<th>Rectangle</th>
<th>Spherical</th>
<th>Rectangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r^2 = x^2 + y^2 )</td>
<td>( x = r \cos \theta )</td>
<td>( \rho^2 = x^2 + y^2 + z^2 )</td>
<td>( x = \rho \sin \phi \cos \theta )</td>
</tr>
<tr>
<td>( \theta = \tan^{-1} \frac{y}{x} )</td>
<td>( y = r \sin \theta )</td>
<td>( \phi = \tan^{-1} \sqrt{\frac{x^2+y^2}{z}} )</td>
<td>( y = \rho \sin \phi \sin \theta )</td>
</tr>
<tr>
<td>( z = z )</td>
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</tbody>
</table>

Example (1) : Describe the graph \( r = 5 \) in cylindrical coordinates.

Solution: As \( z \) and \( \theta \) can take any values, the graph of \( r = 5 \) is an infinite cylinder with \( z \)-axis as the axis of the cylinder, and every point on the graph has distance 5 to the \( z \)-axis. Notice that the straight line \( L : x = \sqrt{5}, y = 0, z = t \) is on this graph, this graph can also be obtained by rotating \( L \) about the \( z \)-axis.

Example (2) : Describe the graph \( \theta = \frac{\pi}{4} \) in cylindrical (or spherical) coordinates.

Solution: As \( z \) and \( r \) can take any values, the graph of \( \theta = \frac{\pi}{4} \) consists of all the points in the space whose \( \theta \) value is \( \frac{\pi}{4} \), and so it is a plane that contains the \( z \)-axis.

Remark As \( \theta \) in cylindrical coordinates represents the same measure as in spherical coordinates, in this example, \( \theta = \frac{\pi}{4} \) in spherical coordinates has the same graph \( y = x \).

Example (3) : Describe the graph \( \phi = \frac{\pi}{6} \) in spherical coordinates.

Solution: As \( \rho \) and \( \theta \) can take any valid values, the graph of \( \phi = \frac{\pi}{6} \) consists of all the points in the space whose \( \phi \) value is \( \frac{\pi}{6} \), and so it is a (two penning) cone with its vertex at the origin, and with its axis being the \( z \)-axis.

Remark As \( \theta \) in cylindrical coordinates represents the same measure as in spherical coordinates, in this example, \( \theta = \frac{\pi}{4} \) in spherical coordinates has the same graph \( y = x \).

One can also use algebraic techniques to see what the graph is like. Apply the formula \( r = z \tan \phi \) and \( r^2 = x^2 + y^2 \), and substitute \( \phi \) by \( \frac{\pi}{6} \) (knowing that \( \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \)). This yields the equation for the graph

\[
\frac{x^2}{3} + \frac{y^2}{3} = \frac{z^2}{3}.
\]

Therefore, the graph can be obtained from the straight line \( z = \sqrt{3}x \) on the \( xz \)-plane by rotating this line about the \( z \)-axis.