Arc length and surface area computing

1. Arc length computing. Using the a line segment to approximate a small arc piece, the length of the small arc piece can be approximated by

\[ ds \simeq \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy. \]

The total length of the whole arc can then be obtained by adding up all lengths of the small arc pieces in the Riemann sum sense under a limiting process, which leads to (with \( x \) bounds given as an example)

\[ \text{Arc length} = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx. \]

2. Surface area computing. The surface generated by rotating a smooth arc \( C \) around an axis. Then area of the corresponding surface generated by a small piece of arc with length \( ds \) equals

\[ dA = 2\pi \left( \text{distance from the arc piece to the ration axis}\right) ds. \]

The area of the whole surface is then (assuming the axis of rotation is parallel to the \( x \)-axis, and the arc \( C \) has \( x \) bounds \( a \) and \( b \))

\[ \text{Area} = 2\pi \int_{a}^{b} \left( \text{distance from the arc piece to the ration axis}\right) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx. \]

If the axis of rotation is parallel to the \( y \)-axis, then changes should be made accordingly.

Example 1 Find the length of an arc \( C \) which is given by \( y = \frac{1}{6} x^3 + \frac{1}{2}x \) from \( x = 1 \) to \( x = 3 \).

Solution: First compute \( \frac{dy}{dx} \), and \( ds \):

\[ \frac{dy}{dx} = \frac{1}{2} x^2 - \frac{1}{2} x^{-2}. \]

Thus

\[ \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2} x^2 - \frac{1}{2} x^{-2}\right)^2 = \frac{1}{4} x^4 - \frac{2}{4} + \frac{1}{4} x^{-4}, \]

and so

\[ ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{\frac{(x^2 + x^{-2})^2}{4}} \, dx = \frac{x^2 + x^{-2}}{2} \, dx \]

It follows that

\[ \text{Arc length} = \int_{1}^{3} \frac{x^2 + x^{-2}}{2} \, dx = \frac{14}{3}. \]
Example 2  Find the length of an arc $C$ which is given by $x = \frac{2}{3}(y - 1)^{\frac{3}{2}}$ from $y = 1$ to $y = 5$.

Solution: First compute $\frac{dx}{dy}$, and $ds$:

\[
\frac{dx}{dy} = \frac{2}{3} \frac{3}{2} (y - 1)^{\frac{1}{2}}, \quad \text{and so} \quad \left(\frac{dx}{dy}\right)^2 = y - 1.
\]

Thus

\[
\text{Arc length} = \int_1^5 \sqrt{1 + (y - 1)}\,dy = \left[\frac{2y^{\frac{3}{2}}}{3}\right]_1^5 = \frac{10\sqrt{5} - 2}{3}.
\]

Example 3  Find the area of the surface of revolution generated by revolving the curve $C$ which is given by $y = x^3$, $1 \leq x \leq 2$ about the $x$-axis.

Solution: First compute $\frac{dy}{dx}$, and $ds$:

\[
\frac{dy}{dx} = 3x^2, \quad \text{and so} \quad ds = \sqrt{1 + 9x^4}\,dx.
\]

For each $x$ with $1 \leq x \leq 2$, the distance from the corresponding arc piece to the axis of rotation is $x^3$. Thus

\[
\text{Surface area} = 2\pi \int_1^2 x^3 \sqrt{1 + 9x^4}\,dx = \frac{\pi}{27}(145\sqrt{145} - 10\sqrt{10}).
\]

Example 4  Set up and simplify the integral that gives the surface area of revolution generated by rotation of the curve $y = x^2$, $0 \leq x \leq 4$ about the $y$-axis. (No need to evaluate the integral.)

Solution: First compute $\frac{dy}{dx}$, and $ds$:

\[
\frac{dy}{dx} = 2x, \quad \text{and so} \quad ds = \sqrt{1 + 4x^2}\,dx.
\]

For each $x$ with $0 \leq x \leq 4$, the distance from the corresponding arc piece to the axis of rotation is $x$. Thus

\[
\text{Surface area} = 2\pi \int_0^4 x \sqrt{1 + 4x^2}\,dx.
\]

Example 5  Set up and simplify the integral that gives the surface area of revolution generated by rotation of the curve $y = x^2$, $0 \leq x \leq 4$ about the $x$-axis. (No need to evaluate the integral.)

Solution: First compute $\frac{dy}{dx}$, and $ds$:

\[
\frac{dy}{dx} = 2x, \quad \text{and so} \quad ds = \sqrt{1 + 4x^2}\,dx.
\]

For each $x$ with $0 \leq x \leq 4$, the distance from the corresponding arc piece to the axis of rotation is $y$ which is $x^2$. Thus

\[
\text{Surface area} = 2\pi \int_0^4 x^2 \sqrt{1 + 4x^2}\,dx.
\]
Example 6  Set up and simplify the integral that gives the surface area of revolution generated by rotation of the curve $y = x^2$, $0 \leq x \leq 4$ about the line $x = 2$. (No need to evaluate the integral.)

Solution: First compute $\frac{dy}{dx}$, and $ds$:

$$\frac{dy}{dx} = 2x,$$ and so $ds = \sqrt{1 + 4x^2}dx$.

For each $x$ with $0 \leq x \leq 4$, the distance from the corresponding arc piece to the axis of rotation is $2 - x$. Thus

$$\text{Surface area} = 2\pi \int_0^4 (2 - x)\sqrt{1 + 4x^2}dx.$$

Example 7  Set up and simplify the integral that gives the surface area of revolution generated by rotation of the curve $y = x^2$, $0 \leq x \leq 4$ about the line $y = 4$. (No need to evaluate the integral.)

Solution: First compute $\frac{dy}{dx}$, and $ds$:

$$\frac{dy}{dx} = 2x,$$ and so $ds = \sqrt{1 + 4x^2}dx$.

For each $x$ with $0 \leq x \leq 4$, the distance from the corresponding arc piece to the axis of rotation is $4 - x^2$. Thus

$$\text{Surface area} = 2\pi \int_0^4 (4 - x^2)\sqrt{1 + 4x^2}dx.$$