Compute unit tangent and unit normal vectors, tangential and normal components (for 2D vectors)

Example: Find the unit tangent and unit normal vectors, tangential and normal components of the curve \( x = t - \sin t, y = 1 - \cos t \) at \( t = \frac{\pi}{2} \).

Solution: The position vector is \( \mathbf{r}(t) = (t - \sin t, 1 - \cos t) \).

(Step 1) Compute the velocity vector \( \mathbf{v}(t) = \mathbf{r}'(t) = (1 - \cos t, \sin t) \), and the speed \( |\mathbf{v}| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t} \).

(Step 2) Compute the unit tangent vector:

\[
\mathbf{T}(t) = \frac{1}{|\mathbf{v}|} \mathbf{v} = \left( \frac{1 - \cos t}{\sqrt{2 - 2\cos t}}, \frac{\sin t}{\sqrt{2 - 2\cos t}} \right).
\]

When \( t = \frac{\pi}{2} \), \( \cos(\frac{\pi}{2}) = 0 \) and \( \sin(\frac{\pi}{2}) = 1 \). Thus \( |\mathbf{v}| = \sqrt{2} \), ans so \( \mathbf{T}(\frac{\pi}{2}) = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \).

(Step 3) Compute the acceleration vector and the tangential component:

\[
\mathbf{a}(t) = \mathbf{v}'(t) = (\sin t, \cos t).
\]

\[
a_T = \frac{d|\mathbf{v}|}{dt} = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = \frac{\sin t - \sin t \cos t + \sin t \cos t}{\sqrt{2 - 2\cos t}} = \frac{\sin t}{\sqrt{2 - 2\cos t}}.
\]

When \( t = \frac{\pi}{2} \), \( \mathbf{a} = (1, 0) \) and \( a_T = \frac{1}{\sqrt{2}} \).

(Step 4) Compute, at \( t = \frac{\pi}{2} \), (view the vectors as 3D vectors) \( \mathbf{v} \times \mathbf{a} = (1, 1, 0) \times (1, 0, 0) = (0, 0, -1) \). Then use it to compute the curvature

\[
\kappa(t) = \frac{1}{|\mathbf{v}|} \frac{\mathbf{T}}{dt} = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} , \ \kappa\left( \frac{\pi}{2} \right) = \frac{1}{(\sqrt{2})^3} ,
\]

and the normal component at \( t = \frac{\pi}{2} \),

\[
a_N = \kappa v^2 = \frac{1}{(\sqrt{2})^3} (\sqrt{2})^2 = \frac{1}{\sqrt{2}} .
\]

(Step 5) Compute the unit normal vector at \( t = \frac{\pi}{2} \),

\[
\mathbf{N} = \frac{1}{a_N} (\mathbf{a} - a_T \mathbf{T}) = \sqrt{2} \left( 1, 0 \right) - \frac{\sqrt{2}}{\sqrt{2}} \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \left( \sqrt{2} - 1, -1 \right) .
\]